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More on Ideal Rothberger Spaces

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Abstract. The aim of this note is to provide an answer to a question posted in a recent paper. In 2018, after introducing the notion of Ideal Rothberger space, author examines some properties of these spaces. Also there has been a comparison of the spaces (X, τ), and (X, τ^*) in terms of being (ideal)Rothberger. According to this, it is shown that if (X, τ^*) is a Rothberger space, then (X, τ) is also Rothberger. Therefore, naturally it is asked that, if one can find some extra conditions for ideal $\mathbf{I}$, then the opposite also holds. Thus, for which ideal $\mathbf{I}$, an $\mathbf{I}$-Rothberger space (X, τ) implies an $\mathbf{I}$-Rothberger space (X, τ^*)? In this work it has been proved that $\mathbf{I}$ is a σ-ideal, and τ is compatible with $\mathbf{I}$, which provides the solution.

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1. Introduction and Preliminaries

Extending topological spaces by adding ideals, has been done since Vaidyanathaswamy [7], and Kuratowski [3]. An ideal $\mathbf{I}$ on a set X is defined to be a nonempty collection of subsets of X, which is closed under the subset and finite union operations. A topological space (X, τ) with an ideal $\mathbf{I}$ defined on X is donoted by $(X, \tau, \mathbf{I})$. By using an ideal on a topological space, a local function is defined in [3] as follows:

Definition 1. *[3] Let (X, τ) be a topological space, and let $\mathbf{I}$ be an ideal on X. Then the local function $A^*(\mathbf{I}, \tau)$ of $A \subset X$ is defined as:*
$A^*(\mathbf{I}, \tau) = \{x \in X \mid A \cap U \notin \mathbf{I} \text{ for every } U \in \tau(x)\}$, *where* $\tau(x) = \{U \in \tau \mid x \in U\}$.

Using the notation of the previous definition, it can easily be seen that, by saying, for every $A \subset X$, $c^*(A) = A \cup A^*(\mathbf{I}, \tau)$, one defines a Kuratowski closure operator c^*.This closure operator induces a topology on X (see [2]).

Definition 2. *For an ideal $\mathbf{I}$ of subsets of a topological space (X, τ), the topology on X induced by the closure operator c^* is denoted by $\tau^*(\mathbf{I}, \tau)$ or for simplicity, τ^* if this does not lead to misunderstanding.*

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On the other hand, examining the covering properties of topological spaces is also an attractive area for topologists. One of these covering properties, which is introduced in [5], [6], is being a Rothberger Space.

Definition 3. *[5], [6] A space X is Rothberger if for every sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n \mid n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} U_n$.*

So, in [1] the notions of ideal topological space and Rothberger space were brought together, and resulted in Ideal Rothberger(**I**-Rothberger) Spaces.

Definition 4. *[1] Let $(X, \tau, \mathbf{I})$ be an ideal topological space. (X, τ) is said to be* **I***-Rothberger or Rothberger with respect to* **I***, if for every sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n \mid n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$, and $X \setminus \bigcup_{n\in\mathbb{N}} U_n \in \mathbf{I}$.*

In [1] besides exploring properties, and weak forms of **I**-Rothberger spaces, a theorem about the transition between (X, τ), and (X, τ^*) is given as following:

Theorem 1. *Let (X, τ) be a topological space,* **I** *be an ideal on X, and (X, τ^*) be a Rothberger space. Then (X, τ) is also Rothberger, and since every Rothberger space is shown to be* **I***-Rothberger, therefore (X, τ) is indeed an* **I***-Rothberger space.*

Note that, the proof is clear via the fact that $\tau \subset \tau^*$.

Following the previous theorem, an example which shows that being **I**-Rothberger for (X, τ) does not imply being Rothberger for (X, τ^*).

Example 1. *[1] Let X be the set of real numbers $\mathbb{R}$, the topology τ be the usual topology of $\mathbb{R}$, and the ideal* **I** *be the power set $\mathcal{P}(\mathbb{R})$. It is clear that, $(\mathbb{R}, \tau)$ is $\mathcal{P}(\mathbb{R})$-Rothberger. On the other hand τ^* on $\mathbb{R}$ is the discrete topology, which does not qualify as Lindelöf, therefore, it is not Rothberger.*

Based on these, a question is posted in [1].

[1] What extra conditions the ideal **I** might have, in order to provide the converse of the previous theorem?

In this paper, the condition for the ideal **I** is provided.

2. Main Result

Given a topology τ on a set X and an ideal **I** of subsets of X, such that **I** is compatible with τ, τ^* denotes the topology on X consisting of all sets of the form $U \setminus A$ where $U \in \tau$ and $A \in \mathbf{I}$. In 2018, the notion of an **I**-Rothberger space was introduced. It is known that if (X, τ^*) is a Rothberger space, so is (X, τ). However, a satisfactory answer to the question under which conditions on $\mathbf{I}$ the reverse implication also holds is still unknown. The main theorem of the article asserts that it holds if $\mathbf{I}$ is a σ-ideal of subsets of X such that $\mathbf{I}$ is compatible with a topology τ on X, and the space (X,τ) is $\mathbf{I}$-Rothberger, so is (X,τ^*). This gives a partial answer to the above-mentioned question.

Before providing this condition, some basic definitions, and theorems to be used are included:

Definition 5. *[2] An ideal $\mathbf{I}$ is said to be a σ-ideal if it is countably additive, that is, if $I_n \in \mathbf{I}$, for each $n \in \mathbb{N}$, then $\bigcup\{I_n | n \in \mathbb{N}\} \in \mathbf{I}$.*

Definition 6. *[4] Let (X,τ) be a topological space with an ideal $\mathbf{I}$. The topology τ is indicated as compatible with the ideal $\mathbf{I}$, which is denoted by $\tau \sim \mathbf{I}$, if the following holds for every $A \subset X$:*
if for every $x \in A$, there exists a set $U \in \tau(x)$ such that $U \cap A \in \mathbf{I}$, then $A \in \mathbf{I}$.

Theorem 2. *[4] Let (X,τ) be a topological space, $\mathbf{I}$ be an ideal on X, and τ be compatible with $\mathbf{I}$. A set is closed in τ^* if and only if it is the union of a set which is closed with respect to τ and a set from the ideal $\mathbf{I}$.*

Corollary 1. *[2] Let (X,τ) be a topological space and $\mathbf{I}$ be an ideal on X, with $\tau \sim \mathbf{I}$. Based on the previous theorem it is known that every closed set K can be written as $K = F \cup I$, where F is closed with respect to τ, and $I \in \mathbf{I}$. Then every τ^*-open set U will be the complement of the sets of this type and hence will have the form:*
$U = G \setminus I$, where $G \in \tau$, and $I \in \mathbf{I}$.

Finally, the answer to the question that was posted:

Theorem 3. *Let (X,τ) be a topological space, $\mathbf{I}$ be a σ-ideal on X, and $\tau \sim \mathbf{I}$. If (X,τ) is $\mathbf{I}$-Rothberger, then (X,τ^*) is also an $\mathbf{I}$-Rothberger space.*

Proof. Let $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ be a sequence of open covers of X with respect to topology τ^*.
So; $\mathcal{U}_n = \{U_n^\alpha \mid \alpha \in \Delta_n\}$; for every $n \in \mathbb{N}$, and $U_n^\alpha \in \tau^*$.
By compatibility:
$U_n^\alpha = G_n^\alpha \setminus I_n^\alpha$, where $G_n^\alpha \in \tau$, and $I_n^\alpha \in \mathbf{I}$.
Clearly $\mathcal{G}_n = \{G_n^\alpha \mid \alpha \in \Delta_n\}$ is an open cover of X, with respect to topology τ, and so $\{\mathcal{G}_n \mid n \in \mathbb{N}\}$ is a sequence of open covers. Since (X,τ) is $\mathbf{I}$-Rothberger, there exists a sequence $\{\alpha_n \mid n \in \mathbb{N}\}$ such that, for every $n \in \mathbb{N}$, $\alpha_n \in \Delta_n$, and $J = X \setminus \bigcup_{n=1}^{\infty} G_n^{\alpha_n} \in \mathbf{I}$. On the other hand, since $\mathbf{I}$ is a σ-ideal, it is accepted that $K = \bigcup_{n=1}^{\infty} I_n^{\alpha_n} \in \mathbf{I}$.

So; $J \cup K = (X \setminus \bigcup_{n=1}^{\infty} G_n^{\alpha_n}) \cup (\bigcup_{n=1}^{\infty} I_n^{\alpha_n}) \in \mathbf{I}$.
And clearly;

$$X \setminus \bigcup_{n=1}^{\infty} U_n^{\alpha_n} \subseteq (X \setminus \bigcup_{n=1}^{\infty} G_n^{\alpha_n}) \cup (\bigcup_{n=1}^{\infty} I_n^{\alpha_n}).$$

Finally;

$X \setminus \bigcup_{n=1}^{\infty} U_n^{\alpha_n} \in \mathbf{I}$, so (X, τ^*) is an **I**-Rothberger space.

3. Conclusions

In this work, a question posted in [1] is partially answered. So, one may still search for a weaker condition for a possible future work.

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