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Moment Generating Function of Current Records Based on Generalized Exponential Distribution with Some Recurrence Relations

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Abstract. This paper tries to find some formulas for calculating the moment generating function for upper and lower current records picked from generalized exponential distributed data and the joint moment generating function between them. After that, some formulas are derived from the previous ones to find the moments of each and the product moments of both upper and lower current records. Then, various recurrence relations are established for most of the mentioned formulas. After that, an integral form of the moments of record range is founded followed by a numerical example with simulated data to clarify the effectiveness of the formulas found in the study and how they can make the calculation process easier and faster. Finally, a conclusion part is added, to sum up what has been done and the results.

2020 Mathematics Subject Classifications: 62G30, 65Q30, 62E99, 60E10

Key Words and Phrases: Record range, lower and upper current records, moment generating function and moments, recurrence relations, generalized exponential distribution

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1. Introduction

Imagine that we have a sequence $\{X_j\}$ that consists of identical and independent random variables such that each of which is distributed with a probability density function(pdf) $f(x)$ and a cumulative distribution function(cdf) $F(x)$ that is absolutely continuous. X_j as an observation is considered to be an upper record if $X_j > X_i$ for every $i < j$. A similar definition has been established for the lower record values. In some situations, we tend to pick the smallest and largest X values detected as new lower or upper record values of either kind take place, and in such a situation, we name it current records. We will symbolize U_n^c as the nth upper current records and L_n^c as the nth lower current records of the sequence X_n, when any kind o the nth records appears. So, $U_{n+1}^c = U_n^c$ if $L_{n+1}^c < L_n^c$ and $L_{n+1}^c = L_n^c$ if $U_{n+1}^c > U_n^c$, for all $n = 1, 2, ...$ where by definition, $L_0^c = U_0^c = X_1$. We can define the record range as $R_n^c = U_n^c - L_n^c$. And of

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course, a new range is recorded at the moment a new upper or lower record has occurred. For example, let's consider the following sequence of observations:

$$3, 2, 2.5, 2.2, 1, 3.7, 2.6, 1.5, 2.7, 4, 0.5, 2.5, 4.7....$$

Then we can determine the upper and lower current records along with the current record range as follows:

$$U_0^c = 3, U_1^c = 3, U_2^c = 3, U_3^c = 3.7, U_4^c = 4, U_5^c = 4, U_6^c = 4.7$$

$$L_0^c = 3, L_1^c = 2, L_2^c = 1, L_3^c = 1, L_4^c = 1, L_5^c = 0.5, L_6^c = 0.5$$

$$R_0^c = 0, R_1^c = 1, R_2^c = 2, R_3^c = 2.7, R_4^c = 3, R_5^c = 3.5, R_6^c = 4.2$$

There are cases in which it is interesting for the current records to be considered in real life. Such as whether data where lower and upper records are being taken together. Also, in the case of outlier detection and when choosing a fitted model in which the record range plays an important role in it (see, Basak [10]). Again, it is very substantial to be used when we want to see if the production process is good enough to fall within the scope of the production's specifications or not. Meaning, if the record range is greater than a specific value, then the products will not fit the specifications and vice versa. And you can see it also in any life test in which we made sequential measurements and recorded only the values that fall below or above a current extreme value like what happens in industrial stress.

Barakat et al. [8] obtained some current record recurrence relations for some distributions and also moments recurrence relations for record range when the data follows the exponential distribution. Once again, Barakat et al. [9] worked on the current record and record range, but at this time, they established a prediction interval for a future value of them. To read more about the record range and current records and the applications titled to them, we pointed to Raqab[20], Ahmadi et al.[3] and Ahmadi and Balakrishnan[1] and [2]. But to read about the record values themselves, you can check Aldallal [4], Abd Elgawad et al. [12], Amany et al. [5], Husseiny et al.[17] and Barakat and Harpy [7].

Some of the data in the lifetime examples mentioned above can be distributed using a distribution introduced by Gupta and Kundu[13] called the generalized exponential distribution (GED) which can be used as a good alternative for gamma or the Weibull model (see, Gupta and Kundu([13],[15], [14] and Mohie El-Din and Sharawy [11]). They also mentioned that the GED has more similarities to a gamma family than a Weibull family regarding the hazard function.

In this paper, we will introduce some new moment generating function (MGF) formulas for U_n^c, L_n^c and the joint MGF between (U_n^c and L_n^c) based on GED along with their i^{th} moments. Also, some recurrence relations are introduced for the MGF of them beside the recurrence relation of their i^{th} moments. Moreover, an integrated form for the m^{th} moments for the record range also follows a GED has been driven. These results can help minimize the direct computation of these MGFs and moments since numerical steps will be applied.

2. Preliminaries Results

Houchens [16] concluded the pdf of U_n^c, L_n^c, R_n^c, and the joint pdf between (U_n^c and L_n^c) when if follows any cdf $F(x)$ respectively by

$$f_{L_n^c}(x) = 2^n f(x)\Big\{1 - F(x)\sum_{k=0}^{n-1}\frac{[-\log F(x)]^k}{k!}\Big\}, \tag{1}$$

$$f_{U_n^c}(x) = 2^n f(x)\Big\{1 - \bar{F}(x)\sum_{k=0}^{n-1}\frac{[-\log \bar{F}(x)]^k}{k!}\Big\}, \tag{2}$$

$$f_{R_n^c}(r) = \frac{2^n}{(n-1)!}\int_{-\infty}^{\infty} f(r+x)f(x)\{-\log[1+F(x)-F(r+x)]\}^{n-1}dx, \quad 0<r<\infty, \tag{3}$$

$$f_{L_n^c,U_n^c}(x,y) = \frac{2^n}{(n-1)!}f(x)f(y)\Big\{-\log[1+F(x)-F(y)]\Big\}^{n-1}, \tag{4}$$

where $\bar{F}(x) = 1 - F(x)$.
Gupta and Kundu[13] established GED with the following pdf

$$f(x;\alpha,\lambda) = \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}, \tag{5}$$

and the coming cdf

$$F(x) = (1-e^{-\lambda x})^{\alpha}. \tag{6}$$

Where α is a shape parameter, λ is a scale parameter and a location parameter $\mu = 0$.

Remark 1. *The followings are some previously introduced expansions:*
1. The logarithmic expansion introduced by Balakrishnan and cohen [6]

$$[-ln(1-t)]^i = \left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^i = \sum_{p=0}^{\infty}a_p(i)t^{i+p}, \quad |t|<1 \tag{7}$$

where $a_p(i)$ is the coefficient of t^{i+p} in the expansion $\left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^i$.
2. We have

$$I_x(a,b) = \frac{\beta_x(a,b)}{\beta(a,b)},$$

where $I_x(a,b)$,$\beta_x(a,b)$ and $\beta(a,b)$ represents the regularized incomplete beta function, the incomplete beta function and the beta function respectively. We can find many series representations for $I_x(a,b)$ in many books and journals like the following which has been introduced by Pearson [18]

$$I_x(a,b) = 1-(1-x)^{a+b-1}\sum_{i=0}^{a-1}\binom{a+b-1}{i}\left(\frac{x}{1-x}\right)^i,$$

from which we conclude that

$$\beta_x(a,b) = \beta(a,b)\Big[1-(1-x)^{a+b-1}\sum_{i=0}^{a-1}\binom{a+b-1}{i}\Big(\frac{x}{1-x}\Big)^i\Big]. \tag{8}$$

Remark 2. *We can conclude the following characterized differential equation*

$$F(x) = \frac{1}{\lambda\alpha}(e^{\lambda x}-1)f(x), \tag{9}$$

which will be used in some of the recurrence relations. And this can be done by replacing the term $(1-e^{-\lambda x})^\alpha$ from (5) by $F(x)$, which will lead to

$$f(x) = \alpha\lambda F(x)\frac{e^{-\lambda x}}{1-e^{-\lambda x}}.$$

Then

$$F(x) = \frac{1}{\alpha\lambda}\frac{1-e^{-\lambda x}}{e^{-\lambda x}}f(x) = \frac{1}{\alpha\lambda}(\frac{1}{e^{-\lambda x}}-1)f(x) = .$$

Remark 3. *The following relations will be used in the proof of some theorems (see, Raqab [19])*

$$\beta_x(a,b) = a^{-1}x^a\,{}_2F_1(a,1-b;a+1;x), \tag{10}$$

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,{}_2F_1(c,d;\rho;u)du = \beta(a,b)\,{}_3F_2(a,c,d;\rho,a+b;1). \tag{11}$$

Where ${}_2F_1$ and ${}_3F_2$ are generalized hypergeometric function's special cases.

3. Moment Generating Functions

In this section, we will introduce some new formulas for calculating the MGF of the lower current record $M_{L_n^c}(t)$, MGF of the upper current record $M_{U_n^c}(t)$ and joint MGF of lower and upper current records $M_{L_n^c,U_n^c}(t_1,t_2)$ all based on GED.

Theorem 1. *For $n \geq 2$*

$$M_{L_n^c}(t) = 2^n\alpha\Big[\beta(\alpha,1-\frac{t}{\lambda})-\beta(2\alpha,1-\frac{t}{\lambda})-\sum_{k=1}^{n-1}\frac{\alpha^k}{k!}\sum_{p=0}^{\infty}a_p(k)\beta(2\alpha,1+k+p-\frac{t}{\lambda})\Big]. \tag{12}$$

While

$$M_{U_n^c}(t) = 2^n\alpha\Big[\beta(2\alpha,1-\frac{t}{\lambda})-\sum_{k=1}^{n-1}\frac{1}{k!}\sum_{p=0}^{\infty}a_p(k)\Big\{\beta((1+k+p)\alpha,1-\frac{t}{\lambda})-\beta((2+k+p)\alpha,1-\frac{t}{\lambda})\Big\}\Big]. \tag{13}$$

And

$$M_{L_n^c,U_n^c}(t_1,t_2) = \frac{2^n\alpha^2}{(n-1)!}\sum_{p=0}^{\infty} a_p(n-1)\sum_{k=0}^{n+p-1}\binom{n+p-1}{k}(-1)^k\beta(\alpha k+1, 1-\frac{t_1}{\lambda})$$
$$\Big[\beta(\alpha,1-\frac{t_2}{\lambda}) - \sum_{i=0}^{\alpha k}\binom{\alpha k-\frac{t_1}{\lambda}+1}{i}\beta((n+p-k)\alpha-1+i, 2+i+\alpha k-\frac{t_1}{\lambda}-\frac{t_2}{\lambda})\Big]. \tag{14}$$

Proof. By substituting (5) and (6) in (1), we get

$$\begin{aligned} f_{L_n^c}(x) &= 2^n\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}\Big\{1-(1-e^{-\lambda x})^{\alpha}\sum_{k=0}^{n-1}\frac{[-\log(1-e^{-\lambda x})^{\alpha}]^k}{k!}\Big\} \\ &= 2^n\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}\Big\{1-(1-e^{-\lambda x})^{\alpha}-(1-e^{-\lambda x})^{\alpha} \\ &\qquad \sum_{k=1}^{n-1}\frac{[-\log(1-e^{-\lambda x})^{\alpha}]^k}{k!}\Big\}. \end{aligned}$$

So

$$\begin{aligned} M_{L_n^c}(t) &= 2^n\alpha\lambda\int_0^{\infty} e^{-(\lambda-t)x}(1-e^{-\lambda x})^{\alpha-1}dx - 2^n\alpha\lambda\int_0^{\infty} e^{-(\lambda-t)x}(1-e^{-\lambda x})^{2\alpha-1}dx \\ &\quad -2^n\alpha\lambda\int_0^{\infty} e^{-(\lambda-t)x}(1-e^{-\lambda x})^{2\alpha-1}\sum_{k=1}^{n-1}\frac{\alpha^k[-\log(1-e^{-\lambda x})]^k}{k!}dx \\ &= 2^n\alpha[\beta(\alpha,1-\frac{t}{\lambda})-\beta(2\alpha,1-\frac{t}{\lambda})]-K(t). \end{aligned}$$

Upon using the substitution $a = e^{-\lambda x}$ in $K(t)$ and then applying the logarithmic expansion from (7), we reach the following result

$$K(t) = 2^n\sum_{k=1}^{n-1}\frac{\alpha^{k+1}}{k!}\sum_{p=0}^{\infty}a_p(k)\beta(2\alpha,1+k+p-\frac{t}{\lambda}),$$

which completes the proof of (12). Now to proof (13), we will substitute (5) and (6) in (2) and with a routine calculation we get

$$\begin{aligned} f_{U_n^c}(x) &= 2^n\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}\Big\{-\sum_{k=1}^{n-1}\frac{[-\log(1-(1-e^{-\lambda x})^{\alpha})]^k}{k!}+(1-e^{-\lambda x})^{\alpha} \\ &\quad +(1-e^{-\lambda x})^{\alpha}\sum_{k=1}^{n-1}\frac{[-\log(1-(1-e^{-\lambda x})^{\alpha})]^k}{k!}\Big\}. \end{aligned}$$

Then

$$M_{U_n^c}(t) = 2^n\alpha\lambda\int_0^\infty e^{-(\lambda-t)x}(1-e^{-\lambda x})^{2\alpha-1}dx - 2^n\alpha\lambda\int_0^\infty e^{-(\lambda-t)x}(1-e^{-\lambda x})^{\alpha-1}$$
$$\sum_{k=1}^{n-1}\frac{[-\log(1-(1-e^{-\lambda x})^\alpha]^k}{k!}dx + 2^n\alpha\lambda\int_0^\infty e^{-(\lambda-t)x}(1-e^{-\lambda x})^{2\alpha-1}$$
$$\sum_{k=1}^{n-1}\frac{[-\log(1-(1-e^{-\lambda x})^\alpha]^k}{k!}dx = 2^n\alpha\beta(2\alpha, 1-\frac{t}{\lambda}) - V(t) + Z(t).$$

By applying the substitution $b = (1-e^{-\lambda x})^\alpha$ then using the logarithmic expansion (7) for both $V(t)$ and $Z(t)$ we reach (13). To prove (14), we will apply (5) and (6) into (4) and this will lead to

$$f_{L_n^c,U_n^c}(x,y) = \frac{2^n}{(n-1)!}\alpha^2\lambda^2(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}(1-e^{-\lambda y})^{\alpha-1}e^{-\lambda y}$$
$$\Big\{-\log[1-(1-e^{-\lambda y})^\alpha + (1-e^{-\lambda x})^\alpha]\Big\}^{n-1}.$$

And

$$M_{L_n^c,U_n^c}(t_1,t_2) = \frac{2^n\alpha^2\lambda^2}{(n-1)!}\int_0^\infty e^{(t_2-\lambda)y}(1-e^{-\lambda y})^{\alpha-1}I(y)dy,$$

where

$$I(y) = \int_0^y e^{(t_1-\lambda)x}(1-e^{-\lambda x})^{\alpha-1}\Big\{-\log[1-(1-e^{-\lambda y})^\alpha + (1-e^{-\lambda x})^\alpha]\Big\}^{n-1}dx.$$

Let $u = (1-e^{-\lambda y})^\alpha - (1-e^{-\lambda x})^\alpha$, then using (7) we get

$$I(y) = \frac{1}{\alpha\lambda}\sum_{p=0}^\infty a_p(n-1)\int_0^{(1-e^{-\lambda y})^\alpha} u^{n+p-1}\Big\{1-[(1-e^{-\lambda y})^\alpha - u]^{\frac{1}{\alpha}}\Big\}^{\frac{-t_1}{\lambda}}du,$$

again using another substitution $w = [(1-e^{-\lambda y})^\alpha - u]^{\frac{1}{\alpha}}$ then a binomial expansion on one of the terms, we get

$$I(y) = \frac{1}{\lambda}\sum_{p=0}^\infty a_p(n-1)\sum_{k=0}^{n+p-1}\binom{n+p-1}{k}(-1)^k(1-e^{-\lambda y})^{\alpha(n+p-1-k)}J(y),$$

where

$$J(y) = \int_0^{1-e^{-\lambda y}}(1-w)^{\frac{-t_1}{\lambda}}w^{\alpha k}dw.$$

By applying (8) on $J(y)$, we get

$$I(y) = \frac{1}{\lambda}\sum_{p=0}^\infty a_p(n-1)\sum_{k=0}^{n+p-1}\binom{n+p-1}{k}(-1)^k(1-e^{-\lambda y})^{\alpha(n+p-1-k)}\beta(\alpha k+1, \frac{-t_1}{\lambda}+1)$$
$$\Big[1-(e^{-\lambda y})^{\alpha k-\frac{t_1}{\lambda}+1}\sum_{i=0}^{\alpha k}\binom{\alpha k-\frac{t_1}{\lambda}+1}{i}\frac{(1-e^{-\lambda y})^i}{(e^{-\lambda y})^i}\Big].$$

Now

$$M_{L_n^c,U_n^c}(t_1,t_2) = \frac{2^n\alpha^2\lambda}{(n-1)!}\sum_{p=0}^{\infty} a_p(n-1)\sum_{k=0}^{n+p-1}\binom{n+p-1}{k}(-1)^k(1-e^{-\lambda y})^{\alpha(n+p-1-k)}$$
$$\beta(\alpha k+1,\frac{-t_1}{\lambda}+1)\Big[Q(t_2)-\sum_{i=0}^{\alpha k}\binom{\alpha k-\frac{t_1}{\lambda}+1}{i}R(t_2)\Big],$$

where

$$Q(t_2) = \int_0^{\infty} e^{(t_2-\lambda)y}(1-e^{-\lambda y})^{\alpha(n+p-k)-1}dy,$$

and

$$R(t_2) = \int_0^{\infty} e^{t_2 y}(e^{-\lambda y})^{2+i+\alpha k-\frac{t_1}{\lambda}}(1-e^{-\lambda y})^{\alpha(n+p-k)-1+i}dy.$$

Substituting $z = e^{-\lambda y}$ in $Q(t_2)$ and $R(t_2)$, and with some routine calculations, we reach (14). And this completes the proof.

Corollary 1. *The following recurrence relation evaluated from recursively differentiating $\beta(\alpha,1-\frac{t}{\lambda})$ with respect to t for (i) times*

$$\beta^{(i)}(\alpha,1-\frac{t}{\lambda}) = \sum_{k=0}^{i-1}(-1)^{i-k-1}(\frac{1}{\lambda})^{i-k}\binom{i-1}{j}\beta^{(k)}(\alpha,1-\frac{t}{\lambda})\Big[\psi^{(i-k-1)}(\alpha+1-\frac{t}{\lambda})-\psi^{(i-k-1)}(1-\frac{t}{\lambda})\Big]. \tag{15}$$

Where $\beta^{(i)}(\alpha,1-\frac{t}{\lambda})$ is the i^{th} derivative of $\beta(\alpha,1-\frac{t}{\lambda})$ and $\psi^{(i)}(\alpha)$ is the i^{th} derivative of the Poly-gamma function $\psi(\alpha)$.

Corollary 2. *By differentiating (12) and (13) i times and substituting $t=0$, we get the i^{th} moment of lower current record $\mu_{L_n^c}^{(i)}$ and i^{th} moment of upper current record $\mu_{U_n^c}^{(i)}$ respectively. Also, by differentiating (14) j times with respect to t_2 and i times with respect to t_1 and then substituting $t_1=t_2=0$ we get the product moment of the lower and upper current record $\mu_{L_n^c,U_n^c}^{(i,j)}$.*

$$\mu_{L_n^c}^{(i)} = 2^n\alpha\Big[\beta^{(i)}(\alpha,1-\frac{t}{\lambda})-\beta^{(i)}(2\alpha,1-\frac{t}{\lambda})-\sum_{k=1}^{n-1}\frac{\alpha^k}{k!}\sum_{p=0}^{\infty}a_p(k)\beta^{(i)}(2\alpha,1+k+p-\frac{t}{\lambda})\Big]\Big|_{t=0}. \tag{16}$$

While

$$\mu_{U_n^c}^{(i)} = 2^n\alpha\Big[\beta^{(i)}(2\alpha,1-\frac{t}{\lambda})-\sum_{k=1}^{n-1}\frac{1}{k!}\sum_{p=0}^{\infty}a_p(k)\Big\{\beta^{(i)}((1+k+p)\alpha,1-\frac{t}{\lambda}) -\beta^{(i)}((2+k+p)\alpha,1-\frac{t}{\lambda})\Big\}\Big]\Big|_{t=0}. \tag{17}$$

4. Some Recurrence Relations

Here we tried to find recurrence relations for $M_{L^c_{n+1}}(t)$ and $M_{U^c_{n+1}}(t)$ based on previous terms $M_{L^c_n}(t)$ and $M_{L^c_n}(t)$ along with some additional terms. Also, the same has been done for $\mu^{(i)}_{L^c_{n+1}}$ and $\mu^{(i)}_{U^c_{n+1}}$. Again, the same has been done for $M_{L^c_{n+1},U^c_{n+1}}(t_1,t_2)$ based on previous function $M_{L^c_n,U^c_n}(t_1,t_2)$ with some added terms and for the product moments $\mu^{(i,j)}_{L^c_{n+1},U^c_{n+1}}$.

Theorem 2. *For $n \geq 2$*

$$M_{L^c_{n+1}}(t) = 2M_{L^c_n}(t) - \frac{(2\alpha)^{n+1}}{n!}\sum_{p=0}^{\infty} a_p(n)\beta(2\alpha, -\frac{t}{\lambda} + n + p + 1) \tag{18}$$

$$M_{U^c_{n+1}}(t) = 2M_{U^c_n}(t) - \frac{2^{n+1}\alpha}{n!}\sum_{p=0}^{\infty} a_p(n)\Big[\beta((1+n+p)\alpha, 1-\frac{t}{\lambda}) - \beta((2+n+p)\alpha, 1-\frac{t}{\lambda})\Big] \tag{19}$$

Proof. By replacing n by $n+1$ in (12) and (13), and by some routine calculations we reach (18) and (19) respectively.

Theorem 3. *Knowing $M_{L^c_n,U^c_n}(t_1,t_2)$ and for $n \geq 2$, we can calculate $M_{L^c_{n+1},U^c_{n+1}}(t_1,t_2)$ using the following*

$$\begin{aligned}
M_{L^c_{n+1},U^c_{n+1}}(t_1,t_2) &= 2M_{L^c_n,U^c_n}(t_1,t_2) + \frac{2^{n+1}}{(n-1)!}\sum_{p=0}^{\infty} a_p(n)\Big\{\alpha^2 \sum_{k=0}^{\infty}\sum_{\omega=0}^{n+p+k-1}(-1)^{\omega}\binom{n+p+k-1}{\omega} \\
&\Big[\sum_{i=0}^{\alpha(n+p+k-\omega+1)-1}(-1)^i\binom{\alpha(n+p+k-\omega+1)-1}{i}\beta(\alpha(\omega+1)+1, i-\frac{t_2}{\lambda}+1) \\
&{}_3F_2\Big(\alpha(\omega+1)+1, \alpha(\omega+1), 1-\frac{t_1}{\lambda}; \alpha(\omega+1)+1, \alpha(\omega+1)+i-\frac{t_2}{\lambda}+2; 1\Big) \\
&-\sum_{l=0}^{\alpha(n+p+k-\omega)-1}(-1)^l\binom{\alpha(n+p+k-\omega)-1}{l}\beta(\alpha(\omega+1)+1, l-\frac{t_2}{\lambda}+1) \\
&{}_3F_2\Big(\alpha(\omega+1)+1, \alpha(\omega+1), 1-\frac{t_1}{\lambda}; \alpha(\omega+1)+1, \alpha(\omega+1)+l-\frac{t_2}{\lambda}+2; 1\Big] \\
&+\frac{t_1}{n\lambda(\lambda+1)}\sum_{h=0}^{p+n}\binom{n+p}{h}(-1)^h\beta(\alpha(p+n+2), \frac{-t_2}{\lambda}+1) \\
&\Big[{}_3F_2\Big(\alpha(p+n+2)), \alpha(h+1), 1+\frac{t_1}{\lambda}; \alpha(h+1)+1, \alpha(p+n+2)+1-\frac{t_2}{\lambda}; 1\Big) \\
&-{}_3F_2\Big(\alpha(p+n+2)), \alpha(h+1), \frac{t_1}{\lambda}; \alpha(h+1)+1, \alpha(p+n+2)+1-\frac{t_2}{\lambda}; 1\Big)\Big]\Big\}.
\end{aligned} \tag{20}$$

Proof. We already know that

$$M_{L^c_{n+1},U^c_{n+1}}(t_1,t_2)=\frac{2^{n+1}}{n!}\int_0^\infty\int_0^y e^{t_1x+t_2y}f(x)f(y)-\log[1+F(x)-F(y)]^n dxdy.$$

Let

$$I(y)=\int_0^y e^{t_1x}f(x)\{-\log[1+F(x)-F(y)]\}^n dx,$$

using integration by parts by putting the deferential part to be $e^{t_1x}\{-\log[1+F(x)-F(y)]\}^n$ and the rest as the integral part we reach

$$\begin{aligned}M_{L^c_{n+1},U^c_{n+1}}(t_1,t_2)=\frac{2^{n+1}}{(n-1)!}\int_0^\infty\int_0^y e^{t_1x+t_2y}\frac{F(x)}{1+F(x)-F(y)}f(x)f(y)\\ \{-\log[1+F(x)-F(y)]\}^{n-1}dxdy-\frac{2^{n+1}}{n!}t_1\int_0^\infty\int_0^y e^{t_1x+t_2y}f(y)F(x)\\ \{-\log[1+F(x)-F(y)]\}^n dxdy,\end{aligned}$$

from which, we can simply conclude that it will became

$$\begin{aligned}M_{L^c_{n+1},U^c_{n+1}}(t_1,t_2)=2M_{L^c_n,U^c_n}(t_1,t_2)+\frac{2^{n+1}}{(n-1)!}\int_0^\infty\int_0^y e^{t_1x+t_2y}\frac{F(y)-1}{1+F(x)-F(y)}f(x)f(y)\\ \{-\log[1+F(x)-F(y)]\}^{n-1}dxdy-\frac{2^{n+1}}{n!}t_1\int_0^\infty\int_0^y e^{t_1x+t_2y}f(y)F(x)\\ \{-\log[1+F(x)-F(y)]\}^n dxdy.\end{aligned} \tag{21}$$

Now choose

$$J(t_1,t_2)=\int_0^\infty\int_0^y e^{t_1x+t_2y}\frac{F(y)-1}{1+F(x)-F(y)}f(x)f(y)\{-\log[1+F(x)-F(y)]\}^{n-1}dxdy,$$

and

$$\zeta(t_1,t_2)=\int_0^\infty\int_0^y e^{t_1x+t_2y}f(y)F(x)\{-\log[1+F(x)-F(y)]\}^n dxdy.$$

From $J(t_1,t_2)$, let

$$K(y)=\int_0^\infty e^{t_1x}\frac{f(x)}{1+F(x)-F(y)}\{-\log[1+F(x)-F(y)]\}^{n-1}dx,$$

now, substituting $u=F(y)-F(x)$ and then using the logarithmic expansion from (7) we reach

$$K(y)=\sum_{p=0}^\infty a_p(n)\int_0^{F(y)}\frac{u^{n-1+p}}{1-u}[1-(F(y)-u)^{\frac{1}{\alpha}}]^{\frac{-t_1}{\lambda}}du,$$

since $-1<u<1$, we can use the well known expansion $\frac{1}{1-u}=\sum_{k=0}^\infty u^k$, then make a substitution of $w=(F(y)-u)^{\frac{1}{\alpha}}$ to reach

$$K(y)=\alpha\sum_{p=0}^\infty a_p(n)\sum_{k=0}^\infty\sum_{\omega=0}^{n+p+k-1}(-1)^\omega\binom{n+p+k-1}{\omega}(F(y))^{n+p+k-\omega-1}\beta_{(F(y))^{\frac{1}{\alpha}}}(\alpha(\omega+1),1-\frac{t_1}{\lambda}).$$

Now $J(t_1,t_2)$ has became

$$J(t_1,t_2)=\alpha\sum_{p=0}^{\infty}a_p(n)\sum_{k=0}^{\infty}\sum_{\omega=0}^{n+p+k-1}(-1)^{\omega}\binom{n+p+k-1}{\omega}\int_0^{\infty}f(y)(F(y)-1)(F(y))^{n+p+k-\omega-1}e^{t_2y}$$
$$\beta_{(F(y))^{\frac{1}{\alpha}}}(\alpha(\omega+1),1-\frac{t_1}{\lambda})dy.$$

By replacing $f(y)$ and $F(y)$ by their original formulas from (5) and (6) and after that we made a substituting $e^{-\lambda y}=u$, we get

$$J(t_1,t_2)=\alpha^2\sum_{p=0}^{\infty}a_p(n)\sum_{k=0}^{\infty}\sum_{\omega=0}^{n+p+k-1}(-1)^{\omega}\binom{n+p+k-1}{\omega}\Big[\sum_{i=0}^{\alpha(n+p+k-\omega+1)-1}(-1)^{i}$$
$$\binom{\alpha(n+p+k-\omega+1)-1}{i}\int_0^1 u^{i-\frac{t_2}{\lambda}}\beta_{1-u}(\alpha(\omega+1),1-\frac{t_1}{\lambda})du$$
$$-\sum_{l=0}^{\alpha(n+p+k-\omega)-1}(-1)^{l}\binom{\alpha(n+p+k-\omega)-1}{l}\int_0^1 u^{l-\frac{t_2}{\lambda}}\beta_{1-u}(\alpha(\omega+1),1-\frac{t_1}{\lambda})du\Big],$$

upon using (10) and (11) in the last integral we reach the final form of the function $J(t_1,t_2)$ as follows

$$J(t_1,t_2)=\alpha^2\sum_{p=0}^{\infty}a_p(n)\sum_{k=0}^{\infty}\sum_{\omega=0}^{n+p+k-1}(-1)^{\omega}\binom{n+p+k-1}{\omega}\Big[\sum_{i=0}^{\alpha(n+p+k-\omega+1)-1}(-1)^{i}$$
$$\binom{\alpha(n+p+k-\omega+1)-1}{i}\beta(\alpha(\omega+1)+1,i-\frac{t_2}{\lambda}+1)$$
$${}_3F_2\Big(\alpha(\omega+1)+1,\alpha(\omega+1),1-\frac{t_1}{\lambda};\alpha(\omega+1)+1,\alpha(\omega+1)+i-\frac{t_2}{\lambda}+2;1\Big) \quad (22)$$
$$-\sum_{l=0}^{\alpha(n+p+k-\omega)-1}(-1)^{l}\binom{\alpha(n+p+k-\omega)-1}{l}\beta(\alpha(\omega+1)+1,l-\frac{t_2}{\lambda}+1)$$
$${}_3F_2\Big(\alpha(\omega+1)+1,\alpha(\omega+1),1-\frac{t_1}{\lambda};\alpha(\omega+1)+1,\alpha(\omega+1)+l-\frac{t_2}{\lambda}+2;1\Big].$$

And for $\zeta(t_1,t_2)$ we begin by taking the following integration out of it

$$Z(y)=\int_0^{y}e^{t_1x}F(x)\{-\log[1+F(x)-F(y)]\}^{n}dx,$$

then making a substitution of $u=F(y)-F(x)$, after that we used equation (9) followed by the logarithmic expansion from (7). Another substitution made by $w=((F(y)-u)^{\frac{1}{\alpha}}$ and some routine calculations leads to the following

$$Z(y)=\frac{1}{\lambda}\sum_{q=0}^{\infty}a_q(n)\sum_{h=0}^{q+n}\binom{n+q}{h}(-1)^{h}(F(y))^{q+n-h}\Big\{\beta_{(F(y))^{\frac{1}{\alpha}}}(\alpha(h+1),\frac{-t_1}{\lambda})-$$
$$\beta_{(F(y))^{\frac{1}{\alpha}}}(\alpha(h+1),\frac{-t_1}{\lambda}+1)\Big\}.$$

Now, we return to the original $\zeta(t_1, t_2)$ double integral and put the final form of $Z(y)$ function in it. After that, we replace $F(y)$ and $f(y)$ as in (5) and (6) respectively and by the substitution $u = 1 - e^{-\lambda y}$ we reach an integral in which we can use (10) and (11) to solve it and reach that final formula of $\zeta(t_1, t_2)$ which is going to be

$$\begin{aligned}\zeta(t_1,t_2) &= \frac{-1}{\lambda(\lambda+1)}\sum_{q=0}^{\infty} a_q(n)\sum_{h=0}^{q+n}\binom{n+q}{h}(-1)^h\beta(\alpha(q+n+2),\frac{-t_2}{\lambda}+1)\\ &\Big\{ {}_3F_2\Big(\alpha(q+n+2)),\alpha(h+1),1+\frac{t_1}{\lambda};\alpha(h+1)+1,\alpha(q+n+2)+1-\frac{t_2}{\lambda};1\Big)\\ &- {}_3F_2\Big(\alpha(q+n+2)),\alpha(h+1),\frac{t_1}{\lambda};\alpha(h+1)+1,\alpha(q+n+2)+1-\frac{t_2}{\lambda};1\Big)\Big\}.\end{aligned} \tag{23}$$

Exchanging (22) and (23) by their equal parts from (21), we get (20).

Lemma 1. *Another recurrence relation for of $M_{L^c_{n+1},U^c_{n+1}}(t_1,t_2)$ can be found by using analogs steps used in Theorem 3 but leads to the following relation*

$$\begin{aligned}(1-\frac{t_1}{\alpha\lambda})M_{L^c_{n+1},U^c_{n+1}}(t_1,t_2) &+ \frac{t_1}{\alpha\lambda}M_{L^c_{n+1},U^c_{n+1}}(t_1+\lambda,t_2) = 2M_{L^c_n,U^c_n}(t_1,t_2) + \frac{2^{n+1}}{(n-1)!}\alpha^2\\ &\sum_{p=0}^{\infty}a_p(n)\sum_{k=0}^{\infty}\sum_{\omega=0}^{n+p+k-1}(-1)^{\omega}\binom{n+p+k-1}{\omega}\\ &\Big[\sum_{i=0}^{\alpha(n+p+k-\omega+1)-1}(-1)^i\binom{\alpha(n+p+k-\omega+1)-1}{i}\beta(\alpha(\omega+1)+1,i-\frac{t_2}{\lambda}+1)\\ &{}_3F_2\Big(\alpha(\omega+1)+1,\alpha(\omega+1),1-\frac{t_1}{\lambda};\alpha(\omega+1)+1,\alpha(\omega+1)+i-\frac{t_2}{\lambda}+2;1\Big)\\ &-\sum_{l=0}^{\alpha(n+p+k-\omega)-1}(-1)^l\binom{\alpha(n+p+k-\omega)-1}{l}\beta(\alpha(\omega+1)+1,l-\frac{t_2}{\lambda}+1)\\ &{}_3F_2\Big(\alpha(\omega+1)+1,\alpha(\omega+1),1-\frac{t_1}{\lambda};\alpha(\omega+1)+1,\alpha(\omega+1)+l-\frac{t_2}{\lambda}+2;1\Big].\end{aligned} \tag{24}$$

Corollary 3. *For $i = 1, 2, 3, ...,$ and $n \geq 2$, by taking the i^{th} derivative of (18) and (19) and substituting $t = 0$ we reach the following relations. Also, by differentiating (20) i times with respect to t_1 and j times with respect to t_2 and then substituting $t_1 = t_2 = 0$ we get the recurrence relation of the product moment between lower and upper current record $\mu^{(i,j)}_{L^c_{n+1},U^c_{n+1}}$.*

$$\mu^{(i)}_{L^c_{n+1}} = 2\mu^{(i)}_{L^c_n}(t) - \frac{(2\alpha)^{n+1}}{n!}\sum_{p=0}^{\infty}a_p(n)\beta^{(i)}(2\alpha,-\frac{t}{\lambda}+n+p+1)\Big|_{\mathbf{t=0}}. \tag{25}$$

$$\mu^{(i)}_{U^c_{n+1}} = 2\mu^{(i)}_{U^c_n}(t) - \frac{2^{n+1}\alpha}{n!}\sum_{p=0}^{\infty}a_p(n)\Big[\beta^{(i)}((1+n+p)\alpha,1-\frac{t}{\lambda})-\beta^{(i)}((2+n+p)\alpha,1-\frac{t}{\lambda})\Big]\Big|_{\mathbf{t=0}}. \tag{26}$$

5. Moments of Record Range

A calculable formula for the m^{th} moments of record range $\mu^{(m)}_{R^c_n}$ will be provided in the next theorem along with its proof. The formula contains an integrated part that can be easily calculated using any mathematical package such as MATHEMATICA.

Theorem 4. *For $n > 2$*

$$\mu^{(m)}_{R^c_n} = \frac{2^n\alpha^2\lambda}{(n-1)!}\sum_{v=0}^{\alpha-1}\sum_{w=0}^{\alpha-1}\sum_{j=0}^{\infty}\binom{\alpha-1}{v}\binom{\alpha-1}{w}\frac{(-1)^{v+w}}{v+w+n+j+1}\int_0^\infty r^m e^{-\lambda(w+1)r} a_j(n-1, e^{-\lambda r}, \alpha)dr. \tag{27}$$

Proof. By substituting (5) and (6) in (3), we get

$$f_{R^c_n}(r) = \frac{2^n\alpha^2\lambda e^{-\lambda r}}{(n-1)!}\lambda I(r), \tag{28}$$

where

$$I(r) = \lambda\int_0^\infty (e^{-\lambda x})^2(1-e^{-\lambda x})^{\alpha-1}(1-e^{-\lambda r}e^{-\lambda x})^{\alpha-1}\Big\{-\log[1-(1-e^{-\lambda r}e^{-\lambda x})^\alpha + (1-e^{-\lambda x})^\alpha]\Big\}^{n-1}dx.$$

By making the substitution $u = e^{\lambda x}$ and then using the binomial expansion on some of the terms, we reach

$$I(r) = \sum_{v=0}^{\alpha-1}\sum_{w=0}^{\alpha-1}\binom{\alpha-1}{v}\binom{\alpha-1}{w}(-1)^{v+w}(e^{-\lambda r})^w\int_0^1 u^{v+w+1}\Big\{-\log[1-(1-e^{-\lambda r}u)^\alpha+(1-u)^\alpha]\Big\}^{n-1}du.$$

The following expansion was created in the same way Balakrishnan and cohen [6] did in (7)

$$\Big\{-\log[1-(1-e^{-\lambda r}u)^\alpha + (1-u)^\alpha]\Big\}^{n-1} = \sum_{j=0}^{\infty} a_j(n-1, e^{-\lambda r}, \alpha)u^{n-1+j}$$

, where $a_j(n-1, e^{-\lambda r}, \alpha)$ is the coefficient of the expansion. Then

$$\begin{aligned} I(r) &= \sum_{v=0}^{\alpha-1}\sum_{w=0}^{\alpha-1}\sum_{j=0}^{\infty}\binom{\alpha-1}{v}\binom{\alpha-1}{w}(-1)^{v+w}(e^{-\lambda r})^w a_j(n-1, e^{-\lambda r}, \alpha)\int_0^1 u^{v+w+1}u^{n-1+j}du \\ &= \sum_{v=0}^{\alpha-1}\sum_{w=0}^{\alpha-1}\sum_{j=0}^{\infty}\binom{\alpha-1}{v}\binom{\alpha-1}{w}(-1)^{v+w}a_j(n-1, e^{-\lambda r}, \alpha)\frac{e^{-\lambda wr}}{v+w+n+j+1}. \end{aligned} \tag{29}$$

Now, substitute (29) in (28) we get

$$f_{R_n^c}(r) = \frac{2^n\alpha^2\lambda}{(n-1)!}\sum_{v=0}^{\alpha-1}\sum_{w=0}^{\alpha-1}\sum_{j=0}^{\infty}\binom{\alpha-1}{v}\binom{\alpha-1}{w}(-1)^{v+w}a_j(n-1,e^{-\lambda r},\alpha)\frac{e^{-\lambda(w+1)r}}{v+w+n+j+1}. \tag{30}$$

To find the moments of the record range we will use the well-known formula

$$\mu_{R_n^c}^{(m)} = \int_0^\infty r^m f_{R_n^c}(r)dr. \tag{31}$$

Substituting (30) in (31) we get reach (27).

6. Numerical Example

To prove the benefits and motivations of using the previous formulas, a numerical simulation study was conducted to illustrate that.

Simulation Study. *Samples of size 20 from the pdf of the upper current records (2) for GED cdf when $\alpha = 5$ and $\lambda = 2$ and at different values of $n = 2, 3, 4, 5\, and\, 6$ was simulated using MATHEMATICA 12.0 and the true mean for each sample was calculated. After that, we used formula (17) to estimate the value of the mean at the same values of $\alpha, \lambda\, and\, n$ when $i = 1$. Then, from (23) we calculated the predicted mean at also the same $\alpha, \lambda, n\, and\, i$. As we can see from the results listed in Table 1, the true mean is very close to the estimated mean and the predicted mean is even closer to the estimated mean indicating the efficiency of the formulas created in this paper. Analogies work has been done for the lower current records formulas and the same good results were found.*
Note: Most of the results calculated from the formulas (16),(17),(18), and (19) were conducted to a maximum of 2000 iterations. And after that, the results did not change.

Table 1: True, Estimate and Predicted Mean

Upper Current Record	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
True Mean	1.73759	2.03974	2.32234	2.52788	2.78866
Estimated Mean	1.75027	2.02074	2.28374	2.54529	2.81821
Predicted Mean		2.02066	2.28373	2.5453	2.81863
Lower Current Record	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
True Mean	0.649044	0.52386	0.474306	0.403313	0.326389
Estimated Mean	0.649046	0.535685	0.452289	0.387566	0.335597
Predicted Mean		0.535681	0.452289	0.387566	0.335597

7. Conclusion

It is difficult to calculate the MGF of the lower or the upper current record or the joint MGF of the lower and upper current record especially when it follows a distribution such as GED. The author tries to solve this by finding the simplest formulas to achieve that. Although, the author establishes some formulas to find the moments of the lower current record, upper current record, record range, and product moments of the lower and upper current record based on the same distribution. Also, the larger the value n grows, the more time it takes and the more difficult it becomes to be calculated. And here comes the role of the recurrence relation formulas created for the moments and MGFs in this paper to make it easier and faster. To demonstrate that, a numerical example is added and most of the formulas are used in it and the desired result occurred by finding that the formulas of the MGF and the moments give values very close to the actual ones. And also, the values that result from the recurrence relations are almost the same values as the ones from its opposite MGF or moments but calculated faster.

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