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On Interval Valued Fuzzy Bi-interior Ideals in Semigroups

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Abstract. In this paper, we study the concept of an interval valued fuzzy bi-interior ideal. We investigate the properties of an interval valued fuzzy bi-interior ideal in semigroups. We characterize a regular semigroup in terms of an interval valued fuzzy bi-interior ideal.

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1. Introduction

Uncertainties cannot be handled using traditional mathematical tools but maybe deal with using a wide range of existing theories such as probability theory, theory of fuzzy sets, interval valued fuzzy sets. In 1975, Zadeh [11] introduced the theory of interval valued fuzzy sets as a generalization of the notion of fuzzy sets. Interval valued fuzzy sets have various applications in several areas like medical science [2], image processing [1], decision making [12], etc. In 2006, Narayanan and Manikantan [9] for the first time employed the theory of interval valued fuzzy subsemigroup and studied types of interval valued fuzzy ideals in semigroups. In 2018, MK. Rao [5] introduced and studied the definition and properties of the bi-interior ideal in the semigroup. In 2019, A. Mahboob et al. [8] characterizations of regular ordered semigroups by $(\varepsilon, \varepsilon \bigvee_{(k,qk)})$-fuzzy quasi ideals. G Muhiuddin et al. discussed a new type of fuzzy semiprime subsets in ordered semigroups. Many researchers studied in interval valued fuzzy semigroup such that in 2020 Ahsan et al. [6] extend the ideals of (m, n)-ideals in semigroups to fuzzy sets in semigroup and they characterize the regular semigroup by using fuzzy (m, n)-ideals. I. Crista et al. [3] studied a new type fuzzy quasi-ideal in ordered semigroups. In 2021 T. Gaketem [4], studied interval valued fuzzy almost (m, n)-bi-ideal in semigroups. A. Mahboob and G. Muhiuddin [7] studied a fuzzy prime subset in ordered semigroups.

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In this work, we establish the concept of an interval valued fuzzy bi-interior ideal. We investigate the properties of an interval valued fuzzy bi-interior ideal in semigroups. Finaly, we characterize a regular semigroup in terms of an interval valued fuzzy bi-interior ideal in semigroups.

2. Preliminaries

In this section, we begin with elementary some fundamental concepts about semigroups, fuzzy sets, and interval valued fuzzy sets that are necessary for this paper.

By a *subsemigroup* of a semigroup S we mean a non-empty subset M of S such that $M^2 \subseteq M$, and by a *left (right) ideal* of S we mean a non-empty subset M of S such that $SM \subseteq M(MS \subseteq M)$. By a *two-sided ideal* or simply an ideal, we mean a non-empty subset of a semigroup S that is both a left and a right ideal of S. A non-empty subset M of S is called a *quasi-ideal* of S if $MS \cap SM \subseteq M$. A subsemigroup M of S is called a *bi-ideal* of S if $MSM \subseteq M$. A subsemigroup M of a semigroup S is called an interior ideal of S if $SMS \subseteq M$. A subsemigroup M of a semigroup S is said to be a *bi-interior ideal* of S if M is a subsemigroup of S and $SMS \cap MSM \subseteq M$.[5]. We note here that the properties is hold:

(1) Every left ideal is a bi-interior ideal of S.

(2) Every right ideal is a bi-interior ideal of S.

(3) Every ideal is a bi-interior ideal of S.

(4) Every quasi ideal is a bi-interior ideal of S.

(5) The arbitrary intersection of bi-interior of S is also bi-interior ideal of S.

(6) If M is a bi-interior ideal of S then MS and SM are bi-interior ideals of S [5].

Definition 1. *[10] A* fuzzy subset η *of a non-empty set* X *is a function* $\eta : X \to [0, 1]$.

For any $\eta_i \in [0, 1]$ where $i \in \mathcal{A}$ define

$$\underset{i\in\mathcal{A}}{\vee}\eta_i := \sup_{i\in\mathcal{A}}\{\eta_i\} \quad \text{and} \quad \underset{i\in\mathcal{A}}{\wedge}\eta_i := \inf_{i\in\mathcal{A}}\{\eta_i\}.$$

We see that for any $\eta_1, \eta_2 \in [0, 1]$, we have

$$\eta_1 \vee \eta_2 = \max\{\eta_1, \eta_2\} \quad \text{and} \quad \eta_1 \wedge \eta_2 = \min\{\eta_1, \eta_2\}.$$

Let $\Omega[0, 1]$ be the set of all closed subintervals of $[0, 1]$, i.e.,

$$\Omega[0, 1] = \{\tilde{p} = [p^-, p^+] \mid 0 \leq p^- \leq p^+ \leq 1\}.$$

Let $\tilde{p} = [p^-, p^+]$ and $\tilde{q} = [q^-, q^+] \in \Omega[0, 1]$. Define the operations $\preceq$, $=$, $\curlywedge$ and $\curlyvee$ as follows:

(1) $\tilde{p} \preceq \tilde{q}$ if and only if $p^- \leq q^-$ and $p^+ \leq q^+$

(2) $\tilde{p} = \tilde{q}$ if and only if $p^- = q^-$ and $p^+ = q^+$

(3) $\tilde{p} \curlywedge \tilde{q} = [(p^- \wedge q^-), (p^+ \wedge q^+)]$

(4) $\tilde{p} \curlyvee \tilde{q} = [(p^- \vee q^-), (p^+ \vee q^+)]$.

If $\tilde{p} \succeq \tilde{q}$, we mean $\tilde{q} \preceq \tilde{p}$.

For each interval $\tilde{p}_i = [p_i^-, p_i^+] \in \mu[0,1]$, $i \in \mathcal{A}$ where $\mathcal{A}$ is an index set, we define

$$\underset{i\in\mathcal{A}}{\curlywedge} \tilde{p}_i = [\underset{i\in\mathcal{A}}{\wedge} p_i^-, \underset{i\in\mathcal{A}}{\wedge} p_i^+] \quad \text{and} \quad \underset{i\in\mathcal{A}}{\curlyvee} \tilde{p}_i = [\underset{i\in\mathcal{A}}{\vee} p_i^-, \underset{i\in\mathcal{A}}{\vee} p_i^+].$$

Definition 2. *[9] Let T be a non-empty set. Then the function $\tilde{\mu} : T \to \Omega[0,1]$ is called an* interval valued fuzzy set *(shortly, IVF set) of T.*

Definition 3. *[9] Let M be a subset of a non-empty set T. An* interval valued characteristic function *of T is defined to be a function $\tilde{\chi}_M : T \to \Omega[0,1]$ by*

$$\tilde{\chi}_M(e) = \begin{cases} [1,1] & \text{if} \quad e \in M, \\ [0,0] & \text{if} \quad e \notin M \end{cases}$$

for all $e \in T$.

For two IVF sets $\tilde{\mu}$ and $\tilde{\varpi}$ of a non-empty set T, define

(1) $\tilde{\mu} \sqsubseteq \tilde{\varpi} \Leftrightarrow \tilde{\mu}(e) \preceq \tilde{\varpi}(e)$ for all $e \in T$,

(2) $\tilde{\mu} = \tilde{\varpi} \Leftrightarrow \tilde{\mu} \sqsubseteq \tilde{\varpi}$ and $\tilde{\varpi} \sqsubseteq \tilde{\mu}$,

(3) $(\tilde{\mu} \sqcap \tilde{\varpi})(e) = \tilde{\mu}(e) \curlywedge \tilde{\varpi}(e)$ for all $e \in T$,

(4) $(\tilde{\mu} \sqcup \tilde{\varpi})(e) = \tilde{\mu}(e) \curlyvee \tilde{\varpi}(e)$ for all $e \in T$.

For two IVF sets $\tilde{\mu}$ and $\tilde{\varpi}$ in a semigroup S, define the product $\tilde{\mu} \circ \tilde{\varpi}$ as follows : for all $e \in S$,

$$(\tilde{\mu} \circ \tilde{\varpi})(e) = \begin{cases} \bigcup\limits_{e=th} \{\tilde{\mu}(t) \curlywedge \tilde{\varpi}(h)\}, \\ [0,0]. \end{cases}$$

Definition 4. *[9] An IVF subset $\tilde{\mu}$ of a semigroup S is said to be*

(1) *an* IVF subsemigroup *of S if $\tilde{\mu}(uv) \succeq \tilde{\mu}(u) \curlywedge \tilde{\mu}(v)$ for all $u, v \in S$,*

(2) *an* IVF left (right) ideal *of S if $\tilde{\mu}(uv) \succeq \tilde{\mu}(v)(\tilde{\mu}(uv) \succeq \tilde{\mu}(u))$ for all $u, v \in S$. An IVF subset $\tilde{\mu}$ of S is called an* IVF ideal *of S if it is both an IVF left ideal and an IVF right ideal of S,*

(3) *an* IVF bi-ideal *of S if $\tilde{\mu}$ is an IVF subsemigroup and $\tilde{\mu}(uvw) \succeq \tilde{\mu}(u) \curlywedge \tilde{\mu}(w)$ for all $u, v, w \in S$,*

(4) *an* IVF interior ideal *of S if $\tilde{\mu}$ is an IVF subsemigroup and $\tilde{\mu}(uav) \succeq \tilde{\mu}(a)$ for all $a, u, v \in S$,*

(5) *an* IVF quasi-ideal *of S if $(\tilde{\mathcal{S}} \circ \tilde{\mu})(u) \curlywedge (\tilde{\mu} \circ \tilde{\mathcal{S}})(u) \preceq \tilde{\mu}(u)$ for all $u \in S$ where $\tilde{\mathcal{S}}$ is an IVF subset of S mapping every element of S on $[1, 1]$.*

Theorem 1. *[9] Let S be a semigroup and let M be non-empty subset of S. Then M is a subsemigroup (left ideals, right ideals, interior ideals, bi-ideals, quasi-ideals) of S if and only if the characteristic set $\tilde{\chi}_M$ is an IVF subsemigroup (left ideals, right ideals, interior ideals, bi-ideals, quasi-ideals) of S.*

3. Interval valued fuzzy bi-interior ideals of Semigroups

In this section, we introduce the notion of an interval valued fuzzy bi-interior ideal and study the properties of interval valued fuzzy bi-interior ideals of semigroups.

Definition 5. *An IVF subsemigroup $\tilde{\mu}$ of a semigroup S is called an IVF bi-interior ideal of S if it satisfies the following condition: $\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S} \sqcap \tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu} \tilde{\subseteq} \tilde{\mu}$,*

Example 1. *Define $\tilde{\mu} : T \to \Omega[0, 1]$ by*

$$\tilde{\mu}(e) = \begin{cases} [1, 1] & \text{if} \quad e \in T, \\ [0, 0] & \text{if} \quad e \notin T \end{cases}$$

Then $\tilde{\mu}$ is an IVF bi-interior ideal of T.

The next Theorems are studies IVF ideals in semigroup are IVF bi-interior ideals of semigroups

Theorem 2. *Every IVF left ideal of a semigroup S is an IVF bi-interior ideal of S.*

Proof. Let $\tilde{\mu}$ be an IVF left ideal of S. Let $x \in S$. Then

$$(\tilde{\chi_S} \circ \tilde{\mu})(x) = \bigcup_{x=yz} \{\tilde{\chi_S}(y) \curlywedge \tilde{\mu}(z)\} = \bigcup_{x=yz} \{\tilde{\mu}(z)\} \subseteq \bigcup_{x=yz} \{\tilde{\mu}(yz)\} = \bigcup_{x=yz} \{\tilde{\mu}(x)\} = \tilde{\mu}(x).$$

We have, $(\tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu})(x) = \bigcup_{x=abc} \{\tilde{\mu}(a) \curlywedge (\tilde{\chi_S} \circ \tilde{\mu})(bc)\} \subseteq \bigcup_{x=abc} \{\tilde{\mu}(a) \curlywedge \tilde{\mu}(bc)\} = \tilde{\mu}(x)$. Now

$$\begin{aligned}(\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S} \sqcap \tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu})(x) &= (\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S})(x) \sqcap (\tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu})(x) \\ &\preceq (\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S})(x) \sqcap \tilde{\mu}(x) \preceq \tilde{\mu}(x).\end{aligned}$$

Therefor $\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S} \sqcap \tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu} \sqsubseteq \tilde{\mu}$. Hence $\tilde{\mu}$ is an IVF bi-interior ideal of S.

Theorem 3. *Every IVF right ideal of a semigroup S is an IVF bi-interior ideal of S.*

Proof. It follows Theorem 2.

Corollary 1. *Every IVF ideal of a semigroup S is an IVF bi-interior ideal of S.*

Definition 6. *Let $\tilde{\mu}$ be an IVF set in a non-empty set X. Define $U(\tilde{\mu};\tilde{t}) = \{x \in X | \tilde{t} \subseteq \tilde{\mu}(x)\}$ where $\bar{t} \in \Omega[0,1]$ is called the IVF level set of $\tilde{\mu}$.*

Theorem 4. *Let S be a semigroup and $\tilde{\mu}$ be a non-empty IVF set of S. An IVF set $\tilde{\mu}$ is an IVF bi-interior ideal of a semigroup S if and only if the IVF level set $U(\tilde{\mu};\tilde{t})$ of S is a bi-interior ideal of a semigroup S for every $\tilde{t} \in \Omega[0,1]$, where $U(\tilde{\mu};\tilde{t}) \neq \emptyset$.*

Proof. Assume that $\tilde{\mu}$ is an IVF bi-interior ideal of S and let $x \in SU(\tilde{\mu};\tilde{t})S \cap U(\tilde{\mu};\tilde{t})SU(\tilde{\mu};\tilde{t})$. Then $x = bau = cde$ where $b,u,d \in S$ and $a,c,e \in U(\tilde{\mu};\tilde{t})$. Then $\tilde{t} \preceq (\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S})(x)$ and $\tilde{t} \preceq (\tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu})(x)$ implies that $\tilde{t} \preceq \tilde{\mu}(x)$ Then $x \in U(\tilde{\mu};\tilde{t})$. Therefore $U(\tilde{\mu};\tilde{t})$ is a bi-interior ideal of S.

Conversely suppose that $U(\tilde{\mu};\tilde{t})$ is a bi-interior ideal of S, for all $\tilde{t} \in Im(\tilde{\mu})$.
Let $x,y \in S$. Then $\tilde{\mu}(x) = \tilde{t}_1, \tilde{\mu}(y) = \tilde{t}_2, \tilde{t}_1 \succeq \tilde{t}_2$. Then $x,y \in U(\tilde{\mu};\tilde{t})$.
Thus $SU(\tilde{\mu};\tilde{l})S \sqcap U(\tilde{\mu};\tilde{l})SU(\tilde{\mu};\tilde{l}) \preceq U(\tilde{\mu};\tilde{l})$, for all $\tilde{l} \in Im(\tilde{\mu})$.
Suppose $\tilde{t} = \min\{Im(\tilde{\mu})\}$. Then $SU(\tilde{\mu};\tilde{t})S \sqcap U(\tilde{\mu};\tilde{t})SU(\tilde{\mu};\tilde{t}) \preceq U(\tilde{\mu};\tilde{t})$.
Therefor $\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S} \sqcap \tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu} \sqsubseteq \tilde{\mu}$. Hence $\tilde{\mu}$ is an IVF bi-interior ideal of S.

Theorem 5. *Let M be a non-empty subset of a semigroup S and $\tilde{\chi}_M$ be the characteristic IVF set of M. Then M is a bi-interior ideal of a semigroup S if and only if $\tilde{\chi}_M$ is an IVF bi-interior ideal of a semigroup S.*

Proof. Suppose M is a bi-interior ideal of S. Then M is a subsemigroup of S. Thus by Theorem 1, $\tilde{\chi}_M$ is an IVF subsemigroup of S. Let $x \in S$. Since M is a bi-interior ideal of S, we have $SMS \cap MSM \subseteq M$. Thus

$$\begin{aligned}(\tilde{\chi}_S \circ \tilde{\chi}_M \circ \tilde{\chi}_S \sqcap \tilde{\chi}_M \circ \tilde{\chi}_S \circ \tilde{\chi}_M)(x) &= (\tilde{\chi}_S \circ \tilde{\chi}_M \circ \tilde{\chi}_S)(x) \curlywedge (\tilde{\chi}_M \circ \tilde{\chi}_S \circ \tilde{\chi}_M)(x)\\ &= \tilde{\chi}_{SMS}(x) \curlywedge \chi_{M\tilde{S}M}(x)\\ &= \tilde{\chi}_{SIS \cap MSM}(x) \preceq \tilde{\chi}_M(x).\end{aligned}$$

Therefore $\tilde{\chi}_S \circ \tilde{\chi}_M \circ \tilde{\chi}_S \sqcap \tilde{\chi}_M \circ \tilde{\chi}_S \circ \tilde{\chi}_M \sqsubseteq \tilde{\chi}_M$.
Hence $\tilde{\chi}_M$ is an IVF bi-interior ideal of S.

Conversely, suppose that $\tilde{\chi}_M$ is an IVF bi-interior ideal of S.
Then $\tilde{\chi}_M$ is an IVF subsemigroup of S. Thus by Theorem 1,
M is a subsemigroup of S. Let $x \in M$. We have

$$\begin{aligned}&(\tilde{\chi}_S \circ \tilde{\chi}_M \circ \tilde{\chi}_S)(x) \curlywedge (\tilde{\chi}_M \circ \tilde{\chi}_S \circ \tilde{\chi}_M)(x) \preceq \tilde{\chi}_M(x)\\ &\Rightarrow \tilde{\chi}_{SMS}(x) \curlywedge \tilde{\chi}_{MSM}(x) \preceq \tilde{\chi}_M(x)\\ &\Rightarrow \tilde{\chi}_{SMS \cap MSM}(x) \preceq \tilde{\chi_M}(x).\end{aligned}$$

Therefore $SMS \cap MSM \subseteq M$. Hence M is a bi-interior ideal of S.

Theorem 6. *If $\tilde{\mu}$ and $\tilde{\lambda}$ are IVF bi-interior ideals of a semigroup S, then $\tilde{\mu} \sqcap \tilde{\lambda}$ is an IVF bi-interior ideal of S.*

Proof. Let $\tilde{\mu}$ and $\tilde{\lambda}$ be IVF bi-interior ideals of S. Then

$$\begin{aligned}
(\tilde{\chi}_S \circ \tilde{\mu} \tilde{\cap} \tilde{\lambda})(x) &= \bigcup_{x=ab} \{\tilde{\chi}_S(a) \curlywedge (\tilde{\mu} \tilde{\cap} \tilde{\lambda})(b)\} \\
&= \bigcup_{x=ab} \{\tilde{\chi}_S(a) \curlywedge \tilde{\mu}(b) \curlywedge \tilde{\lambda}(b)\} \\
&= \bigcup_{x=ab} \{\{\tilde{\chi}_S(a) \curlywedge \tilde{\mu}(b)\} \curlywedge \{\tilde{\chi}_S(a) \cap \tilde{\lambda}(b)\}\} \\
&= \bigcup_{x=ab} \{\tilde{\chi}_S(a) \curlywedge \tilde{\mu}(b)\} \cap \bigcup_{x=ab} \{\tilde{\chi}_S(a) \cap \tilde{\lambda}(b)\} \\
&= (\tilde{\chi}_S \circ \tilde{\mu})(x) \curlywedge (\tilde{\chi}_S \circ \tilde{\lambda})(x) \\
&= (\tilde{\chi}_S \circ \tilde{\mu} \tilde{\cap} \tilde{\chi}_S \circ \tilde{\lambda})(x).
\end{aligned}$$

Therefore $\tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\lambda} = \tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\chi}_S \circ \tilde{\lambda}$.

$$\begin{aligned}
(\tilde{\mu} \tilde{\cap} \tilde{\lambda} \circ \tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\lambda})(x) &= \bigcup_{x=abc} \{(\tilde{\mu} \tilde{\cap} \tilde{\lambda})(a) \curlywedge (\tilde{\chi}_S \circ \tilde{\mu} \tilde{\cap} \tilde{\lambda})(bc)\} \\
&= \bigcup_{x=abc} \{(\tilde{\mu} \sqcap \tilde{\lambda})(a) \curlywedge \{(\tilde{\chi}_S \circ \tilde{\mu} \tilde{\cap} \tilde{\chi}_S \circ \tilde{\lambda})(bc)\}\} \\
&= \bigcup_{x=abc} \{(\tilde{\mu} \sqcap \tilde{\lambda})(a) \curlywedge \{(\tilde{\chi}_S \circ \tilde{\mu})(bc) \curlywedge (\tilde{\chi}_S \circ \tilde{\lambda})(bc)\}\} \\
&= \bigcup_{x=abc} \{\{\tilde{\mu}(a) \curlywedge (\tilde{\chi}_S \circ \tilde{\mu})(bc)\} \cap \{\tilde{\lambda}(a) \curlywedge (\tilde{\chi}_S \circ \tilde{\lambda})(bc)\}\} \\
&= (\tilde{\mu} \circ \tilde{\chi}_S \circ \tilde{\mu})(x) \curlywedge (\tilde{\lambda} \circ \tilde{\chi}_S \circ \tilde{\lambda})(x) \\
&= (\tilde{\mu} \circ \tilde{\chi}_S \circ \tilde{\mu} \tilde{\cap} \tilde{\lambda} \circ \tilde{\chi}_S \circ \tilde{\lambda})(x).
\end{aligned}$$

Therefore $\tilde{\mu} \sqcap \tilde{\lambda} \circ \tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\lambda} = \tilde{\mu} \circ \tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\lambda} \circ \tilde{\chi}_S \circ \tilde{\lambda}$.

Then

$$\begin{aligned}
&(\tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\lambda} \circ \tilde{\chi}_S)(x) \curlywedge (\tilde{\mu} \sqcap \tilde{\lambda} \circ \tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\lambda})(x) \\
&= (\tilde{\chi}_S \circ \tilde{\mu} \circ \tilde{\chi}_S)(x) \curlywedge (\tilde{\mu} \circ \tilde{\chi}_S \circ \tilde{\mu})(x) \curlywedge (\tilde{\chi}_S \circ \tilde{\lambda} \circ \tilde{\mu}_{\chi_S})(x) \curlywedge (\tilde{\lambda} \circ \tilde{\chi}_S \circ \tilde{\lambda})(x) \\
&\preceq (\tilde{\mu} \sqcap \tilde{\lambda})(x).
\end{aligned}$$

Therefore $(\tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\lambda} \circ \tilde{\chi}_S) \sqcap (\tilde{\mu} \sqcap \tilde{\lambda} \circ \tilde{\chi}_S \circ \tilde{\mu} \sqcap \tilde{\lambda}) \sqsubseteq \tilde{\mu} \sqcap \tilde{\lambda}$.

Hence $\tilde{\mu} \tilde{\cap} \tilde{\lambda}$ is an IVF bi-interior ideal of a semigroup S.

We know that every IVF ideal is an IVF bi-interior ideal then the following theorem holds.

Theorem 7. *If $\tilde{\mu}$ and $\tilde{\lambda}$ are IVF right ideals and an IVF left ideal of a semigroup S respectively. Then $\tilde{\mu} \sqcap \tilde{\lambda}$ is an IVF bi-interior ideal of S.*

Proof. Assume that $\tilde{\mu}$ and $\tilde{\lambda}$ are IVF right ideals and an IVF left ideal of S respectively. Then by Theorems 2 and 3, we have $\tilde{\mu}$ and $\tilde{\lambda}$ are IVF bi-interior ideals of S. By Theorem 6 we have $\tilde{\mu} \sqcap \tilde{\lambda}$ is an IVF bi-interior ideal of S.

The following are tools the converse of an IVF bi-interior ideals is IVF ideals on semigroups.

Definition 7. *A semigroup S is called regular if for all $a \in S$ there exists $x \in S$ such that $a = axa$.*

Theorem 8. *If $\tilde{\mu}$ be an IVF quasi-ideal of a regular semigroup S.Then $\tilde{\mu}$ is an IVF ideal of a semigroup S.*

Proof. Assume that $\tilde{\mu}$ is an IVF quasi-ideal of S and let $x, y \in S$. Then

$$\begin{aligned} \tilde{\mu}(xy) &\succeq (\tilde{\mu} \circ \tilde{\chi_S})(xy) \curlywedge (\tilde{\chi_S} \circ \tilde{\mu})(xy) \\ &= \bigcup_{xy=ab} \{\tilde{\mu}(a) \curlywedge \tilde{\chi_S}(b)\} \curlywedge \bigcup_{xy=ij} \{\tilde{\chi_S}(i) \curlywedge \tilde{\mu}(j)\} \\ &\succeq \tilde{\mu}(x) \cap \tilde{\chi_S}(y) \curlywedge \tilde{\chi_S}(x) \curlywedge \tilde{\mu}(y) \\ &= (\tilde{\mu}(x) \curlywedge [1,1]) \curlywedge ([1,1] \curlywedge \tilde{\mu}(y)) = \tilde{\mu}(x) \cap \tilde{\mu}(y). \end{aligned}$$

Thus $\tilde{\mu}(xy) \succeq \tilde{\mu}(x) \cap \tilde{\mu}(y)$. Hence $\tilde{\mu}$ is an IVF subsemigroup of S. Let $x, y, z \in S$. Then

$$\begin{aligned} \tilde{\mu}(xyz) &\succeq (\tilde{\mu} \circ \tilde{\chi_S})(xyz) \curlywedge (\tilde{\chi_S} \circ \tilde{\mu})(xyz) \\ &= \bigcup_{xyz=ab} \{\tilde{\mu}(a) \curlywedge \tilde{\chi_S}(b)\} \curlywedge \bigcup_{xyz=ij} \{\tilde{\chi_S}(i) \curlywedge \tilde{\mu}(j)\} \\ &\succeq \tilde{\mu}(x) \curlywedge \tilde{\chi_S}(yz) \curlywedge \tilde{\chi_S}(xy) \curlywedge \tilde{\mu}(z) \\ &= (\tilde{\mu}(x) \curlywedge [1,1]) \curlywedge ([1,1] \curlywedge \tilde{\mu}(z)) = \tilde{\mu}(x) \curlywedge \tilde{\mu}(z). \end{aligned}$$

Thus $\tilde{\mu}(xyz) \succeq \tilde{\mu}(x) \curlywedge \tilde{\mu}(z)$. Hence $\tilde{\mu}$ is an IVF bi-ideal of S. Since S is regular, $\tilde{\mu}$ is an IVF bi-ideal of S and $x, y \in S$ we have $xy \in (xSx)S \subseteq xSx$. Thus there exists $k \in S$ such that $xy = xkx$. So $\tilde{\mu}(xy) = \tilde{\mu}(xkx) \succeq \tilde{\mu}(x) \curlywedge \tilde{\mu}(x) = \tilde{\mu}(x)$. Similarly, we can show that $\tilde{\mu}(xy) \succeq \tilde{\mu}(y)$. Thus $\tilde{\mu}$ is an IVF left ideal of S. Hence $\tilde{\mu}$ is an IVF ideal of S.

Theorem 9. *Let S be a regular semigroup. Then $\tilde{\mu}$ is an IVF bi-interior ideal of S if and only if $\tilde{\mu}$ is an IVF quasi-ideal of S.*

Proof. Let $\tilde{\mu}$ be an IVF bi-interior ideal of S and $x \in S$. Then
$(\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S})(x) \curlywedge (\tilde{\mu} \circ \tilde{\mu}_{\chi_S} \circ \tilde{\mu})(x) \succeq \tilde{\mu}(x)$. Suppose $(\tilde{\chi_S} \circ \tilde{\mu})(x) \succeq \tilde{\mu}(x)$. Since S is regular, there exists $y \in S$ such that $x = xyx$. Then

$$(\tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu})(x) = \bigcup_{x=xyx} \{\tilde{\mu}(xy) \curlywedge (\tilde{\chi_S} \circ \tilde{\mu})(x)\} \supseteq \bigcup_{x=xyx} \{\tilde{\mu}(x) \curlywedge \tilde{\mu}(x)\} = \tilde{\mu}(x).$$

Which is a contradiction. Therefore $\tilde{\mu}$ is an IVF quasi-ideal of S. By Theorem 8, converse is true.

Theorem 10. *If $\tilde{\mu}$ be an IVF bi-interior ideal of a regular semigroup S. Then $\tilde{\mu}$ is an IVF ideal of a semigroup S.*

Proof. Suppose that $\tilde{\mu}$ is an IVF bi-interior ideal of S. Then by Theorem 9, $\tilde{\mu}$ is an IVF quasi-ideal of S. Thus by Theorem 8, $\tilde{\mu}$ is an IVF ideal of S. Hence the theorem is complete.

The following theorems are a tool in characterization regular semigroup in terms of IVF bi-interior ideals on semigroups.

Theorem 11. *[4] For non-empty subsets G and H of a semigroup S, we have*

(1) $\tilde{\chi_G} \circ \tilde{\chi_H} = \tilde{\chi_{GH}}$,

(2) $\tilde{\chi_G} \sqcap \tilde{\chi_H} = \tilde{\chi_{G \cap H}}$.

Proof. It is straightforward.

Theorem 12. *Let S be a semigroup. Then S is a regular semigroup if and only if $B = SBS \cap SBS$, for every bi-interior ideal of S.*

The following theorems are characterization regular semigroup in terms of IVF bi-interior ideals in semigroups.

Theorem 13. *Let S be a semigroup. Then S is a regular if and only if $\tilde{\mu} = \tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S} \sqcap \tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu}$ for every IVF bi-interior ideal of a semigroup S.*

Proof. Let $\tilde{\mu}$ be an IVF bi-interior ideal of the regular semigroup S and let $x \in S$. Since S is regular, there exists $a \in S$ such that $x = xax$. Thus

$$
\begin{aligned}
(\tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu})(x) &= \bigcup_{x=xax} \{\tilde{\mu}(x) \curlywedge (\tilde{\chi_S} \circ \tilde{\mu})(ax)\} \\
&= \bigcup_{x=xax} \{\tilde{\mu}(x) \curlywedge \bigcup_{ax=yz} \{\tilde{\chi_S}(y) \curlywedge \tilde{\mu}(z)\}\} \\
&\supseteq \bigcup_{x=xax} \{\tilde{\mu}(x) \curlywedge \tilde{\mu}(x)\} = \tilde{\mu}(x).
\end{aligned}
$$

Similarly, $(\tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S})(x) \preceq \tilde{\mu}(x)$. Therefore $\tilde{\mu} = \tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S} \sqcap \tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu}$.

Conversely, suppose that B is a bi-interior ideal of a semigroup S. Then by Theorem 5, $\tilde{\chi_B}$ is an IVF bi-interior ideal of the semigroup S. Thus by Theorem 11,

$$
\begin{aligned}
\tilde{\mu}_{\chi_B}(x) &= \tilde{\chi_S} \circ \tilde{\mu}_{\chi_B} \circ \tilde{\chi_S}(x) \curlywedge \tilde{\mu}_{\chi_B} \circ \tilde{\chi_S} \circ \tilde{\mu}_{\chi_B}(x) \\
&= \tilde{\mu}_{\chi_{SBS}}(x) \curlywedge \tilde{\mu}_{\chi_{BSB}}(x) = \tilde{\mu}_{\chi_{SBS \cap BSB}}(x).
\end{aligned}
$$

Therefore $B = SBS \cap BSB$. By Theorem 12, S is a regular semigroup.

Theorem 14. *Let S be a semigroup. Then S is regular if and only if $\tilde{\mu} \sqcap \tilde{\lambda} \tilde{\subseteq} \tilde{\lambda} \circ \tilde{\mu} \circ \tilde{\lambda} \sqcap \tilde{\mu} \circ \tilde{\lambda} \circ \tilde{\mu}$ for every IVF bi-interior ideal $\tilde{\mu}$ and every IVF ideal $\tilde{\lambda}$ of S.*

Proof. Let $\tilde{\mu}$ be an IVF bi-interior ideal and $\tilde{\lambda}$ be an IVF ideal of a regular semigroup S and let $x \in S$. Then there exists $y \in S$ such that $x = xyx$.

$$\begin{aligned}(\tilde{\mu} \circ \tilde{\lambda} \circ \tilde{\mu})(x) &= \bigcup_{x=xyx} \{(\tilde{\mu} \circ \tilde{\lambda})(xy) \curlywedge \tilde{\mu}(x)\} \\ &= \bigcup_{x=xyx} \{ \bigcup_{xy=xyxy} \{\tilde{\mu}(x) \cap \tilde{\lambda}(yxy)\} \curlywedge \tilde{\mu}(x)\} \\ &\succeq \{\tilde{\mu}(x) \curlywedge \tilde{\lambda}(x)\} \curlywedge \tilde{\mu}(x) = \tilde{\mu}(x) \curlywedge \tilde{\lambda}(x) = (\tilde{\mu} \sqcap \tilde{\lambda})(x).\end{aligned}$$

$$(\tilde{\lambda} \circ \tilde{\mu})(x) = \bigcup_{x=xyx} \{\tilde{\lambda}(xy) \curlywedge \tilde{\mu}(x)\} \succeq \{\tilde{\lambda}(x) \curlywedge \tilde{\mu}(x)\} = (\tilde{\mu} \sqcap \tilde{\lambda})(x).$$

Therefore $\tilde{\mu} \circ \tilde{\lambda} \circ \tilde{\mu} \sqsubseteq \tilde{\mu} \sqcap \tilde{\lambda}$. Similary, we can prove $\tilde{\lambda} \circ \tilde{\mu} \circ \tilde{\lambda} \sqsupseteq \tilde{\mu} \sqcap \tilde{\lambda}$.
Hence $\tilde{\mu} \sqcap \tilde{\lambda} \sqsubseteq \tilde{\lambda} \circ \tilde{\mu} \circ \tilde{\lambda} \sqcap \tilde{\mu} \circ \tilde{\lambda} \circ \tilde{\mu}$.

Conversely, suppose that the condition holds. Let $\tilde{\mu}$ be an IVF bi-interior ideal. We have $\tilde{\mu} \sqcap \tilde{\chi_S} \sqsubseteq \tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S} \sqcap \tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu}$ and implies that $\tilde{\mu} \sqsubseteq \tilde{\chi_S} \circ \tilde{\mu} \circ \tilde{\chi_S} \sqcap \tilde{\mu} \circ \tilde{\chi_S} \circ \tilde{\mu}$. By Theorem 13, S is a regular semigroup.

4. Conclusion

In this paper, we give the concept of IVF bi-interior ideals in semigroups and we study properties of IVF bi-interior ideals in semigroups. Moreover, we prove relationship between IVF bi-interior ideals and bi-interior ideals. In the future we study other kinds of IVF bi-quasi interior ideals in semigroup or algebric system.

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References

[1] J. Aranzazu, S. Antonio, P. Daniel, F. Javier, and B. Humberto. Interval valued fuzzy sets for color image super-resolution. *Advances in Artificial Intelligence*, pages 373–382, 2011.

[2] H. Bustince. Indicator of inclusion grade for interval valued fuzzy sets. application to approximate reasoning based on interval valued fuzzy sets. *International Journal of Approximate Reasoning*, 23:137–209, 1998.

[3] I. Cristea, A. Mahboob, and N. Mohammad Khan. A new type fuzzy quasi-ideals of ordered semigrouops. *Journal of Multiple-Valued Logic and Soft Computing*, 17(3):739–752, 2020.

[4] T. Gaketem. On interval valued fuzzy almost (m,n)-bi-ideal in semigroups. *J. Math. Comput. Sci.*, 11(6):6657–6665, 2021.

[5] M. Krishna and M. Rao. Bi-interior ideals of semigroups. *Discussiones Mathematicae General algebra and applications*, 38:69–78, 2018.

[6] A. Mahboob, B. Davvaz, and N. M. Khan. Fuzzy (m, n)-ideals in semigroups. *Computational and applied mathematics*, 38(189):1–18, 2019.

[7] A. Mahboob and G. Muhiuddin. A new type of fuzzy prime subset in ordered semigroups. *New Mathematics and Natural Computing*, 17(3):739–752, 2021.

[8] A. Mahboob, A. Salam, Md. F. Ali, and N. M. Khan. Characterizations of regular ordered semigroups by $(\varepsilon, \varepsilon \bigvee_{(k,qk)})$-fuzzy quasi ideals. *Mathematics*, 7:1–16, 2019.

[9] AL. Narayanan and T. Manikantan. Interval valued fuzzy ideals generated by an interval valued fuzzy subset in semigroups. *Journal of Applied Mathematics and Computing*, 20(1-2):455–464, 2006.

[10] L.A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

[11] L.A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning. *Information Sciences*, 8:199–249, 1975.

[12] M. Zulquanain and M. Saeed. A new decision making method on interval value fuzzy soft matrix. *British journal of mathematics and computer science*, 20(5):1–17, 2017.