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Single-valued Neutrosophic Soft sets in Hyper *UP*-Algebra

Allan N. Cano[1,*], Gaudencio C. Petalcorin, Jr.[1]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. In this paper, the notions of *SVN* hyper *UP*-algebra and *SVNS* hyper *UP*-algebra are introduced, and some of their structural properties are investigated. Moreover, the Cartesian product of *SVNS* hyper *UP*-algebra is discussed and proved to be a *SVNS* hyper *UP*- algebra. Finally, the homomorphic image and preimage of *SVNS* hyper *UP*-algebra under *SVNS* functions are studied and showed also to be *SVNS* hyper *UP*-algebra.

2020 Mathematics Subject Classifications: 08A30

Key Words and Phrases: Hyper *UP*-algebra, single-valued neutrosophic set, single-valued neutrosophic soft set

1. Introduction

The concept of fuzzy sets and fuzzy logic has been used widely in many applications involving uncertainties. Such concept was initiated by L. Zadeh [10]. Resulting from vagueness or partial belongingness of an element in a set, fuzzy set is successful in handling uncertainties. However, there are still some situations which it cannot cover like problems involving incomplete information. Motivated by this, a lot of researchers extended this concept and presented a different theories regarding uncertainty which include intuitionistic fuzzy set theory [3], interval-valued intuitionistic fuzzy set theory [9] and so on. Later on, Smarandache [18] generalized intuitionistic fuzzy set theory by introducing the concept of neutrosophic set in 1998. Neutrosophic set is a part of neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. It is a powerful general formal framework that has been recently proposed. To have its real life application in engineering and science, neutrosophic set needs to be specified from a technical point of view. That is why single valued neutrosophic set was introduced by Wang et al. [22] together with its various properties. Single-valued neutrosophic set has been developing rapidly due to its wide range of

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Email addresses: allan.cano@g.msuiit.edu.ph (A. Cano), gaudencio.petalcorin@g.msuiit.edu.ph (G. Petalcorin)

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theoretical elegance and application areas. The reader may refer to the following articles [7, 8, 16, 17, 19, 20] as references. In 1999, Molodtsov [14] studied another mathematical theory called soft set theory by giving parameterized approach to uncertainties. On the other hand, Maji [12] unified the fundamental theories of neutrosophic set and soft set, and came up with the concept of neutrosophic soft set. Some theoretical advancement and applications have been reported in the following literatures [1, 4, 5, 11].

The hyper algebraic structure theory was introduced in 1934 by F. Marty [13] at the 8th congress of Scandinavian Mathematicians. This theory is then applied by Y. B. Jun et al. [21] to BCK-algebras to produce the notion of hyper BCK-algebras as a generalization of the BCK-algebras. After that, many researchers have been inspired to generalize some existing algebras and one of them is D. Romano [15]. He has come up with the concept of hyper *UP*-algebras to generalize *UP*-algebras.

In this paper, we utilize the notions of single-valued neutrosophic sets and single-valued neutrosophic soft sets to hyper *UP*-algebra to generate *SVN* hyper *UP*-algebra and *SVNS* hyper *UP*-algebra. Several of their basic properties are studied. In addition, we define the Cartesian product of *SVNS* hyper *UP*-algebra, and image and preimage of *SVNS* hyper *UP*-algebra under *SVNS* function. Each of them is discussed and illustrated with corresponding examples.

2. Preliminary Concepts

Definition 1. [15] Let $\mathcal{P}(H)$ to be the power set of H. Consider $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}$. A *hyperoperation* on a nonempty set H is a function $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$. The image of $(x, y) \in H \times H$ under $\circ$ is denoted by $x \circ y$. If $x \in H$ and A, B are nonempty subsets of H, then we define

(*i*) $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$;

(*ii*) $A \circ x = A \circ \{x\}$; and

(*iii*) $x \circ B = \{x\} \circ B$.

Definition 2. [15] Let $x, y \in H$ and $A, B \subseteq H$. Then

(*i*) $x \ll y$ if and only if $0 \in x \circ y$; and

(*ii*) $A \ll B$ if and only if for any $a \in A$, there exists $b \in B$ such that $a \ll b$.

We call $\ll$ a hyperorder on H.

Remark 1. [15] For all $A, B \subseteq H$, $A \ll B$ implies $0 \in A \circ B$.

Definition 3. [15] Let X be a nonempty set such that $0 \in X$ and $(X, \circ, \ll, 0)$ be a hyperstructure. Then $(X, \circ, \ll, 0)$ is called a *hyper UP-algebra* if the following formulas are valid: $\forall x, y, z \in X$,

$$(\mathbf{HUP1}) \qquad y \circ z \ll (x \circ y) \circ (x \circ z),$$

(**HUP2**) $x \circ 0 = \{0\}$,
(**HUP3**) $0 \circ x = \{x\}$, and
(**HUP4**) $x \ll y \wedge y \ll x \Longrightarrow x = y$.

Example 1. Let $X = \{0, r, s, t\}$ be a set. If we define a hyper operation "$\circ$" as following:

$\circ$	0	r	s	t
0	$\{0\}$	$\{r\}$	$\{s\}$	$\{t\}$
r	$\{0\}$	$\{0, r\}$	$\{0, s\}$	$\{r, s\}$
s	$\{0\}$	$\{r, s\}$	$\{0, s\}$	$\{r\}$
t	$\{0\}$	$\{0, r, s, t\}$	$\{s, t\}$	$\{0\}$

,

then the routine calculation will show that $(X, \circ, \ll, 0)$ is a hyper UP-algebra.

Example 2. Let $X = \{0, u, v\}$. Define a hyper operation "$\circ$" as follows:

$\circ$	0	u	v
0	$\{0\}$	$\{u\}$	$\{v\}$
u	$\{0\}$	$\{0, u\}$	$\{0, v\}$
v	$\{0\}$	$\{u, v\}$	$\{0, v\}$

.

By routine calculation, $(X, \circ, \ll, 0)$ is a hyper UP-algebra.

Example 3. Let $X = \{0, a, b\}$. Define a hyper operation "$\circ$" as follows:

$\circ$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{0\}$	$\{0, a, b\}$	$\{0, b\}$
b	$\{0\}$	$\{0, a, b\}$	$\{0\}$

.

Observe that $a \ll b$ and $b \ll a$. But $a \neq b$. Thus, $(X, \circ, \ll, 0)$ does not satisfy $(HUP4)$ and so it is not a hyper UP-algebra.

Proposition 1. [15] *Let $(H, \circ, \ll, 0)$ be a hyper UP-algebra. Then the following hold for all $x, y, z \in H$ and for every nonempty subsets $A, B, C \subseteq H$:*

(i) $A \subseteq B$ *implies* $A \ll B$
(ii) $0 \circ 0 = \{0\}$
(iii) $x \ll 0$
(iv) $x \ll x$
(v) $z \ll x \circ z$
(vi) $A \circ 0 = \{0\}$
(vii) $0 \circ A = A$
(viii) $(0 \circ 0) \circ x = \{x\}$

Proposition 2. [15] *Let S be a nonempty subset of a hyper UP-algebra $(X, \circ, \ll, 0)$. Then S is a hyper UP-subalgebra of X if and only if $\forall x, y \in S$, $x \circ y \subseteq S$.*

Definition 4. [15] Let $(X_1, \circ_1, \ll_1, 0_1)$ and $(X_2, \circ_2, \ll_2, 0_2)$ be hyper UP-algebras. A mapping $f : X_1 \longrightarrow X_2$ is called a *hyper homomorphism* if for all $a, b \in X_1$,

(i) $f(0_1) = 0_2$ and

(ii) $f(a \circ_1 b) = f(a) \circ_2 f(b)$.

Definition 5. [2] Let $f : (X_1, \circ_1, \ll_1, 0_1) \longrightarrow (X_2, \circ_2, \ll_2, 0_2)$ be a hyper homomorphism. We say that f is a *hyper monomorphism* if f is one-to-one and f is a *hyper epimorphism* if f is onto. We also say that f is a *hyper isomorphism* if f is both one-to-one and onto. In this case, X_1 and X_2 are hyper isomorphic which is denoted as $X_1 \cong_{\mathcal{H}} X_2$.

Definition 6. [2] Let $(X_1, \circ_1, \ll_1, 0_1)$ and $(X_2, \circ_2, \ll_2, 0_2)$ be hyper UP-algebras. Define a set $X_1 \times X_2$ by

$$X_1 \times X_2 = \{(a, b) : a \in X_1 \text{ and } b \in X_2\}.$$

with a hyperoperation " $\circ$ " on $X_1 \times X_2$ given by

$$(a, b) \circ (c, d) = (a \circ_1 c, b \circ_2 d)$$

and a hyperorder "$\ll$" given by

$$(a, b) \ll (c, d) \iff a \ll_1 c \text{ and } b \ll_2 d$$

for all $(a, b), (c, d) \in X_1 \times X_2$.. Then $(X_1 \times X_2, \circ, \ll, (0_1, 0_2))$ is called the *hyper product of* X_1 *and* X_2.

Definition 7. [18] Let U be the universe. A *neutrosophic set* A is characterized by a truth membership function $\mathcal{T}_A$, an indeterminacy membership function $\mathcal{I}_A$, and a falsity membership function $\mathcal{F}_A$ where $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A$ are real standard or non-standard elements of $]^-0, 1^+[$ with $^-0 = 0 - \epsilon$ and $1^+ = 1 + \epsilon$ for any infinitesimal number ϵ. It can be written as

$$A = \{\langle x, (\mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x))\rangle \,|x \in U\}$$

where $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A : U \longrightarrow]^-0, 1^+[$ and $^-0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3^+$.

However, it is difficult to use a neutrosophic set with values from real standard or non-standard subsets of $]^-0, 1^+[$ in real life application especially scientific and engineering problem [22]. So, this paper considers the neutrosophic set which takes values from the interval $[0, 1]$.

Definition 8. [22] Let X be a space of points (objects), with a generic element in X denoted by x. A *single valued neutrosophic set* ($SVNS$) A in X is characterized by truth-membership function $\mathcal{T}_A$, indeterminacy-membership function $\mathcal{I}_A$ and falsity-membership function $\mathcal{F}_A$. For each point $x \in X$, $\mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x) \in [0, 1]$.

Definition 9. [14] Given an initial universe set U and set E of parameters or attributes with respect to U, let $\mathcal{P}(U)$ denote the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \longrightarrow \mathcal{P}(U)$.

For any $\epsilon \in A$, $F(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (F, A).

The concept of neutrosophic soft set was first defined by Maji [12] and later on, it was modified by Deli and Broumi [6] as given below:

Definition 10. Let U be an initial universe set and E be a set of parameters. Let $\mathcal{N}(U)$ denote the set of all neutrosophic sets of U. Then a *neutrosophic soft set* (F, E) *over* U is a set defined by a set valued function F representing a mapping $F : E \longrightarrow \mathcal{N}(U)$ where F is called approximate function of the neutrosophic soft set (F, E).

In other words, the neutrosophic soft set is a parameterized family of some elements of the set $\mathcal{N}(U)$ and therefore it can be written as a set of ordered pairs

$$(F, E) = \{(e, \{\langle x, (\mathcal{T}_{F(e)}(x), \mathcal{I}_{F(e)}(x), \mathcal{F}_{F(e)}(x))\rangle\})|x \in U, e \in E\}$$

where $\mathcal{T}_{F(e)}(x), \mathcal{I}_{F(e)}(x), \mathcal{F}_{F(e)}(x) \in [0, 1]$, respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $F(e)$. Since supremum of each $\mathcal{T}, \mathcal{I}, \mathcal{F}$ is 1 so the inequality $0 \leq \mathcal{T}_{F(e)}(x) + \mathcal{I}_{F(e)}(x) + \mathcal{F}_{F(e)}(x) \leq 3$ is obvious.

Definition 11. [6] The *complement of a neutrosophic soft set* (F, E) *over* U is denoted by $(F, E)^c$ and is defined by

$$(F, E)^c = \{(e, \{\langle x, (\mathcal{F}_{F(e)}(x), 1 - \mathcal{I}_{F(e)}(x), \mathcal{T}_{F(e)}(x))\rangle\})|x \in U, e \in E\}.$$

Definition 12. [6] Let (H, E) and (G, E) be two neutrosophic soft sets over the common universe U. Then (H, E) is said to be *neutrosophic soft subset* of (G, E) if $\forall e \in E$ and $\forall x \in U$, $\mathcal{T}_{H(e)}(x) \leq \mathcal{T}_{G(e)}(x), \mathcal{I}_{H(e)}(x) \geq \mathcal{I}_{G(e)}(x), \mathcal{F}_{H(e)}(x) \geq \mathcal{F}_{G(e)}(x)$. We write $(H, E) \subseteq (G, E)$ and (G, E) is a neutrosophic soft superset of (H, E).

Definition 13. [5] A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is *continuous t-norm* if $*$ satisfies the following conditions :

(*i*) $*$ is commutative and associative.

(*ii*) $*$ is continuous.

(*iii*) $a * 1 = 1 * a = a,\ \forall a \in [0, 1]$.

(*iv*) $a * b \leq c * d$ if $a \leq c$, $b \leq d$ with $a, b, c, d \in [0, 1]$.

A few examples of continuous t-norm are $a * b = ab$, $a * b = \min\{a, b\}$, $a * b = \max\{a + b - 1, 0\}$.

Definition 14. [5] A binary operation $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is *continuous t-conorm* $(s - norm)$ if $\diamond$ satisfies the following conditions :

(*i*) $\diamond$ is commutative and associative.

(*ii*) $\diamond$ is continuous.

(*iii*) $a \diamond 0 = 0 \diamond a = a,\ \forall a \in [0, 1]$.

(*iv*) $a \diamond b \leq c \diamond d$ if $a \leq c$, $b \leq d$ with $a, b, c, d \in [0, 1]$.

A few examples of continuous s-norm are $a \diamond b = a + b - ab$, $a \diamond b = \max\{a, b\}$, $a \diamond b = \min\{a + b, 1\}$.

Definition 15. [6] Let (H, E) and (G, E) be two neutrosophic soft sets over the common universe U.

(*i*) Then the *union of* (H, E) *and* (G, E) is denoted by $(H, E) \cup (G, E) = (K, E)$ and is defined by:

$$(K, E) = \{(e, \{\langle x, (\mathcal{T}_{K(e)}(x), \mathcal{I}_{K(e)}(x), \mathcal{F}_{K(e)}(x))\rangle\}) | x \in U, e \in E\}$$

where

$$\begin{aligned} \mathcal{T}_{K(e)}(x) &= \mathcal{T}_{H(e)}(x) \diamond \mathcal{T}_{G(e)}(x) \\ \mathcal{I}_{K(e)}(x) &= \mathcal{I}_{H(e)}(x) * \mathcal{I}_{G(e)}(x) \\ \mathcal{F}_{K(e)}(x) &= \mathcal{F}_{H(e)}(x) * \mathcal{F}_{G(e)}(x). \end{aligned}$$

(*ii*) Then the *intersection of* (H, E) *and* (G, E) is denoted by $(H, E) \cap (G, E) = (F, E)$ and is defined by:

$$(F, E) = \{(e, \{\langle x, (\mathcal{T}_{F(e)}(x), \mathcal{I}_{F(e)}(x), \mathcal{F}_{F(e)}(x))\rangle\}) | x \in U, e \in E\}$$

where

$$\begin{aligned} \mathcal{T}_{F(e)}(x) &= \mathcal{T}_{H(e)}(x) * \mathcal{T}_{G(e)}(x) \\ \mathcal{I}_{F(e)}(x) &= \mathcal{I}_{H(e)}(x) \diamond \mathcal{I}_{G(e)}(x) \\ \mathcal{F}_{F(e)}(x) &= \mathcal{F}_{H(e)}(x) \diamond \mathcal{F}_{G(e)}(x). \end{aligned}$$

In this paper, we use the minimality and maximality as binary operations $*$ and $\diamond$ respectively to define the union and intersection of two NSS sets.

Example 4. Consider $U = \{s_1, s_2, s_3\}$ be the set of all students and $E = \{a_1, a_2\}$ be the set of parameters where

a_1 stands for the parameter 'brilliant',
a_2 stands for the parameter 'healthy'.

Define a mapping $H : E \longrightarrow \mathcal{N}(U)$ by

$$\begin{aligned} H(a_1) &= \{\langle s_1, (0.1, 0.5, 0.4)\rangle, \langle s_2, (0.6, 0.6, 0.7)\rangle, \langle s_3, (0.5, 0.6, 0.4)\rangle\} \\ H(a_2) &= \{\langle s_1, (0.8, 0.4, 0.5)\rangle, \langle s_2, (0.7, 0.7, 0.3)\rangle, \langle s_3, (0.7, 0.5, 0.6)\rangle\}. \end{aligned}$$

and a mapping $G : E \longrightarrow \mathcal{N}(U)$ by

$$G(a_1) = \{\langle s_1, (0.8, 0.5, 0.6)\rangle, \langle s_2, (0.5, 0.7, 0.6)\rangle, \langle s_3, (0.4, 0.7, 0.5)\rangle\},$$

$$G(a_2) = \{\langle s_1, (0.7, 0.6, 0.5)\rangle, \langle s_2, (0.6, 0.8, 0.4)\rangle, \langle s_3, (0.5, 0.8, 0.6)\rangle\}.$$

Then the neutrosophic soft sets (H, E) and (G, E) are collections of approximations as below:

$$\begin{aligned}(H, E) = \{&(a_1, \{\langle s_1, (0.1, 0.5, 0.4)\rangle, \langle s_2, (0.6, 0.6, 0.7)\rangle, \langle s_3, (0.5, 0.6, 0.4)\rangle\})\\ &(a_2, \{\langle s_1, (0.8, 0.4, 0.5)\rangle, \langle s_2, (0.7, 0.7, 0.3)\rangle, \langle s_3, (0.7, 0.5, 0.6)\rangle\})\}\end{aligned}$$

and

$$\begin{aligned}(G, E) = \{&(a_1, \{\langle s_1, (0.8, 0.5, 0.6)\rangle, \langle s_2, (0.5, 0.7, 0.6)\rangle, \langle s_3, (0.4, 0.7, 0.5)\rangle\})\\ &(a_2, \{\langle s_1, (0.7, 0.6, 0.5)\rangle, \langle s_2, (0.6, 0.8, 0.4)\rangle, \langle s_3, (0.5, 0.8, 0.6)\rangle\})\}.\end{aligned}$$

Thus, their union and intersection are

$$\begin{aligned}(H, E) \cup (G, E) = \{&(a_1, \{\langle s_1, (0.8, 0.5, 0.4)\rangle, \langle s_2, (0.6, 0.6, 0.6)\rangle, \langle s_3, (0.5, 0.6, 0.4)\rangle\})\\ &(a_2, \{\langle s_1, (0.8, 0.4, 0.5)\rangle, \langle s_2, (0.7, 0.7, 0.3)\rangle, \langle s_3, (0.7, 0.5, 0.6)\rangle\})\}\end{aligned}$$

and

$$\begin{aligned}(H, E) \cap (G, E) = \{&(a_1, \{\langle s_1, (0.1, 0.5, 0.6)\rangle, \langle s_2, (0.5, 0.7, 0.7)\rangle, \langle s_3, (0.4, 0.7, 0.5)\rangle\})\\ &(a_2, \{\langle s_1, (0.7, 0.6, 0.5)\rangle, \langle s_2, (0.6, 0.8, 0.4)\rangle, \langle s_3, (0.5, 0.8, 0.6)\rangle\})\},\end{aligned}$$

respectively.

3. Main Results

3.1. Single-Valued Neutrosophic Hyper *UP*-subalgebra

In this section, we introduce the concept of single-valued neutrosophic hyper *UP*-subalgebra and prove some of its basic properties. From here onwards, we simply denote a hyper *UP*-algebra $(X, \circ, \ll, 0)$ by X.

Given a single-valued neutrosophic set $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ in a hyper *UP*-algebra X and a subset S of X, we denote the following:

$$\begin{aligned}{}^*\mathcal{T}_A(S) &= \sup_{y\in S} \mathcal{T}_A(y) \text{ and } {}_*\mathcal{T}_A(S) = \inf_{y\in S} \mathcal{T}_A(y);\\ {}^*\mathcal{I}_A(S) &= \sup_{y\in S} \mathcal{I}_A(y) \text{ and } {}_*\mathcal{I}_A(S) = \inf_{y\in S} \mathcal{I}_A(y);\\ {}^*\mathcal{F}_A(S) &= \sup_{y\in S} \mathcal{F}_A(y) \text{ and } {}_*\mathcal{F}_A(S) = \inf_{y\in S} \mathcal{F}_A(y).\end{aligned}$$

Definition 16. Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic set in a hyper *UP*-algebra X. Then A is said to be a *single-valued neutrosophic (SVN) hyper UP-subalgebra of* X if for all $x, y \in X$,

$${}_*\mathcal{T}_A(x \circ y) \geq \min\{\mathcal{T}_A(x), \mathcal{T}_A(y)\},$$

$$\begin{aligned} {}^{*}\mathcal{I}_A(x \circ y) &\leq \max\{\mathcal{I}_A(x), \mathcal{I}_A(y)\}, \text{ and} \\ {}^{*}\mathcal{F}_A(x \circ y) &\leq \max\{\mathcal{F}_A(x), \mathcal{F}_A(y)\}. \end{aligned}$$

Example 5. Consider a hyper UP-algebra $(X, \circ, \ll, 0)$ of Example 1 where $X = \{0, r, s, t\}$. Also, define a single-valued neutrosophic set $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ in X by the following:

$$\begin{aligned} \mathcal{T}_A(x) &= \begin{pmatrix} 0 & r & s & t \\ 0.87 & 0.42 & 0.56 & 0.29 \end{pmatrix}, \\ \mathcal{I}_A(x) &= \begin{pmatrix} 0 & r & s & t \\ 0.39 & 0.79 & 0.76 & 0.94 \end{pmatrix}, \text{and} \\ \mathcal{F}_A(x) &= \begin{pmatrix} 0 & r & s & t \\ 0.49 & 0.83 & 0.53 & 0.95 \end{pmatrix}. \end{aligned}$$

By routine calculation, A is a SVN hyper UP-subalgebra of X.

Example 6. Consider $X = \mathbb{N} \cup \{0\}$ and a hyperoperation "$\circ$" on X defined by

$$x \circ y = \begin{cases} \{0\} & \text{if } y = 0, \\ \{0, y\} & \text{if } y = x, y \neq 0, \\ \{y\} & \text{otherwise.} \end{cases}$$

By thorough inspection, X is a hyper UP-algebra. Define a single-valued neutrosophic set $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ in X by

$$\mathcal{T}_A(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x \neq 0. \end{cases}$$

$$\mathcal{I}_A(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.5 & \text{if } x \neq 0. \end{cases}$$

$$\mathcal{F}_A(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.5 & \text{if } x \neq 0. \end{cases}$$

Again, by thorough inspection, A is a SVN hyper UP-subalgebra of X.

Proposition 3. *Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a SVN hyper UP-subalgebra of X. Then for all $x, y \in X$,*

(i) $\quad \mathcal{T}_A(0) \geq \mathcal{T}_A(x)$
$\quad \mathcal{I}_A(0) \leq \mathcal{I}_A(x)$
$\quad \mathcal{F}_A(0) \leq \mathcal{F}_A(x)$.

(ii) $\quad {}_{*}\mathcal{T}_A(0 \circ x) = \mathcal{T}_A(x)$
$\quad {}^{*}\mathcal{I}_A(0 \circ x) = \mathcal{I}_A(x)$
$\quad {}^{*}\mathcal{F}_A(0 \circ x) = \mathcal{F}_A(x)$

(iii) $\begin{array}{l} {}_*\mathcal{T}_A(x \circ 0) = \mathcal{T}_A(0) \\ {}^*\mathcal{I}_A(x \circ 0) = \mathcal{I}_A(0) \\ {}^*\mathcal{F}_A(x \circ 0) = \mathcal{F}_A(0) \end{array}$

(iv) *If* $\begin{array}{l} \mathcal{T}_A(x) = \mathcal{T}_A(0) \\ \mathcal{I}_A(x) = \mathcal{I}_A(0) \\ \mathcal{F}_A(x) = \mathcal{F}_A(0) \end{array}$ *, then* $\begin{array}{l} {}_*\mathcal{T}_A(x \circ y) \geq \mathcal{T}_A(y) \\ {}^*\mathcal{I}_A(x \circ y) \leq \mathcal{I}_A(y) \\ {}^*\mathcal{F}_A(x \circ y) \leq \mathcal{F}_A(y) \end{array}$.

(v) *If* $\begin{array}{l} \mathcal{T}_A(y) = \mathcal{T}_A(0) \\ \mathcal{I}_A(y) = \mathcal{I}_A(0) \\ \mathcal{F}_A(y) = \mathcal{F}_A(0) \end{array}$ *, then* $\begin{array}{l} {}_*\mathcal{T}_A(x \circ y) \geq \mathcal{T}_A(x) \\ {}^*\mathcal{I}_A(x \circ y) \leq \mathcal{I}_A(x) \\ {}^*\mathcal{F}_A(x \circ y) \leq \mathcal{F}_A(x) \end{array}$.

(vi) *If* $\begin{array}{l} {}_*\mathcal{T}_A(x \circ y) = \mathcal{T}_A(x) \\ {}^*\mathcal{I}_A(x \circ y) = \mathcal{I}_A(x) \\ {}^*\mathcal{F}_A(x \circ y) = \mathcal{F}_A(x) \end{array}$ *, then* $\begin{array}{l} \mathcal{T}_A(x) = \mathcal{T}_A(0) \\ \mathcal{I}_A(x) = \mathcal{I}_A(0) \\ \mathcal{F}_A(x) = \mathcal{F}_A(0) \end{array}$ *and* $\begin{array}{l} \mathcal{T}_A(y) = \mathcal{T}_A(0) \\ \mathcal{I}_A(y) = \mathcal{I}_A(0) \\ \mathcal{F}_A(y) = \mathcal{F}_A(0) \end{array}$.

Proof. Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a *SVN* hyper *UP*-subalgebra in X and let $x, y \in X$

(i) Note that $x \ll x$. Then $0 \in x \circ x$. By hypothesis,

$$\begin{array}{rcl} \mathcal{T}_A(0) & \geq & {}_*\mathcal{T}_A(x \circ x) \geq \min\{\mathcal{T}_A(x), \mathcal{T}_A(x)\} = \mathcal{T}_A(x), \\ \mathcal{I}_A(0) & \leq & {}^*\mathcal{I}_A(x \circ x) \leq \max\{\mathcal{I}_A(x), \mathcal{I}_A(x)\} = \mathcal{I}_A(x), \text{ and} \\ \mathcal{F}_A(0) & \leq & {}^*\mathcal{F}_A(x \circ x) \leq \max\{\mathcal{F}_A(x), \mathcal{F}_A(x)\} = \mathcal{F}_A(x). \end{array}$$

$(ii\text{-}iii)$ The proofs are straightforward since $0 \circ x = \{x\}$ and $x \circ 0 = \{0\}$.

(iv) Assume that $\mathcal{T}_A(x) = \mathcal{T}_A(0)$. By hypothesis and by (i), ${}_*\mathcal{T}_A(x \circ y) \geq \min\{\mathcal{T}_A(0), \mathcal{T}_A(y)\} = \mathcal{T}_A(y)$. Using similar routine, $\mathcal{I}_A(x) = \mathcal{I}_A(0)$ implies that ${}^*\mathcal{I}_A(x \circ y) \leq \mathcal{I}_A(y)$ and $\mathcal{F}_A(x) = \mathcal{F}_A(0)$ implies that ${}^*\mathcal{F}_A(x \circ y) \leq \mathcal{F}_A(y)$.

(v) Using similar arguments from (iv), the claim is true.

(vi) Assume that ${}_*\mathcal{T}_A(x \circ y) = \mathcal{T}_A(x)$. Taking $x = 0$, we have ${}_*\mathcal{T}_A(0 \circ y) = \mathcal{T}_A(0)$. By (ii), $\mathcal{T}_A(y) = {}_* \mathcal{T}_A(0 \circ y) = \mathcal{T}_A(0)$. Similarly, ${}^*\mathcal{I}_A(x \circ y) = \mathcal{I}_A(x)$ implies that $\mathcal{I}_A(y) = \mathcal{I}_A(0)$ and ${}^*\mathcal{F}_A(x \circ y) = \mathcal{F}_A(x)$ implies that $\mathcal{F}_A(y) = \mathcal{F}_A(0)$. On the other hand, if we take $y = 0$, we get ${}_*\mathcal{T}_A(x \circ 0) = \mathcal{T}_A(x)$. By (iii), $\mathcal{T}_A(0) = {}_* \mathcal{T}_A(x \circ 0) = \mathcal{T}_A(x)$. Also, $\mathcal{I}_A(x) = \mathcal{I}_A(0)$ and $\mathcal{F}_A(x) = \mathcal{F}_A(0)$ will follow. □

Proposition 4. *If* $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ *is a SVN hyper UP-subalgebra of* X*, then the set* $K = \{x \in X | \mathcal{T}_A(x) = \mathcal{T}_A(0), \mathcal{I}_A(x) = \mathcal{I}_A(0), \mathcal{F}_A(x) = \mathcal{F}_A(0)\}$ *is a hyper UP-subalgebra of* X*.*

Proof. Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a *SVN* hyper *UP*-subalgebra of X and let $K = \{x \in X | \mathcal{T}_A(x) = \mathcal{T}_A(0), \mathcal{I}_A(x) = \mathcal{I}_A(0), \mathcal{F}_A(x) = \mathcal{F}_A(0)\}$. Note that $K \neq \varnothing$ since $0 \in K$.

Now, suppose $x, y \in K$ and $z \in x \circ y$. Then $\mathcal{T}_A(x) = \mathcal{T}_A(0) = \mathcal{T}_A(y), \mathcal{I}_A(x) = \mathcal{I}_A(0) = \mathcal{I}_A(y), \mathcal{F}_A(x) = \mathcal{F}_A(0) = \mathcal{F}_A(y)$. By Proposition 3$(i)$ and by hypothesis, we get

$$\begin{aligned}\mathcal{T}_A(z) &\geq {}_*\mathcal{T}_A(x \circ y)\\ &\geq \min\{\mathcal{T}_A(x), \mathcal{T}_A(y)\}\\ &= \min\{\mathcal{T}_A(0), \mathcal{T}_A(0)\}\\ &= \mathcal{T}_A(0),\end{aligned}$$

$$\begin{aligned}\mathcal{I}_A(z) &\leq {}^*\mathcal{I}_A(x \circ y)\\ &\leq \max\{\mathcal{I}_A(x), \mathcal{I}_A(y)\}\\ &= \max\{\mathcal{I}_A(0), \mathcal{I}_A(0)\}\\ &= \mathcal{I}_A(0),\end{aligned}$$

and similarly,

$$\mathcal{F}_A(z) \leq \mathcal{F}_A(0).$$

Thus, $\mathcal{T}_A(z) = \mathcal{T}_A(0), \mathcal{I}_A(z) = \mathcal{I}_A(0)$, and $\mathcal{F}_A(z) = \mathcal{F}_A(0)$. That is, $z \in K$ and so $x \circ y \subseteq K$. By Proposition 2, K is a hyper UP-subalgebra of X. $\square$

We define the following α, β, γ-level subsets of X and their intersection:

$$\begin{aligned}T_A^\alpha &= \{x \in X : \mathcal{T}_A(x) \geq \alpha\},\\ I_A^\beta &= \{x \in X : \mathcal{I}_A(x) \leq \beta\},\\ F_A^\gamma &= \{x \in X : \mathcal{F}_A(x) \leq \gamma\}, \text{ and}\\ A^{(\alpha,\beta,\gamma)} &= T_A^\alpha \cap I_A^\beta \cap F_A^\gamma.\end{aligned}$$

where $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ is a SVN set in X and $\alpha, \beta, \gamma \in [0, 1]$.

Theorem 1. *Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a SVN set in X. Then A is a SVN hyper UP-subalgebra of X if and only if $A^{(\alpha,\beta,\gamma)}$ is a hyper UP-subalgebra of X for all $\alpha, \beta, \gamma \in [0, 1]$.*

Proof. Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a SVN set in X.

($\Rightarrow$) Assume that A is a SVN hyper UP-subalgebra of X. Note that $\mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x) \in [0, 1]$ $\forall x \in X$. Then take $\alpha = \mathcal{T}_A(x)$, $\beta = \mathcal{I}_A(x)$ and $\gamma = \mathcal{F}_A(x)$. By Proposition 3$(i)$, $\mathcal{T}_A(0) \geq \mathcal{T}_A(x) = \alpha, \mathcal{I}_A(0) \leq \mathcal{I}_A(x) = \beta$, and $\mathcal{F}_A(0) \leq \mathcal{F}_A(x) = \gamma$. Hence, $0 \in T_A^\alpha \cap I_A^\alpha \cap F_A^\alpha = A^{(\alpha,\beta,\gamma)}$ and so $A^{(\alpha,\beta,\gamma)} \neq \varnothing$. Now, we let $x, y \in A^{(\alpha,\beta,\gamma)}$ for all $\alpha, \beta, \gamma \in [0, 1]$. Dealing first with T_A^α, we have $x, y \in T_A^\alpha$. Let $z \in x \circ y$. Then $\mathcal{T}_A(x) \geq \alpha$, $\mathcal{T}_A(y) \geq \alpha$, and $\mathcal{T}_A(z) \geq_* \mathcal{T}_A(x \circ y)$. By assumption,

$$\begin{aligned}\mathcal{T}_A(z) &\geq {}_*\mathcal{T}_A(x \circ y)\\ &\geq \min\{\mathcal{T}_A(x), \mathcal{T}_A(y)\}\\ &= \alpha.\end{aligned}$$

Thus, $z \in T_A^\alpha$ and so $x \circ y \subseteq T_A^\alpha$. For I_A^β, we have $x, y \in I_A^\beta$. Then $\mathcal{I}_A(x) \leq \beta$, $\mathcal{I}_A(y) \leq \beta$, and $\mathcal{I}_A(z) \leq^* \mathcal{I}_A(x \circ y)$. By assumption,

$$\begin{aligned} \mathcal{I}_A(z) &\leq {}^*\mathcal{I}_A(x \circ y) \\ &\leq \max\{\mathcal{I}_A(x), \mathcal{I}_A(y)\} \\ &= \beta. \end{aligned}$$

Thus, $z \in I_A^\beta$ and so $x \circ y \subseteq I_A^\beta$. Using similar arguments, $x \circ y \subseteq F_A^\gamma$ for $x, y \in F_A^\gamma$. Now, it follows that $x \circ y \subseteq T_A^\alpha \cap I_A^\beta \cap F_A^\gamma = A^{(\alpha,\beta,\gamma)}$. By Proposition 2, $A^{(\alpha,\beta,\gamma)}$ is a hyper *UP*-subalgebra.

($\Leftarrow$) Assume that $A^{(\alpha,\beta,\gamma)}$ is a hyper *UP*-subalgebra of X for all $\alpha, \beta, \gamma \in [0,1]$ and let $x, y \in X$. Note that $\mathcal{T}_A(x), \mathcal{T}_A(y), \mathcal{I}_A(x), \mathcal{I}_A(y), \mathcal{F}_A(x), \mathcal{F}_A(y) \in [0,1]$. Then take $\alpha = \min\{\mathcal{T}_A(x), \mathcal{T}_A(y)\}$, $\beta = \max\{\mathcal{I}_A(x), \mathcal{I}_A(y)\}$, and $\gamma = \max\{\mathcal{T}_A(x), \mathcal{T}_A(y)\}$ and so we have $\mathcal{T}_A(x) \geq \alpha$, $\mathcal{T}_A(y) \geq \alpha$, $\mathcal{I}_A(x) \leq \beta$, $\mathcal{I}_A(y) \leq \beta$, $\mathcal{F}_A(x) \leq \gamma$, and $\mathcal{F}_A(y) \leq \gamma$. Thus, $x, y \in T_A^\alpha \cap I_A^\beta \cap F_A^\gamma = A^{(\alpha,\beta,\gamma)}$. By assumption, $x \circ y \subseteq A^{(\alpha,\beta,\gamma)}$. This means that

$$\begin{aligned} {}_*\mathcal{T}_A(x \circ y) &\geq \alpha = \min\{\mathcal{T}_A(x), \mathcal{T}_A(y)\}, \\ {}^*\mathcal{I}_A(x \circ y) &\leq \beta = \max\{\mathcal{I}_A(x), \mathcal{I}_A(y)\}, \text{ and} \\ {}^*\mathcal{F}_A(x \circ y) &\leq \gamma = \max\{\mathcal{F}_A(x), \mathcal{F}_A(y)\}. \end{aligned}$$

Hence, A is a *SVN* hyper *UP*-subalgebra of X. $\square$

Corollary 1. *Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a SVN hyper UP-subalgebra of X. If $0 \leq \alpha \leq \alpha' \leq 1$, $0 \leq \beta \leq \beta' \leq 1$, and $0 \leq \gamma \leq \gamma' \leq 1$, then $A^{(\alpha',\beta,\gamma)}$ is a hyper UP-subalgebra of $A^{(\alpha,\beta',\gamma')}$.*

Proof. Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a *SVN* hyper *UP*-subalgebra of X and let $0 \leq \alpha \leq \alpha' \leq 1$, $0 \leq \beta \leq \beta' \leq 1$, and $0 \leq \gamma \leq \gamma' \leq 1$. By Theorem 1, $A^{(\alpha',\beta,\gamma)}$ and $A^{(\alpha,\beta',\gamma')}$ are both hyper *UP*-subalgebra of X. We are left to show that $A^{(\alpha',\beta,\gamma)} \subseteq A^{(\alpha,\beta',\gamma')}$. Let $y \in T_A^{\alpha'}$. Then $\mathcal{T}_A(y) \geq \alpha' \geq \alpha$. Thus, $y \in T_A^\alpha$ and so $T_A^{\alpha'} \subseteq T_A^\alpha$. Next, let $z \in I_A^\beta$. Then $\mathcal{I}_A(z) \leq \beta \leq \beta'$. Thus, $z \in I_A^{\beta'}$ and so $I_A^\beta \subseteq I_A^{\beta'}$. Similarly, $F_A^\gamma \subseteq F_A^{\gamma'}$. Hence, $A^{(\alpha',\beta,\gamma)} = T_A^{\alpha'} \cap I_A^\beta \cap F_A^\gamma \subseteq T_A^\alpha \cap I_A^{\beta'} \cap F_A^{\gamma'} = A^{(\alpha,\beta',\gamma')}$. Consequently, $A^{(\alpha',\beta,\gamma)}$ is a hyper *UP*-subalgebra of $A^{(\alpha,\beta',\gamma')}$. $\square$

For fixed numbers $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in [0,1]$ such that $\alpha_1 \geq \alpha_2$, $\beta_1 \geq \beta_2$, $\gamma_1 \geq \gamma_2$, and a nonempty subset G of X, we define a *SVN* set

$$A_G \begin{bmatrix} \alpha_1, \beta_2, \gamma_2 \\ \alpha_2, \beta_1, \gamma_1 \end{bmatrix} = \left(\mathcal{T}_{A_G} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \mathcal{I}_{A_G} \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix}, \mathcal{F}_{A_G} \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \right)$$

where

$$\mathcal{T}_{A_G} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (x) = \begin{cases} \alpha_1 & \text{if } x \in G \\ \alpha_2 & \text{otherwise} \end{cases},$$

$$\mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(x)=\begin{cases}\beta_2 & \text{if } x\in G\\ \beta_1 & \text{otherwise}\end{cases},$$

and

$$\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(x)=\begin{cases}\gamma_2 & \text{if } x\in G\\ \gamma_1 & \text{otherwise}\end{cases}.$$

Theorem 2. *Let G be a nonempty subset of X and $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ be a SVN set in X. Then $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ satisfies Proposition 3(i) if and only if $0\in G$.*

Proof. Let G be a nonempty subset of X and $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ be a SVN set in X.

($\Rightarrow$) Assume that $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ satisfies Proposition 3(i). Since $G\neq\varnothing$, there exists $g\in G$. Thus, $\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(g)=\alpha_1$. Now,

$$\begin{aligned}\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(0) &\geq \mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(g)\\ &= \alpha_1\\ &\geq \mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(0).\end{aligned}$$

That is, $\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(0)=\alpha_1$. Hence, $0\in G$.

($\Leftarrow$) Assume that $0\in G$. Then $\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(0)=\alpha_1$, $\mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(0)=\beta_2$ and $\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(0)=\gamma_2$. For all $x\in X$,

$$\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(0)=\alpha_1\geq\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(x),$$

$$\mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(0)=\beta_2\leq\mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(x)$$

and

$$\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(0)=\gamma_2\leq\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(x).$$

Hence, $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ satisfies Proposition 3(i). □

Theorem 3. *Let* $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ *be a SVN set in* X. *Then* $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ *is a SVN hyper UP-subalgebra of* X *if and only if a nonempty subset of* G *is a hyper UP-subalgebra of* X.

Proof. Let $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ be a SVN set in X.

($\Rightarrow$) Assume that $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ is a SVN hyper UP-subalgebra of X. Since $G\neq\varnothing$, we let $x,y\in G$ and $z\in x\circ y$. Then

$$\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(x)=\alpha_1=\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(y).$$

By assumption,

$$\begin{aligned}\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(z) &\geq {}_*\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(x\circ y)\\ &\geq \min\left\{\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(x),\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(y)\right\}\\ &= \alpha_1\\ &\geq \mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(z).\end{aligned}$$

That is, $\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(z)=\alpha_1$. Thus, $z\in G$ and so $x\circ y\subseteq G$. By Proposition 2, G is a hyper UP-subalgebra of X.

($\Leftarrow$) Assume that G is a hyper UP-subalgebra of X and suppose $x,y\in X$. Consider the following cases:

Case 1. $x,y\in G$

By assumption, $x\circ y\subseteq G$. Thus,

$${}_*\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(x\circ y)=\alpha_1\geq\alpha_1=\min\left\{\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(x),\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(y)\right\},$$

$${}^*\mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(x\circ y)=\beta_2\leq\beta_2=\max\left\{\mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(x),\mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(y)\right\},$$

and

$${}^*\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(x\circ y)=\gamma_2\leq\gamma_2=\max\left\{\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(x),\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(y)\right\}.$$

Case 2. $x\in G$ and $y\notin G$

So we have

$${}_*\mathcal{T}_{A_G}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(x\circ y)\quad\geq\quad\alpha_2$$

$$\begin{aligned} &= \min\{\alpha_1, \alpha_2\} \\ &= \min\left\{\mathcal{T}_{A_G}\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right](x), \mathcal{T}_{A_G}\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right](y)\right\}, \end{aligned}$$

$$\begin{aligned} {}^*\mathcal{I}_{A_G}\left[\begin{array}{c}\beta_2\\ \beta_1\end{array}\right](x\circ y) &\leq \beta_1 \\ &= \max\{\beta_2, \beta_1\} \\ &= \max\left\{\mathcal{I}_{A_G}\left[\begin{array}{c}\beta_2\\ \beta_1\end{array}\right](x), \mathcal{I}_{A_G}\left[\begin{array}{c}\beta_2\\ \beta_1\end{array}\right](y)\right\}, \end{aligned}$$

and

$$\begin{aligned} {}^*\mathcal{F}_{A_G}\left[\begin{array}{c}\gamma_2\\ \gamma_1\end{array}\right](x\circ y) &\leq \gamma_1 \\ &= \max\{\gamma_2, \gamma_1\} \\ &= \max\left\{\mathcal{F}_{A_G}\left[\begin{array}{c}\gamma_2\\ \gamma_1\end{array}\right](x), \mathcal{F}_{A_G}\left[\begin{array}{c}\gamma_2\\ \gamma_1\end{array}\right](y)\right\} \end{aligned}$$

.

Case 3. $x \notin G$ and $y \in G$

Using similar routine done in case 2, we have

$${}_*\mathcal{T}_{A_G}\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right](x\circ y) \geq \min\left\{\mathcal{T}_{A_G}\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right](x), \mathcal{T}_{A_G}\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right](y)\right\},$$

$${}^*\mathcal{I}_{A_G}\left[\begin{array}{c}\beta_2\\ \beta_1\end{array}\right](x\circ y) \leq \max\left\{\mathcal{I}_{A_G}\left[\begin{array}{c}\beta_2\\ \beta_1\end{array}\right](x), \mathcal{I}_{A_G}\left[\begin{array}{c}\beta_1\\ \beta_2\end{array}\right](y)\right\},$$

and

$${}^*\mathcal{F}_{A_G}\left[\begin{array}{c}\gamma_2\\ \gamma_1\end{array}\right](x\circ y) \leq \max\left\{\mathcal{F}_{A_G}\left[\begin{array}{c}\gamma_2\\ \gamma_1\end{array}\right](x), \mathcal{F}_{A_G}\left[\begin{array}{c}\gamma_2\\ \gamma_1\end{array}\right](y)\right\}.$$

Case 4. $x \notin G$ and $y \notin G$

Now,

$$\begin{aligned} {}_*\mathcal{T}_{A_G}\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right](x\circ y) &\geq \alpha_2 \\ &= \min\{\alpha_2, \alpha_2\} \\ &= \min\left\{\mathcal{T}_{A_G}\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right](x), \mathcal{T}_{A_G}\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right](y)\right\}, \end{aligned}$$

$$\begin{aligned} {}^*\mathcal{I}_{A_G}\left[\begin{array}{c}\beta_2\\ \beta_1\end{array}\right](x\circ y) &\leq \beta_1 \\ &= \max\{\beta_1, \beta_1\} \end{aligned}$$

$$= \max\left\{\mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(x), \mathcal{I}_{A_G}\begin{bmatrix}\beta_2\\ \beta_1\end{bmatrix}(y)\right\},$$

and

$$\begin{aligned}{}^{*}\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(x\circ y) &\leq \gamma_1\\ &= \max\{\gamma_1,\gamma_1\}\\ &= \max\left\{\mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(x), \mathcal{F}_{A_G}\begin{bmatrix}\gamma_2\\ \gamma_1\end{bmatrix}(y)\right\}.\end{aligned}$$

Hence, $A_G\begin{bmatrix}\alpha_1,\beta_2,\gamma_2\\ \alpha_2,\beta_1,\gamma_1\end{bmatrix}$ is a SVN hyper UP-subalgebra of X. □

Theorem 4. *Let G be a hyper UP-subalgebra of X. Then there exists a SVN hyper UP-subalgebra $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ of X such that $A^{(\alpha,\beta,\gamma)} = G$ for $\alpha, \beta, \gamma \in [0, 1]$.*

Proof. Let G be a hyper UP-subalgebra of X. For fixed $\alpha, \beta, \gamma \in (0, 1]$, consider $A = A_G\begin{bmatrix}\alpha,0,0\\ 0,\beta,\gamma\end{bmatrix}$. Since G be a hyper UP-subalgebra of X, $A_G\begin{bmatrix}\alpha,0,0\\ 0,\beta,\gamma\end{bmatrix}$ is a SVN hyper UP-subalgebra of X by Theorem 3. Now, let $x \in G$. Then

$$\mathcal{T}_A(x) = \mathcal{T}_{A_G}\begin{bmatrix}\alpha\\ 0\end{bmatrix}(x) = \alpha \geq \alpha,$$

$$\mathcal{I}_A(x) = \mathcal{I}_{A_G}\begin{bmatrix}0\\ \beta\end{bmatrix}(x) = 0 \leq \beta,$$

and

$$\mathcal{F}_A(x) = \mathcal{F}_{A_G}\begin{bmatrix}0\\ \gamma\end{bmatrix}(x) = 0 \leq \gamma.$$

Thus, $x \in T_A^{\alpha} \cap I_A^{\beta} \cap F_A^{\gamma} = A^{(\alpha,\beta,\gamma)}$ and so $G \subseteq A^{(\alpha,\beta,\gamma)}$. Also, let $y \in A^{(\alpha,\beta,\gamma)}$. Then $\mathcal{T}_A(y) \geq \alpha$, $\mathcal{I}_A(y) \leq \beta$, and $\mathcal{F}_A(y) \leq \gamma$. Suppose that $y \notin G$. Then $0 = \mathcal{T}_{A_G}\begin{bmatrix}\alpha\\ 0\end{bmatrix}(y) = \mathcal{T}_A(y) \geq \alpha$. It follows that $\alpha = 0$. This is a contradiction since $\alpha \in (0, 1]$. Thus, $y \in G$ and so $A^{(\alpha,\beta,\gamma)} \subseteq G$. Consequently, $A^{(\alpha,\beta,\gamma)} = G$. □

Theorem 5. *Given a chain of hyper UP-subalgebras of X:*

$$A_0 \subset A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_n = X.$$

Then there exists a SVN hyper UP- subalgebra $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ of X such that $A^{(\alpha_k,\beta_k,\gamma_k)} = A_k$ where $\alpha_k, \beta_k, \gamma_k \in [0, 1]$ for $0 \leq k \leq n$.

Proof. Let $\{\alpha_k | k = 0, 1, \ldots, n\}$ be a finite decreasing sequence and $\{\beta_k | k = 0, 1, \ldots, n\}$, $\{\gamma_k | k = 0, 1, \ldots, n\}$ be finite increasing sequences such that $\alpha_k, \beta_k, \gamma_k \in [0, 1]$ for $1 \leq k \leq n$. Define a *SVN* set $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ in X by $\mathcal{T}_A(A_0) = \alpha_0$, $\mathcal{I}_A(A_0) = \beta_0$, $\mathcal{F}_A(A_0) = \gamma_0$,$\mathcal{T}_A(A_k \setminus A_{k-1}) = \alpha_k$, $\mathcal{I}_A(A_k \setminus A_{k-1}) = \beta_k$, and $\mathcal{F}_A(A_k \setminus A_{k-1}) = \gamma_k$ for $1 \leq k \leq n$. We will show that A is a *SVN* hyper *UP*-subalgebra of X. Let $a, b \in X$. Consider the following cases:

Case 1. $a, b \in A_k \setminus A_{k-1}$
Then $\mathcal{T}_A(a) = \alpha_k = \mathcal{T}_A(b)$, $\mathcal{I}_A(a) = \beta_k = \mathcal{I}_A(b)$, and $\mathcal{F}_A(a) = \gamma_k = \mathcal{F}_A(b)$. Since A_k is a hyper *UP*-subalgebra of X, $a \circ b \subseteq A_k$.

Subcase 1.1. $a \circ b \subseteq A_k \setminus A_{k-1}$

$$_*\mathcal{T}_A(a \circ b) = \alpha_k \geq \alpha_k = \min\{\mathcal{T}_A(a), \mathcal{T}_A(b)\},$$

$$^*\mathcal{I}_A(a \circ b) = \beta_k \leq \beta_k = \max\{\mathcal{I}_A(a), \mathcal{I}_A(b)\},$$

and

$$^*\mathcal{F}_A(a \circ b) = \gamma_k \leq \gamma_k = \max\{\mathcal{F}_A(a), \mathcal{F}_A(b)\}.$$

Subcase 1.2. $a \circ b \subseteq A_{k-1}$
For some $r \in [0, k-1]$, we have

$$_*\mathcal{T}_A(a \circ b) = \alpha_{k-1-r} \geq \alpha_k = \min\{\mathcal{T}_A(a), \mathcal{T}_A(b)\},$$

$$^*\mathcal{I}_A(a \circ b) = \beta_{k-1-r} \leq \beta_k = \max\{\mathcal{I}_A(a), \mathcal{I}_A(b)\},$$

and

$$^*\mathcal{F}_A(a \circ b) = \gamma_{k-1-r} \leq \gamma_k = \max\{\mathcal{F}_A(a), \mathcal{F}_A(b)\}.$$

Subcase 1.3. $a \circ b = [(a \circ b) \cap (A_k \setminus A_{k-1})] \cup [(a \circ b) \cap A_{k-1}]$ where $(a \circ b) \cap (A_k \setminus A_{k-1}) \neq \varnothing$ and $(a \circ b) \cap A_{k-1} \neq \varnothing$

$$_*\mathcal{T}_A(a \circ b) = \alpha_k \geq \alpha_k = \min\{\mathcal{T}_A(a), \mathcal{T}_A(b)\},$$

$$^*\mathcal{I}_A(a \circ b) = \beta_k \leq \beta_k = \max\{\mathcal{I}_A(a), \mathcal{I}_A(b)\},$$

and

$$^*\mathcal{F}_A(a \circ b) = \gamma_k \leq \gamma_k = \max\{\mathcal{F}_A(a), \mathcal{F}_A(b)\}.$$

Case 2. $a \in A_i \setminus A_{i-1}$ and $b \in A_j \setminus A_{j-1}$ for $i > j > 0$
Then $\mathcal{T}_A(a) = \alpha_i$, $\mathcal{T}_A(b) = \alpha_j$, $\mathcal{I}_A(a) = \beta_i$, $\mathcal{I}_A(b) = \beta_j$, $\mathcal{F}_A(a) = \gamma_i$, and $\mathcal{F}_A(b) = \gamma_j$. Since A_i is a hyper *UP*-subalgebra of X and $A_j \subset A_i$, we have $a \circ b \subseteq A_i$.

Subcase 2.1. $a \circ b \subseteq A_i \setminus A_j$
For some $r \in [0, i-j-1]$, we have

$$_*\mathcal{T}_A(a \circ b) = \alpha_{i-r} \geq \alpha_i = \min\{\mathcal{T}_A(a), \mathcal{T}_A(b)\},$$

$$^{*}\mathcal{I}_A(a \circ b) = \beta_{i-r} \leq \beta_i = \max\{\mathcal{I}_A(a), \mathcal{I}_A(b)\},$$

and

$$^{*}\mathcal{F}_A(a \circ b) = \gamma_{i-r} \leq \gamma_i = \max\{\mathcal{F}_A(a), \mathcal{F}_A(b)\}.$$

Subcase 2.2. $a \circ b \subseteq A_j$
For some $r \in [0, j]$, we have

$$_{*}\mathcal{T}_A(a \circ b) = \alpha_{j-r} \geq \alpha_i = \min\{\mathcal{T}_A(a), \mathcal{T}_A(b)\},$$

$$^{*}\mathcal{I}_A(a \circ b) = \beta_{j-r} \leq \beta_i = \max\{\mathcal{I}_A(a), \mathcal{I}_A(b)\},$$

and

$$^{*}\mathcal{F}_A(a \circ b) = \gamma_{j-r} \leq \gamma_i = \max\{\mathcal{F}_A(a), \mathcal{F}_A(b)\}.$$

Subcase 2.3. $a \circ b = [(a \circ b) \cap (A_i \setminus A_j)] \cup [(a \circ b) \cap A_j]$ where $(a \circ b) \cap (A_i \setminus A_j) \neq \varnothing$ and $(a \circ b) \cap A_j \neq \varnothing$
For some $r \in [0, i-j-1]$, we have

$$_{*}\mathcal{T}_A(a \circ b) = \alpha_{i-r} \geq \alpha_i = \min\{\mathcal{T}_A(a), \mathcal{T}_A(b)\},$$

$$^{*}\mathcal{I}_A(a \circ b) = \beta_{i-r} \leq \beta_i = \max\{\mathcal{I}_A(a), \mathcal{I}_A(b)\},$$

and

$$^{*}\mathcal{F}_A(a \circ b) = \gamma_{i-r} \leq \gamma_i = \max\{\mathcal{F}_A(a), \mathcal{F}_A(b)\}.$$

Thus, A is a SVN hyper UP-subalgebra of X. Furthermore, note that

$$T_A^{\alpha_0} = \{s \in X | \mathcal{T}_A(s) \geq \alpha_0\} = A_0,$$

$$I_A^{\beta_0} = \{s \in X | \mathcal{I}_A(s) \leq \beta_0\} = A_0,$$

and

$$F_A^{\gamma_0} = \{s \in X | \mathcal{F}_A(s) \leq \gamma_0\} = A_0.$$

Thus, $A_0 = T_A^{\alpha_0} \cap I_A^{\beta_0} \cap F_A^{\gamma_0} = A^{(\alpha_0,\beta_0,\gamma_0)}$. For $0 < k \leq n$, let $x \in A_k$. Then $x \in A_{k-i} \setminus A_{k-i-1}$ $\exists 0 \leq i \leq k-1$. Thus,

$$\mathcal{T}_A(x) = \alpha_{k-i} \geq \alpha_k,$$

$$\mathcal{I}_A(x) = \beta_{k-i} \leq \beta_k$$

and

$$\mathcal{F}_A(x) = \gamma_{k-i} \leq \gamma_k$$

$\exists 0 \leq i \leq k-1$. So we have $x \in T_A^{\alpha_k} \cap I_A^{\beta_k} \cap F_A^{\gamma_k} = A^{(\alpha_k,\beta_k,\gamma_k)}$ and $A_k \subseteq A^{(\alpha_k,\beta_k,\gamma_k)}$. Also, let $y \in A^{(\alpha_k,\beta_k,\gamma_k)}$. Then $\mathcal{T}_A(y) \geq \alpha_k$, $\mathcal{I}_A(y) \leq \beta_k$, and $\mathcal{F}_A(y) \leq \gamma_k$. The values of $\mathcal{T}_A(y)$, $\mathcal{I}_A(y)$, and $\mathcal{F}_A(y)$ that will make the three inequalities true are $\mathcal{T}_A(y) = \alpha_t$, $\mathcal{I}_A(y) = \beta_t$, and $\mathcal{F}_A(y) = \gamma_t$ $\exists 0 < t \leq k$. This implies that $y \in A_{k-i} \setminus A_{k-i-1}$ $\exists 0 \leq i \leq k-1$. That is, $y \in A_k$ since $(A_{k-i} \setminus A_{k-i-1}) \subseteq A_k$ $\exists 0 \leq i \leq k-1$. Hence, $A^{(\alpha_k,\beta_k,\gamma_k)} \subseteq A_k$. Consequently, $A^{(\alpha_k,\beta_k,\gamma_k)} = A_k$. □

3.2. Single-Valued Neutrosophic Soft Hyper *UP*-subalgebra

In this section, we define the single-valued neutrosophic soft hyper UP-subalgebra and prove some related properties.

Definition 17. Let (Δ, E) be a single-valued neutrosophic soft set over a hyper UP-algebra X. Then (Δ, E) is said to be *single-valued neutrosophic soft* $(SVNS)$ *hyper UP-subalgebra of* X if for all $x, y \in X$ and $e \in E$,

$$\begin{aligned} {}_{*}\mathcal{T}_{\Delta(e)}(x \circ y) &\geq \min\{\mathcal{T}_{\Delta(e)}(x), \mathcal{T}_{\Delta(e)}(y)\}, \\ {}^{*}\mathcal{I}_{\Delta(e)}(x \circ y) &\leq \max\{\mathcal{T}_{\Delta(e)}(x), \mathcal{T}_{\Delta(e)}(y)\}, \text{ and} \\ {}^{*}\mathcal{F}_{\Delta(e)}(x \circ y) &\leq \max\{\mathcal{T}_{\Delta(e)}(x), \mathcal{T}_{\Delta(e)}(y)\}; \end{aligned}$$

that is, $\Delta(e)$ is a SVN hyper UP-subalgebra of X.

Example 7. Consider the hyper UP-algebra $(X, \circ, \ll, 0)$ of Example 2 where $X = \{0, u, v\}$. Let $E = \{e_1, e_2\}$ be the set of parameters and let $\Delta : E \longrightarrow \mathcal{N}(X)$ be defined by

$$\begin{aligned} \Delta(e_1) &= \{\langle 0, (0.9, 0.2, 0.45)\rangle, \langle u, (0.74, 0.57, 0.7)\rangle, \langle v, (0.8, 0.42, 0.52)\rangle\} \text{ and} \\ \Delta(e_2) &= \{\langle 0, (0.8, 0.2, 0.4)\rangle, \langle u, (0.4, 0.45, 0.5)\rangle, \langle v, (0.67, 0.3, 0.5)\rangle\}. \end{aligned}$$

Then

$$\begin{aligned} (\Delta, E) = \; &\{(e_1, \{\langle 0, (0.9, 0.2, 0.45)\rangle, \langle u, (0.74, 0.57, 0.7)\rangle, \langle v, (0.8, 0.42, 0.52)\rangle\}), \\ &(e_2, \{\langle 0, (0.8, 0.2, 0.4)\rangle, \langle u, (0.4, 0.45, 0.5)\rangle, \langle v, (0.67, 0.3, 0.5)\rangle\})\}. \end{aligned}$$

is a $SVNS$ set over X. By routine calculation, (Δ, E) is a $SVNS$ hyper UP-subalgebra of X.

Proposition 5. *Let (Δ, E) be a SVNS hyper UP-subalgebra of X. Then for all $x, y \in X$ and $e \in E$,*

(i) $$\begin{array}{l} \mathcal{T}_{\Delta(e)}(x) \leq \mathcal{T}_{\Delta(e)}(0) \\ \mathcal{I}_{\Delta(e)}(x) \geq \mathcal{I}_{\Delta(e)}(0) \\ \mathcal{F}_{\Delta(e)}(x) \geq \mathcal{F}_{\Delta(e)}(0) \end{array},$$

(ii) $$\begin{array}{l} {}_{*}\mathcal{T}_{\Delta(e)}(0 \circ x) = \mathcal{T}_{\Delta(e)}(x). \\ {}^{*}\mathcal{I}_{\Delta(e)}(0 \circ x) = \mathcal{I}_{\Delta(e)}(x) \\ {}^{*}\mathcal{F}_{\Delta(e)}(0 \circ x) = \mathcal{F}_{\Delta(e)}(x). \end{array}$$

(iii) $$\begin{array}{l} {}_{*}\mathcal{T}_{\Delta(e)}(x \circ 0) = \mathcal{T}_{\Delta(e)}(0) \\ {}^{*}\mathcal{I}_{\Delta(e)}(x \circ 0) = \mathcal{I}_{\Delta(e)}(0) \\ {}^{*}\mathcal{F}_{\Delta(e)}(x \circ 0) = \mathcal{F}_{\Delta(e)}(0) \end{array}$$

(iv) *If* $$\begin{array}{l} \mathcal{T}_{\Delta(e)}(x) = \mathcal{T}_{\Delta(e)}(0) \\ \mathcal{I}_{\Delta(e)}(x) = \mathcal{I}_{\Delta(e)}(0) \\ \mathcal{F}_{\Delta(e)}(x) = \mathcal{F}_{\Delta(e)}(0) \end{array}, \textit{ then } \begin{array}{l} {}_{*}\mathcal{T}_{\Delta(e)}(x \circ y) \geq \mathcal{T}_{\Delta(e)}(y) \\ {}^{*}\mathcal{I}_{\Delta(e)}(x \circ y) \leq \mathcal{I}_{\Delta(e)}(y) \\ {}^{*}\mathcal{F}_{\Delta(e)}(x \circ y) \leq \mathcal{F}_{\Delta(e)}(y) \end{array}.$$

$$(v)\ \textit{If}\ \begin{array}{l}\mathcal{T}_{\Delta(e)}(y)=\mathcal{T}_{\Delta(e)}(0)\\ \mathcal{I}_{\Delta(e)}(y)=\mathcal{I}_{\Delta(e)}(0)\\ \mathcal{F}_{\Delta(e)}(y)=\mathcal{F}_{\Delta(e)}(0)\end{array},\ \textit{then}\ \begin{array}{l}{}_{*}\mathcal{T}_{\Delta(e)}(x\circ y)\geq \mathcal{T}_{\Delta(e)}(x)\\ {}^{*}\mathcal{I}_{\Delta(e)}(x\circ y)\leq \mathcal{I}_{\Delta(e)}(x)\\ {}^{*}\mathcal{F}_{\Delta(e)}(x\circ y)\leq \mathcal{F}_{\Delta(e)}(x)\end{array}.$$

$$(vi)\ \textit{If}\ \begin{array}{l}{}_{*}\mathcal{T}_{\Delta(e)}(x\circ y)=\mathcal{T}_{\Delta(e)}(x)\\ {}^{*}\mathcal{I}_{\Delta(e)}(x\circ y)=\mathcal{I}_{\Delta(e)}(x)\\ {}^{*}\mathcal{F}_{\Delta(e)}(x\circ y)=\mathcal{F}_{\Delta(e)}(x)\end{array},\ \textit{then}\ \begin{array}{l}\mathcal{T}_{\Delta(e)}(x)=\mathcal{T}_{\Delta(e)}(0)\\ \mathcal{I}_{\Delta(e)}(x)=\mathcal{I}_{\Delta(e)}(0)\\ \mathcal{F}_{\Delta(e)}(x)=\mathcal{F}_{\Delta(e)}(0)\end{array}\ \textit{and}\ \begin{array}{l}\mathcal{T}_{\Delta(e)}(y)=\mathcal{T}_{\Delta(e)}(0)\\ \mathcal{I}_{\Delta(e)}(y)=\mathcal{I}_{\Delta(e)}(0)\\ \mathcal{F}_{\Delta(e)}(y)=\mathcal{F}_{\Delta(e)}(0)\end{array}.$$

Proof. Using similar arguments from Proposition 3, this proposition is valid. □

Theorem 6. *Let (Δ_1, E) and (Δ_2, E) be two SVNS hyper UP-subalgebras of X. Then*

(i) *$(\Delta_1, E) \cap (\Delta_2, E)$ is a SVNS hyper UP-subalgebra of X.*

(ii) *$(\Delta_1, E) \cup (\Delta_2, E)$ is not generally a SVNS hyper UP-subalgebra of X.*

Proof. Let (Δ_1, E) and (Δ_2, E) be two *SVNS* hyper *UP*-subalgebras of X.

(i) Let $(\Delta, E) = (\Delta_1, E) \cap (\Delta_2, E)$, $x, y \in X$ and $e \in E$. Then we have

$$\begin{aligned}
{}_{*}\mathcal{T}_{\Delta(e)}(x\circ y) &= \inf_{a\in x\circ y}\mathcal{T}_{\Delta(e)}(a)\\
&= \inf_{a\in x\circ y}\min\{\mathcal{T}_{\Delta_1(e)}(a),\mathcal{T}_{\Delta_2(e)}(a)\}\\
&\geq \min\{\inf_{a\in x\circ y}\mathcal{T}_{\Delta_1(e)}(a),\inf_{a\in x\circ y}\mathcal{T}_{\Delta_2(e)}(a)\}\\
&= \min\{{}_{*}\mathcal{T}_{\Delta_1(e)}(x\circ y),{}_{*}\mathcal{T}_{\Delta_2(e)}(x\circ y)\}\\
&\geq \min\{\min\{\mathcal{T}_{\Delta_1(e)}(x),\mathcal{T}_{\Delta_1(e)}(y)\},\min\{\mathcal{T}_{\Delta_2(e)}(x),\mathcal{T}_{\Delta_2(e)}(y)\}\}\\
&= \min\{\min\{\mathcal{T}_{\Delta_1(e)}(x),\mathcal{T}_{\Delta_2(e)}(x)\},\min\{\mathcal{T}_{\Delta_1(e)}(y),\mathcal{T}_{\Delta_2(e)}(y)\}\}\\
&= \min\{\mathcal{T}_{\Delta(e)}(x),\mathcal{T}_{\Delta(e)}(y)\}.
\end{aligned}$$

Also,

$$\begin{aligned}
{}^{*}\mathcal{I}_{\Delta(e)}(x\circ y) &= \sup_{a\in x\circ y}\mathcal{I}_{\Delta(e)}(a)\\
&= \sup_{a\in x\circ y}\max\{\mathcal{I}_{\Delta_1(e)}(a),\mathcal{I}_{\Delta_2(e)}(a)\}\\
&\leq \max\{\sup_{a\in x\circ y}\mathcal{I}_{\Delta_1(e)}(a),\sup_{a\in x\circ y}\mathcal{I}_{\Delta_2(e)}(a)\}\\
&= \max\{{}^{*}\mathcal{I}_{\Delta_1(e)}(x\circ y),{}^{*}\mathcal{I}_{\Delta_2(e)}(x\circ y)\}\\
&\leq \max\{\max\{\mathcal{I}_{\Delta_1(e)}(x),\mathcal{I}_{\Delta_1(e)}(y)\},\max\{\mathcal{I}_{\Delta_2(e)}(x),\mathcal{I}_{\Delta_2(e)}(y)\}\}\\
&= \max\{\max\{\mathcal{I}_{\Delta_1(e)}(x),\mathcal{I}_{\Delta_2(e)}(x)\},\max\{\mathcal{I}_{\Delta_1(e)}(y),\mathcal{I}_{\Delta_2(e)}(y)\}\}\\
&= \max\{\mathcal{I}_{\Delta(e)}(x),\mathcal{I}_{\Delta(e)}(y)\}.
\end{aligned}$$

Similarly,

$${}^{*}\mathcal{F}_{\Delta(e)}(x\circ y)\leq \max\{\mathcal{F}_{\Delta(e)}(x),\mathcal{F}_{\Delta(e)}(y)\}.$$

Thus, (Δ, E) is a *SVNS* hyper *UP*-subalgebra of X.

(ii) Using the hyper UP-algebra $(X, \circ, \ll, 0)$ of Example 1 where $X = \{0, r, s, t\}$, we consider the two *SVNS* hyper UP-subalgebras (Δ_1, E) and (Δ_2, E) of X given by: for $e \in E$,

$$\mathcal{T}_{\Delta_1(e)}(x) = \begin{cases} 0.5 & \text{if } x \in \{0, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{I}_{\Delta_1(e)}(x) = \begin{cases} 0 & \text{if } x \in \{0, s\}, \\ 0.5 & \text{otherwise.} \end{cases}$$

$$\mathcal{F}_{\Delta_1(e)}(x) = \begin{cases} 0 & \text{if } x \in \{0, s\}, \\ 0.5 & \text{otherwise.} \end{cases}$$

and

$$\mathcal{T}_{\Delta_2(e)}(x) = \begin{cases} 0.7 & \text{if } x \in \{0, t\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{I}_{\Delta_2(e)}(x) = \begin{cases} 0 & \text{if } x \in \{0, t\}, \\ 0.7 & \text{otherwise.} \end{cases}$$

$$\mathcal{F}_{\Delta_2(e)}(x) = \begin{cases} 0 & \text{if } x \in \{0, t\}, \\ 0.7 & \text{otherwise.} \end{cases},$$

respectively. Let $(\Delta, E) = (\Delta_1, E) \cup (\Delta_2, E)$. For all $e \in E$, taking $x = s$ and $y = t$ gives

$$\begin{aligned} {}_*\mathcal{T}_{\Delta(e)}(x \circ y) &= {}_*\mathcal{T}_{\Delta(e)}(s \circ t) \\ &= {}_*\mathcal{T}_{\Delta(e)}(\{r\}) \\ &= \mathcal{T}_{\Delta(e)}(r) \\ &= \max\{T_{\Delta_1(e)}(r), T_{\Delta_2(e)}(r)\} \\ &= \max\{0, 0\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \min\{\mathcal{T}_{\Delta(e)}(x), \mathcal{T}_{\Delta(e)}(y)\} &= \min\{\mathcal{T}_{\Delta(e)}(s), \mathcal{T}_{\Delta(e)}(t)\} \\ &= \min\{\max\{T_{\Delta_1(e)}(s), T_{\Delta_2(e)}(s)\}, \max\{T_{\Delta_1(e)}(t), T_{\Delta_2(e)}(t)\}\} \\ &= \min\{\max\{0.5, 0\}, \max\{0, 0.7\}\} \\ &= \min\{0.5, 0.7\} \\ &= 0.5. \end{aligned}$$

That is,

$${}_*\mathcal{T}_{\Delta(e)}(x \circ y) = 0 < 0.5 = \min\{\mathcal{T}_{\Delta(e)}(x), \mathcal{T}_{\Delta(e)}(y)\}.$$

Thus, (Δ, E) is not a *SVNS* hyper UP-subalgebra of X. $\square$

3.3. Cartesian Product of *SVNS* Hyper *UP*-subalgebra

In this section, we define the Cartesian product of *SVNS* hyper *UP*-subalgebra and prove that it is also a *SVNS* hyper *UP*-subalgebra.

Definition 18. Let (Δ_1, E) and (Δ_2, E) be two *SVNS* hyper *UP*-subalgebras of X_1 and X_2, respectively. Then their *Cartesian product* is $(\Delta, E \times E) = (\Delta_1, E) \times (\Delta_2, E)$, where $\Delta(a,b) = \Delta_1(a) \times \Delta_2(b)$ for $(a,b) \in E \times E$. Analytically,

$$\Delta(a,b) = \{\langle (x,y), (\mathcal{T}_{\Delta(a,b)}(x,y), \mathcal{I}_{\Delta(a,b)}(x,y), \mathcal{F}_{\Delta(a,b)}(x,y)) \rangle \,|(x,y) \in X_1 \times X_2\}$$

where

$$\begin{aligned} \mathcal{T}_{\Delta(a,b)}(x,y) &= \min\{\mathcal{T}_{\Delta_1(a)}(x), \mathcal{T}_{\Delta_2(b)}(y)\}, \\ \mathcal{I}_{\Delta(a,b)}(x,y) &= \max\{\mathcal{I}_{\Delta_1(a)}(x), \mathcal{I}_{\Delta_2(b)}(y)\}, \text{ and} \\ \mathcal{F}_{\Delta(a,b)}(x,y) &= \max\{\mathcal{F}_{\Delta_1(a)}(x), \mathcal{F}_{\Delta_2(b)}(y)\}. \end{aligned}$$

for $(a,b) \in E \times E$.

Example 8. Using $E = \{e_1, e_2\}$ as the set of parameters, consider the hyper *UP*-algebra $X = \{0_1, r, s, t\}$ of Example 1 as X_1 with its hyperoperation "$\circ_1$" and its *SVNS* hyper *UP*-subalgebra (Δ_1, E) given by

$$\mathcal{T}_{\Delta_1(e)}(x) = \begin{cases} 0.5 & \text{if } x \in \{0_1, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{I}_{\Delta_1(e)}(x) = \begin{cases} 0 & \text{if } x \in \{0_1, s\}, \\ 0.5 & \text{otherwise.} \end{cases}$$

$$\mathcal{F}_{\Delta_1(e)}(x) = \begin{cases} 0 & \text{if } x \in \{0_1, s\}, \\ 0.5 & \text{otherwise.} \end{cases}$$

for $e \in E$. Consider also hyper *UP*-algebra $X = \{0_2, u, v\}$ of Example 2 as X_2 with its hyper operation "$\circ_2$" and its *SVNS* hyper *UP*-subalgebra (Δ_2, E) given by

$$\begin{aligned} (\Delta_2, E) = &\{(e_1, \{\langle 0_2, (0.9, 0.2, 0.45)\rangle, \langle u, (0.74, 0.57, 0.7)\rangle, \langle v, (0.8, 0.42, 0.52)\rangle\}), \\ &(e_2, \{\langle 0_2, (0.8, 0.2, 0.4)\rangle, \langle u, (0.4, 0.45, 0.5)\rangle, \langle v, (0.67, 0.3, 0.5)\rangle\})\}. \end{aligned}$$

Then the Cartesian product of (Δ_1, E) and (Δ_2, E) is

$$\begin{aligned} (\Delta, E) = &\{((e_1, e_1), \{\langle (0_1, 0_2), (0.5, 0.2, 0.45)\rangle, \langle (0_1, u), (0.5, 0.57, 0.7)\rangle, \\ &\langle (0_1, v), (0.5, 0.42, 0.52)\rangle, \langle (r, 0_2), (0, 0.5, 0.5)\rangle, \langle (r, u), (0, 0.57, 0.7)\rangle, \\ &\langle (r, v), (0, 0.5, 0.52)\rangle, \langle (s, 0_2), (0.5, 0.2, 0.45)\rangle, \langle (s, u), (0.5, 0.57, 0.7)\rangle, \\ &\langle (s, v), (0.5, 0.42, 0.52)\rangle, \langle (t, 0_2), (0, 0.5, 0.5)\rangle, \langle (t, u), (0, 0.57, 0.7)\rangle, \\ &\langle (t, v), (0, 0.5, 0.52)\rangle\}), ((e_1, e_2), \{\langle (0_1, 0_2), (0.5, 0.2.0.4)\rangle, \end{aligned}$$

$\langle(0_1,u),(0.4,0.45,0.5)\rangle, \langle(0_1,v),(0.5,0.3,0.5)\rangle, \langle(r,0_2),(0,0.5,0.5)\rangle,$
$\langle(r,u),(0,0.5,0.5)\rangle, \langle(r,v),(0,0.5,0.5)\rangle, \langle(s,0_2),(0.5,0.2,0.4)\rangle,$
$\langle(s,u),(0.4,0.45,0.5)\rangle, \langle(s,v),(0.5,0.3,0.5)\rangle, \langle(t,0_2),(0,0.5,0.5)\rangle,$
$\langle(t,u),(0,0.5,0.5)\rangle, \langle(t,v),(0,0.5,0.5)\rangle\}),((e_2,e_1),\{\langle(0_1,0_2),(0.5,0.2,0.45)\rangle,$
$\langle(0_1,u),(0.5,0.57,0.7)\rangle, \langle(0_1,v),(0.5,0.42,0.52)\rangle, \langle(r,0_2),(0,0.5,0.5)\rangle,$
$\langle(r,u),(0,0.57,0.7)\rangle, \langle(r,v),(0,0.5,0.52)\rangle, \langle(s,0_2),(0.5,0.2,0.45)\rangle,$
$\langle(s,u),(0.5,0.57,0.7)\rangle, \langle(s,v),(0.5,0.42,0.52)\rangle, \langle(t,0_2),(0,0.5,0.5)\rangle,$
$\langle(t,u),(0,0.57,0.7)\rangle, \langle(t,v),(0,0.5,0.52)\rangle\}),((e_2,e_2),\{\langle(0_1,0_2),(0.5,0.2.0.4)\rangle,$
$\langle(0_1,u),(0.4,0.45,0.5)\rangle, \langle(0_1,v),(0.5,0.3,0.5)\rangle, \langle(r,0_2),(0,0.5,0.5)\rangle,$
$\langle(r,u),(0,0.5,0.5)\rangle, \langle(r,v),(0,0.5,0.5)\rangle, \langle(s,0_2),(0.5,0.2,0.4)\rangle,$
$\langle(s,u),(0.4,0.45,0.5)\rangle, \langle(s,v),(0.5,0.3,0.5)\rangle, \langle(t,0_2),(0,0.5,0.5)\rangle,$
$\langle(t,u),(0,0.5,0.5)\rangle, \langle(t,v),(0,0.5,0.5)\rangle\})\}$

Theorem 7. *Let (Δ_1, E) and (Δ_2, E) be two SVNS hyper UP-subalgebras of $(X_1, \circ_1, \ll_1, 0_1)$ and $(X_2, \circ_2, \ll_2, 0_2)$, respectively. Then their Cartesian product $(\Delta_1, E) \times (\Delta_2, E)$ is a SVNS hyper UP-subalgebra of $(X_1 \times X_2, \circ, \ll, (0_1, 0_2))$.*

Proof. Let (Δ_1, E) and (Δ_2, E) be two *SVNS* hyper *UP*-subalgebras of X_1 and X_2, respectively and let $(\Delta, E\times E) = (\Delta_1, E)\times(\Delta_2, E)$, where $\Delta(a,b) = \Delta_1(a)\times\Delta_2(b)$ for $(a,b)\in E\times E$. For $(u,v),(x,y)\in X_1\times X_2$, we have

$$\begin{aligned}
{}_*\mathcal{T}_{\Delta(a,b)}((u,v)\circ(x,y)) &= {}_*\mathcal{T}_{\Delta(a,b)}(u\circ_1 x, v\circ_2 y)\\
&= \inf_{(r,t)\in(u\circ_1 x)\times(v\circ_2 y)} \mathcal{T}_{\Delta(a,b)}(r,t)\\
&= \inf_{(r,t)\in(u\circ_1 x)\times(v\circ_2 y)} \min\{\mathcal{T}_{\Delta_1(a)}(r), \mathcal{T}_{\Delta_2(b)}(t)\}\\
&\geq \min\{\inf_{r\in u\circ_1 x} \mathcal{T}_{\Delta_1(a)}(r), \inf_{t\in v\circ_2 y} \mathcal{T}_{\Delta_2(b)}(t)\}\\
&= \min\{{}_*\mathcal{T}_{\Delta_1(a)}(u\circ_1 x), {}_*\mathcal{T}_{\Delta_2(b)}(v\circ_2 y)\}\\
&\geq \min\{\min\{\mathcal{T}_{\Delta_1(a)}(u), \mathcal{T}_{\Delta_1(a)}(x)\}, \min\{\mathcal{T}_{\Delta_2(b)}(v), \mathcal{T}_{\Delta_2(b)}(y)\}\}\\
&= \min\{\min\{\mathcal{T}_{\Delta_1(a)}(u), \mathcal{T}_{\Delta_2(b)}(v)\}, \min\{\mathcal{T}_{\Delta_1(a)}(x), \mathcal{T}_{\Delta_2(b)}(y)\}\}\\
&= \min\{\mathcal{T}_{\Delta(a,b)}(u,v), \mathcal{T}_{\Delta(a,b)}(x,y)\}.
\end{aligned}$$

Also,

$$\begin{aligned}
{}^*\mathcal{I}_{\Delta(a,b)}((u,v)\circ(x,y)) &= {}^*\mathcal{I}_{\Delta(a,b)}(u\circ_1 x, v\circ_2 y)\\
&= \sup_{(r,t)\in(u\circ_1 x)\times(v\circ_2 y)} \mathcal{I}_{\Delta(a,b)}(r,t)\\
&= \sup_{(r,t)\in(u\circ_1 x)\times(v\circ_2 y)} \max\{\mathcal{I}_{\Delta_1(a)}(r), \mathcal{I}_{\Delta_2(b)}(t)\}\\
&\leq \max\{\sup_{r\in u\circ_1 x} \mathcal{I}_{\Delta_1(a)}(r), \sup_{t\in v\circ_2 y} \mathcal{I}_{\Delta_2(b)}(t)\}\\
&= \max\{{}^*\mathcal{I}_{\Delta_1(a)}(u\circ_1 x), {}^*\mathcal{I}_{\Delta_2(b)}(v\circ_2 y)\}\\
&\leq \max\{\max\{\mathcal{I}_{\Delta_1(a)}(u), \mathcal{I}_{\Delta_1(a)}(x)\}, \max\{\mathcal{I}_{\Delta_2(b)}(v), \mathcal{I}_{\Delta_2(b)}(y)\}\}\\
&= \max\{\max\{\mathcal{I}_{\Delta_1(a)}(u), \mathcal{I}_{\Delta_2(b)}(v)\}, \max\{\mathcal{I}_{\Delta_1(a)}(x), \mathcal{I}_{\Delta_2(b)}(y)\}\}
\end{aligned}$$

$$= \max\{\mathcal{I}_{\Delta(a,b)}(u,v), \mathcal{I}_{\Delta(a,b)}(x,y)\}.$$

Similarly,

$$^{*}\mathcal{F}_{\Delta(a,b)}((u,v)\circ(x,y)) = \max\{\mathcal{F}_{\Delta(a,b)}(u,v), \mathcal{F}_{\Delta(a,b)}(x,y)\}.$$

Hence, $(\Delta, E\times E)$ is a *SVNS* hyper *UP*-subalgebra of $X_1 \times X_2$. □

3.4. Homomorphism of *SVNS* Hyper *UP*-subalgebra

In this section, we define the image and preimage of *SVNS* hyper *UP*-subalgebra and prove that they are *SVNS* hyper *UP*-subalgebra under *SVNS* homomorphic function.

Definition 19. Let $(X_1, \circ_1, \ll_1, 0_1)$ and $(X_2, \circ_2, \ll_2, 0_2)$ be two hyper *UP*-algebras and $(\Delta_1, E), (\Delta_2, E)$ be two *SVNS* hyper *UP*-subalgebra of X_1 and X_2, respectively. Then the pair (φ, ρ) is called a *SVNS function from* X_1 *to* X_2 where $\varphi : X_1 \longrightarrow X_2$ and $\rho : E \longrightarrow E$.

Definition 20. Under the *SVNS* function (φ, ρ),

(i) The *image of* (Δ_1, E) is denoted by $(\varphi,\rho)(\Delta_1, E)$ and is defined by

$$(\varphi,\rho)(\Delta_1, E) = (\varphi(\Delta_1), \rho(E)) = \{(b, \varphi(\Delta_1)(b)) | b \in \rho(E)\}$$

where for all $b \in \rho(E)$ and $y \in X_2$,

$$\begin{aligned}
\mathcal{T}_{\varphi(\Delta_1)(b)}(y) &= \begin{cases} \max\limits_{\varphi(x)=y} \max\limits_{\rho(a)=b} \mathcal{T}_{\Delta_1(a)}(x) & \text{if } x \in \varphi^{-1}(y), \\ 0 & \text{otherwise,} \end{cases} \\
\mathcal{I}_{\varphi(\Delta_1)(b)}(y) &= \begin{cases} \min\limits_{\varphi(x)=y} \min\limits_{\rho(a)=b} \mathcal{I}_{\Delta_1(a)}(x) & \text{if } x \in \varphi^{-1}(y), \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \\
\mathcal{F}_{\varphi(\Delta_1)(b)}(y) &= \begin{cases} \min\limits_{\varphi(x)=y} \min\limits_{\rho(a)=b} \mathcal{F}_{\Delta_1(a)}(x) & \text{if } x \in \varphi^{-1}(y), \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

(ii) The *preimage* (Δ_2, E) is denoted by $(\varphi,\rho)^{-1}(\Delta_2, E)$ and defined by

$$(\varphi,\rho)^{-1}(\Delta_2, E) = (\varphi^{-1}(\Delta_2), \rho^{-1}(E)) = \{(a, \varphi^{-1}(\Delta_2)(a)) | a \in \rho^{-1}(E)\}$$

where for all $a \in \rho^{-1}(E)$ and $x \in X_1$,

$$\begin{aligned}
\mathcal{T}_{\varphi^{-1}(\Delta_2)(a)}(x) &= \mathcal{T}_{\Delta_2(\rho(a))}(\varphi(x)), \\
\mathcal{I}_{\varphi^{-1}(\Delta_2)(a)}(x) &= \mathcal{I}_{\Delta_2(\rho(a))}(\varphi(x)), \text{ and} \\
\mathcal{F}_{\varphi^{-1}(\Delta_2)(a)}(x) &= \mathcal{F}_{\Delta_2(\rho(a))}(\varphi(x)).
\end{aligned}$$

Definition 21. Let the pair (φ, ρ) be a *SVNS* function from X_1 into X_2, then (φ, ρ) is called a *SVNS homomorphism* if φ is a hyper homomorphism from X_1 to X_2 and is said to be a *SVNS isomorphism* if φ is a hyper isomorphism from X_1 to X_2 and ρ is an injective map from E to E.

Example 9. Let $X_1 = \{0_1, r, s\}$ with hyperoperation given by

$\circ$	0_1	r	s
0_1	$\{0_1\}$	$\{r\}$	$\{s\}$
r	$\{0_1\}$	$\{0_1\}$	$\{s\}$
s	$\{0_1\}$	$\{r\}$	$\{0_1\}$

.

Then X_1 is hyper UP-algebra by thorough inspection. Considering $X_2 = \{0_2, u, v\}$ as the second hyper UP-algebra of Example 2 and $E = \mathbb{N}$ as the set of parameters, we define mappings $\varphi : X_1 \longrightarrow X_2$ by

$$\begin{aligned} \varphi(0_1) &= 0_2 \\ \varphi(r) &= v \\ \varphi(s) &= u; \end{aligned}$$

and $\rho : E \longrightarrow E$ by $\rho(a) = 2a$. Let (Δ_1, E) be a *SVNS* set over X_1 given by

$$\begin{aligned} \mathcal{T}_{\Delta_1(a)}(x) &= \begin{cases} \frac{1}{2a} & \text{if } x \in \{0_1, s\}, \\ 0 & \text{otherwise.} \end{cases} \\ \mathcal{I}_{\Delta_1(a)}(x) &= \begin{cases} 0 & \text{if } x \in \{0_1, s\}, \\ 1 - \frac{1}{a} & \text{otherwise.} \end{cases} \\ \mathcal{F}_{\Delta_1(a)}(x) &= \begin{cases} 0 & \text{if } x \in \{0_1, s\}, \\ \frac{1}{2a+1} & \text{otherwise.} \end{cases} \end{aligned}$$

for $a \in E$. By inspection, (Δ_1, E) is a *SVNS* hyper UP-subalgebra of X_1. Thus, the image of (Δ_1, E) under (φ, ρ) is

$$\begin{aligned} \mathcal{T}_{\varphi(\Delta_1)(b)}(y) &= \begin{cases} 0 & \text{if } y \in \{v\}, \\ \frac{1}{b} & \text{otherwise.} \end{cases} \\ \mathcal{I}_{\varphi(\Delta_1)(b)}(y) &= \begin{cases} 1 - \frac{2}{b} & \text{if } y \in \{v\}, \\ 0 & \text{otherwise.} \end{cases} \\ \mathcal{F}_{\varphi(\Delta_1)(b)}(y) &= \begin{cases} \frac{1}{b+1} & \text{if } y \in \{v\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

for $b \in 2\mathbb{N}$ and $y \in X_2$.

Example 10. Consider the two hyper *UP*-algebras X_1 and X_2 of Example 9. Define $E = \{e_1, e_2\}$ as set of parameters and mappings $\varphi : X_1 \longrightarrow X_2$ by

$$\begin{aligned}\varphi(0_1) &= 0_2\\ \varphi(r) &= v\\ \varphi(s) &= u;\end{aligned}$$

and $\rho : E \longrightarrow E$ by

$$\begin{aligned}\rho(e_1) &= e_2\\ \rho(e_2) &= e_1.\end{aligned}$$

Also, consider the *SVNS* hyper *UP*-subalgebra (Δ, E) of X_2 from Example 7 which is given by

$$\begin{aligned}\Delta(e_1) &= \{\langle 0_2, (0.9, 0.2, 0.45)\rangle, \langle u, (0.74, 0.57, 0.7)\rangle, \langle v, (0.8, 0.42, 0.52)\rangle\} \text{ and}\\ \Delta(e_2) &= \{\langle 0_2, (0.8, 0.2, 0.4)\rangle, \langle u, (0.4, 0.45, 0.5)\rangle, \langle v, (0.67, 0.3, 0.5)\rangle\}.\end{aligned}$$

Thus, the preimage of (Δ, E) under (φ, ρ) is given by

$$\begin{aligned}
\mathcal{T}_{\varphi^{-1}(\Delta)(e_1)}(0_1) &= \mathcal{T}_{\Delta(\rho(e_1))}(\varphi(0_1)) = \mathcal{T}_{\Delta(e_2)}(0_2) = 0.8\\
\mathcal{I}_{\varphi^{-1}(\Delta)(e_1)}(0_1) &= \mathcal{I}_{\Delta(\rho(e_1))}(\varphi(0_1)) = \mathcal{I}_{\Delta(e_2)}(0_2) = 0.2\\
\mathcal{F}_{\varphi^{-1}(\Delta)(e_1)}(0_1) &= \mathcal{F}_{\Delta(\rho(e_1))}(\varphi(0_1)) = \mathcal{F}_{\Delta(e_2)}(0_2) = 0.4\\
\mathcal{T}_{\varphi^{-1}(\Delta)(e_1)}(r) &= \mathcal{T}_{\Delta(\rho(e_1))}(\varphi(r)) = \mathcal{T}_{\Delta(e_2)}(v) = 0.67\\
\mathcal{I}_{\varphi^{-1}(\Delta)(e_1)}(r) &= \mathcal{I}_{\Delta(\rho(e_1))}(\varphi(r)) = \mathcal{I}_{\Delta(e_2)}(v) = 0.3\\
\mathcal{F}_{\varphi^{-1}(\Delta)(e_1)}(r) &= \mathcal{F}_{\Delta(\rho(e_1))}(\varphi(r)) = \mathcal{F}_{\Delta(e_2)}(v) = 0.5\\
\mathcal{T}_{\varphi^{-1}(\Delta)(e_1)}(s) &= \mathcal{T}_{\Delta(\rho(e_1))}(\varphi(s)) = \mathcal{T}_{\Delta(e_2)}(u) = 0.4\\
\mathcal{I}_{\varphi^{-1}(\Delta)(e_1)}(s) &= \mathcal{I}_{\Delta(\rho(e_1))}(\varphi(s)) = \mathcal{I}_{\Delta(e_2)}(u) = 0.45\\
\mathcal{F}_{\varphi^{-1}(\Delta)(e_1)}(s) &= \mathcal{F}_{\Delta(\rho(e_1))}(\varphi(s)) = \mathcal{F}_{\Delta(e_2)}(u) = 0.5\\
\mathcal{T}_{\varphi^{-1}(\Delta)(e_2)}(0_1) &= \mathcal{T}_{\Delta(\rho(e_2))}(\varphi(0_1)) = \mathcal{T}_{\Delta(e_1)}(0_2) = 0.9\\
\mathcal{I}_{\varphi^{-1}(\Delta)(e_2)}(0_1) &= \mathcal{I}_{\Delta(\rho(e_2))}(\varphi(0_1)) = \mathcal{I}_{\Delta(e_1)}(0_2) = 0.2\\
\mathcal{F}_{\varphi^{-1}(\Delta)(e_2)}(0_1) &= \mathcal{F}_{\Delta(\rho(e_2))}(\varphi(0_1)) = \mathcal{F}_{\Delta(e_1)}(0_2) = 0.45\\
\mathcal{T}_{\varphi^{-1}(\Delta)(e_2)}(r) &= \mathcal{T}_{\Delta(\rho(e_2))}(\varphi(r)) = \mathcal{T}_{\Delta(e_1)}(v) = 0.8\\
\mathcal{I}_{\varphi^{-1}(\Delta)(e_2)}(r) &= \mathcal{I}_{\Delta(\rho(e_2))}(\varphi(r)) = \mathcal{I}_{\Delta(e_1)}(v) = 0.42\\
\mathcal{F}_{\varphi^{-1}(\Delta)(e_2)}(r) &= \mathcal{F}_{\Delta(\rho(e_2))}(\varphi(r)) = \mathcal{F}_{\Delta(e_1)}(v) = 0.52\\
\mathcal{T}_{\varphi^{-1}(\Delta)(e_2)}(s) &= \mathcal{T}_{\Delta(\rho(e_2))}(\varphi(s)) = \mathcal{T}_{\Delta(e_1)}(u) = 0.74.\\
\mathcal{I}_{\varphi^{-1}(\Delta)(e_2)}(s) &= \mathcal{I}_{\Delta(\rho(e_2))}(\varphi(s)) = \mathcal{I}_{\Delta(e_1)}(u) = 0.57.\\
\mathcal{F}_{\varphi^{-1}(\Delta)(e_2)}(s) &= \mathcal{F}_{\Delta(\rho(e_2))}(\varphi(s)) = \mathcal{F}_{\Delta(e_1)}(u) = 0.7.
\end{aligned}$$

Theorem 8. *Let (φ, ρ) be a SVNS homomorphism from $(X_1, \circ_1, \ll_1, 0_1)$ to $(X_2, \circ_2, \ll_2, 0_2)$. If (Δ_1, E) is a SVNS hyper UP-subalgebra of X_1, then $(\varphi, \rho)(\Delta_1, E)$ is a SVNS hyper UP-subalgebra of X_2.*

Proof. Let (φ, ρ) be a $SVNS$ homomorphism from X_1 to X_2, (Δ_1, E) is a $SVNS$ hyper UP-subalgebra of X_1, $b \in \rho(E)$, and $x, y \in X_2$.

(i) For $\varphi^{-1}(x) = \varnothing$ or $\varphi^{-1}(y) = \varnothing$, the proof is straightforward.

(ii) Assume that there exist $x_0, y_0 \in X_1$ such that $\varphi(x_0) = x$ and $\varphi(y_0) = y$. Then $x \circ_2 y = \varphi(x_0) \circ_2 \varphi(y_0) = \varphi(x_0 \circ_1 y_0)$. Now,

$$\begin{aligned}
{}_*\mathcal{T}_{\varphi(\Delta_1)(b)}(x \circ_2 y) &= \inf_{z \in x \circ_2 y} \mathcal{T}_{\varphi(\Delta_1)(b)}(z) \\
&= \inf_{z_0 \in x_0 \circ_1 y_0} \left[\max_{\varphi(z_0)=z} \max_{\rho(a)=b} \mathcal{T}_{\Delta_1(a)}(z_0) \right] \\
&\geq \inf_{z_0 \in x_0 \circ_1 y_0} \left[\max_{\rho(a)=b} \mathcal{T}_{\Delta_1(a)}(z_0) \right] \\
&= \max_{\rho(a)=b} \left[\inf_{z_0 \in x_0 \circ_1 y_0} \mathcal{T}_{\Delta_1(a)}(z_0) \right] \\
&= \max_{\rho(a)=b} \left[{}_*\mathcal{T}_{\Delta_1(a)}(x_0 \circ_1 y_0) \right] \\
&\geq \max_{\rho(a)=b} \left[\min\{\mathcal{T}_{\Delta_1(a)}(x_0), \mathcal{T}_{\Delta_1(a)}(y_0)\} \right] \\
&= \min\{ \max_{\rho(a)=b} \mathcal{T}_{\Delta_1(a)}(x_0), \max_{\rho(a)=b} \mathcal{T}_{\Delta_1(a)}(y_0)\}
\end{aligned}$$

Since the inequality is satisfied for all $x_0, y_0 \in X_1$ satisfying $\varphi(x_0) = x$ and $\varphi(y_0) = y$, it follows that

$$\begin{aligned}
{}_*\mathcal{T}_{\varphi(\Delta_1)(b)}(x \circ_2 y) &\geq \min\{ \max_{\varphi(x_0)=x} \max_{\rho(a)=b} \mathcal{T}_{\Delta_1(a)}(x_0), \max_{\varphi(y_0)=y} \max_{\rho(a)=b} \mathcal{T}_{\Delta_1(a)}(y_0)\} \\
&= \min\{\mathcal{T}_{\varphi(\Delta_1)(b)}(x), \mathcal{T}_{\varphi(\Delta_1)(b)}(y)\}.
\end{aligned}$$

Also,

$$\begin{aligned}
{}^*\mathcal{I}_{\varphi(\Delta_1)(b)}(x \circ_2 y) &= \sup_{z \in x \circ_2 y} \mathcal{I}_{\varphi(\Delta_1)(b)}(z) \\
&= \sup_{z_0 \in x_0 \circ_1 y_0} \left[\min_{\varphi(z_0)=z} \min_{\rho(a)=b} \mathcal{I}_{\Delta_1(a)}(z_0) \right] \\
&\leq \sup_{z_0 \in x_0 \circ_1 y_0} \left[\min_{\rho(a)=b} \mathcal{I}_{\Delta_1(a)}(z_0) \right] \\
&= \min_{\rho(a)=b} \left[\sup_{z_0 \in x_0 \circ_1 y_0} \mathcal{I}_{\Delta_1(a)}(z_0) \right] \\
&= \min_{\rho(a)=b} \left[{}_*\mathcal{I}_{\Delta_1(a)}(x_0 \circ_1 y_0) \right] \\
&\leq \min_{\rho(a)=b} \left[\max\{\mathcal{I}_{\Delta_1(a)}(x_0), \mathcal{I}_{\Delta_1(a)}(y_0)\} \right] \\
&= \max\{ \min_{\rho(a)=b} \mathcal{I}_{\Delta_1(a)}(x_0), \min_{\rho(a)=b} \mathcal{I}_{\Delta_1(a)}(y_0)\}
\end{aligned}$$

Since the inequality is satisfied for all $x_0, y_0 \in X_1$ satisfying $\varphi(x_0) = x$ and $\varphi(y_0) = y$, it follows that

$$\begin{aligned} {}^*\mathcal{I}_{\varphi(\Delta_1)(b)}(x \circ_2 y) &\leq \max\{\min_{\varphi(x_0)=x} \min_{\rho(a)=b} \mathcal{I}_{\Delta_1(a)}(x_0), \max_{\varphi(y_0)=y} \max_{\rho(a)=b} \mathcal{I}_{\Delta_1(a)}(y_0)\} \\ &= \max\{\mathcal{I}_{\varphi(\Delta_1)(b)}(x), \mathcal{I}_{\varphi(\Delta_1)(b)}(y)\}. \end{aligned}$$

Similarly,

$$^*\mathcal{F}_{\varphi(\Delta_1)(b)}(x \circ_2 y) \leq \max\{\mathcal{F}_{\varphi(\Delta_1)(b)}(x), \mathcal{F}_{\varphi(\Delta_1)(b)}(y)\}.$$

Hence, $(\varphi, \rho)(\Delta_1, E)$ is a *SVNS* hyper *UP*-subalgebra of X_2. □

Theorem 9. *Let (φ, ρ) be a SVNS homomorphism from $(X_1, \circ_1, \ll_1, 0_1)$ to $(X_2, \circ_2, \ll_2, 0_2)$. If (Δ_2, E) is a SVNS hyper UP-subalgebra of X_2, then $(\varphi, \rho)^{-1}(\Delta_2, E)$ is a SVNS hyper UP-subalgebra of X_1.*

Proof. Let (φ, ρ) be a *SVNS* homomorphism from X_1 to X_2, (Δ_2, E) be a *SVNS* hyper *UP*-subalgebra of X_2, $a \in \rho^{-1}(E)$, $x, y \in X_1$. Now,

$$\begin{aligned} {}_*\mathcal{T}_{\varphi^{-1}(\Delta_2)(a)}(x \circ_1 y) &= {}_*\mathcal{T}_{\Delta_2(\rho(a))}(\varphi(x \circ_1 y)) \\ &= {}_*\mathcal{T}_{\Delta_2(\rho(a))}(\varphi(x) \circ_2 \varphi(y)) \\ &\geq \min\{\mathcal{T}_{\Delta_2(\rho(a))}(\varphi(x)), \mathcal{T}_{\Delta_2(\rho(a))}(\varphi(y))\} \\ &= \min\{\mathcal{T}_{\varphi^{-1}(\Delta_2)(a)}(x), \mathcal{T}_{\varphi^{-1}(\Delta_2)(a)}(y)\}, \end{aligned}$$

$$\begin{aligned} {}^*\mathcal{I}_{\varphi^{-1}(\Delta_2)(a)}(x \circ_1 y) &= {}^*\mathcal{I}_{\Delta_2(\rho(a))}(\varphi(x \circ_1 y)) \\ &= {}^*\mathcal{I}_{\Delta_2(\rho(a))}(\varphi(x) \circ_2 \varphi(y)) \\ &\leq \max\{\mathcal{I}_{\Delta_2(\rho(a))}(\varphi(x)), \mathcal{I}_{\Delta_2(\rho(a))}(\varphi(y))\} \\ &= \max\{\mathcal{I}_{\varphi^{-1}(\Delta_2)(a)}(x), \mathcal{I}_{\varphi^{-1}(\Delta_2)(a)}(y)\}, \end{aligned}$$

and similarly,

$$^*\mathcal{F}_{\varphi^{-1}(\Delta_2)(a)}(x \circ_1 y) = \max\{\mathcal{F}_{\varphi^{-1}(\Delta_2)(a)}(x), \mathcal{F}_{\varphi^{-1}(\Delta_2)(a)}(y)\}.$$

Thus, $(\varphi, \rho)^{-1}(\Delta_2, S_2)$ is a *SVNS* hyper *UP*-subalgebra of X_1. □

4. Conclusions

In this paper, we have introduced the *SVN* and *SVNS* hyper *UP*-subalgebra together with their properties. Aside from that, the concept of Cartesian product of *SVNS* hyper *UP*-subalgebra and the homomorphic image and preimage of *SVNS* hyper *UP*-subalgebra have been investigated. This study contributes to the development of the notion of hyper *UP*-algebra under neutrosophic soft environment. It also opens a door for further study by establishing *SVNS* hyper *UP*-filter, *SVNS* hyper *UP*-ideals and some of their variations.

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