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Stable Locating-Dominating Sets in the Edge Corona and Lexicographic Product of Graphs

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Abstract. A set $S \subseteq V(G)$ of an undirected graph G is a locating-dominating set of G if for each $v \in V(G) \setminus S$, there exists $w \in S$ such tha $vw \in E(G)$ and $N_G(x) \cap S \neq N_G(y) \cap S$ for any two distinct vertices x and y in $V(G) \setminus S$. S is a stable locating-dominating set of G if it is a locating-dominating set of G and $S \setminus \{v\}$ is a locating-dominating set of G for each $v \in S$. The minimum cardinality of a stable locating-dominating set of G, denoted by $\gamma_{SLD}(G)$, is called the stable locating-domination number of G. In this paper, we investigate this concept and the corresponding parameter for edge corona and lexicographic product of graphs.

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Key Words and Phrases: Locating, stable, domination, edge corona, lexicographic product

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1. Introduction

Let $G = (V(G), E(G))$ be an undirected graph. The distance between two vertices u and v of G, denoted by $d_G(u, v)$, is equal to the length of a shortest path connecting u and v. Any path connecting u and v of length $d_G(u, v)$ is called a *u-v geodesic*.The neighborhood of $v \in V(G)$ is the set $N_G(v)= \{x \in V(G) : xv \in E(G)\}$. The degree of $v \in V(G)$, denoted by $deg_G(v)$, is equal to the cardinality of $N_G(v)$ and the *maximum degree* of G is $\Delta(G)=$ max $\{deg_G(x) : x \in E(G)\}$. A vertex v of G is a *leaf* if $deg_G(v) = 1$. A vertex u of G is a *support* if $uv \in E(G)$ for some leaf v of G. A connected graph G of order $n \geq 3$ is *point distinguishing* if for any two distinct vertices u and v of G, $N_G[u] \neq N_G[v]$. It is *totally point determining* if for any two distinct vertices u and v of G, $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$. These concepts are defined and studied in [7] and [21].

A subset S of $V(G)$ is a *dominating* set of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $xv \in E(G)$. S is a *locating set* in G if $N_G(u) \cap S \neq N_G(v) \cap S$ for every

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two distinct vertices $u, v \in V(G) \setminus S$. A locating set S is said to be a *strictly locating set* if $N_G(u) \cap S \neq S$ for all $u \in V(G) \setminus S$. A locating set (strictly locating set) S is a *stable locating set* (resp. *stable strictly locating set*) if $S \setminus \{v\}$ is a locating (resp. strictly locating) set for each $v \in S$. A locating (resp. strictly locating) set S of $V(G)$ which is also a dominating set is called a *locating-dominating* (resp. *strictly locating-dominating*) set of G. A locating-dominating (strictly locating-dominating) set S is a *stable locating-dominating* (resp. *stable strictly locating-dominating*) set of G if $S \setminus \{v\}$ is a locating-dominating (resp. strictly locating-dominating) set of G for each $v \in S$. The minimum cardinality of a locating (strictly locating, stable locating, stable strictly locating) set of G is denoted by $ln(G)$ (resp. $sln(G)$, $sbln(G)$, $sbsln(G)$). Any locating (strictly locating, stable locating, stable strictly locating) set of G with cardinality $ln(G)$ (resp. $sln(G)$, $sbln(G)$, $sbsln(G)$) is called an ln-set (resp. sln-set, $sbln$-set, $sbsln$-set) of G. The minimum cardinality of a locating-dominating (resp. strictly locating-dominating, stable locating-dominating, stable strictly locating-dominating) set of G is denoted by $\gamma_L(G)$ (resp. $\gamma_{SL}(G)$, $\gamma_{sL}(G)$, $\gamma_{sSL}(G)$). Any locating-dominating (strictly locating-dominating, stable locating-dominating, stable strictly locating-dominating) set of G with cardinality $\gamma_L(G)$ (resp. $\gamma_{SL}(G)$, $\gamma_{sL}(G)$, $\gamma_{sSL}(G)$) is called an γ_L-set (resp. γ_{SL}-set, γ_{sL}-set, γ_{sSL}-set) of G.

Domination and some variations of the concept are found in the book by Haynes et al. (see [9]). Other variations of domination can be found in [2], [3], [4], [5], [11], [12], [16], and [18]. The concepts of locating, stricly locating, locating-dominating, and strictly locating-dominating, and the associated parameters are studied in [6], [8], [10], [13], [14], [15], [17], [19], [20]. The concept of stable locating-dominating and related concepts are studied in [1].

Let G and H be any two graphs. The *edge corona* $G \diamond H$ is the graph obtained by taking one copy of G and $|E(G)|$ copies H and joining each end vertices u and v of every edge uv to every vertex of the copy H^{uv} of H (i.e. forming the join $\langle\{u, v\}\rangle + H^{uv}$ for each $uv \in E(G)$). The *lexicographic product* $G[H]$ is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$. It is easily observed that for any non-empty subset C of $V(G[H]) = V(G) \times V(H)$, this set can be expressed as $C = \cup_{x \in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Set $S = C_G = \{x \in V(G) : (x, a) \in C$ for some $a \in V(H)\}$ is called the G-projection of C. Moreover, for each $x \in S$, $T_x = \{a \in V(H) : (x, a) \in C\}$.

2. Results

Throughout, we denote by $\mathcal{L}(G)$ the set containing all leaves of a graph G.

The first result is found in [1].

Theorem 1. *Let G be graph without isolated vertices. Then G has a stable strictly locating set if and only if $\gamma(G) \neq 1$.*

Theorem 2. Let G be a connected graph of order $m \geq 3$ and let H be any non-trivial connected graph. Then C is a locating-dominating set of $G \diamond H$ if and only if $C = A \cup [\cup_{uv \in E(G)} S_{uv}]$ and satisfies the following conditions:

(*i*) $A \subseteq V(G)$.

(*ii*) For each $uv \in E(G)$,

(*a*) S_{uv} is a locating set of H^{uv};

(*b*) S_{uv} is a locating-dominating set of H^{uv} whenever $u, v \notin A$;

(*c*) S_{uv} is a strictly locating set of H^{uv} for each $v \in \mathcal{L}(G)$ with $v \notin A$; and

(*d*) S_{uv} is a strictly locating-dominating set of H^{uv} whenever $u, v \notin A$ and $\{u, v\} \cap L(G) \neq \varnothing$.

(*iii*) For each $uv \in E(G)$ with $v \in A$ and $u \notin A$, if $x \in V(H^{uv}) \setminus S_{uv}$ and $N_{H^{uv}}(x) \cap S_{uv} = \varnothing$, then for each $w \in N_G(v) \setminus \{u\}$ and for each $y \in V(H^{wv}) \setminus S_{wv}$, it holds that $w \in A$ or $N_{H^{wv}}(y) \cap S_{wv} \neq \varnothing$.

Proof. Suppose C is a locating-dominating set of $G \diamond H$. Let $A = C \cap V(G)$ and $S_{uv} = C \cap V(H^{uv})$ for each $uv \in E(G)$. Then $C = A \cup [\cup_{uv \in E(G)} S_{uv}]$ and (*i*) holds. Let $uv \in E(G)$. Since C is a locating set of $G \diamond H$, $S_{uv} \neq \varnothing$. Let $x, y \in V(H^{uv}) \setminus S_{uv}$ with $x \neq y$ and let $S = A \cap \{u, v\}$. Since C is a locating set,

$$[N_{H^{uv}}(x) \cap S_{uv}] \cup S = N_{G \diamond H}(x) \cap C \neq N_{G \diamond H}(y) \cap C = [N_{H^{uv}}(y) \cap S_{uv}] \cup S.$$

This implies that $N_{H^{uv}}(x) \cap S_{uv} \neq N_{H^{uv}}(y) \cap S_{uv}$, showing that S_{uv} is a locating set of H^{uv}. Suppose $u, v \notin A$. Since C is a dominating set of $G \diamond H$, S_{uv} is a dominating set of $V(H^{uv}$. Hence, (*a*) and (*b*) hold. Next, suppose that $uv \in E(G)$ and $v \in L(G) \setminus A$. Let $S^* = A \cap \{u\}$ and let $z \in V(H^{uv}) \setminus S_{uv}$. Again, since C is a locating set,

$$[N_{H^{uv}}(z) \cap S_{uv}] \cup S^* = N_{G \diamond H}(z) \cap C \neq N_{G \diamond H}(v) \cap C = S_{uv} \cup S^*.$$

This implies that $[N_{H^{uv}}(z) \cap S_{uv}] \neq S_{uv}$, showing that S_{uv} is a strictly locating set of H^{uv}. If $S^* = \varnothing$ (that is, $u \notin A$), then S_{uv} is a dominating set of $V(H^{uv})$. Thus, (*c*) and (*d*) hold.

Finally, let $uv \in E(G)$ with $v \in A$ and $u \notin A$. Suppose $x \in V(H^{uv}) \setminus S_{uv}$ and $N_{H^{uv}}(x) \cap S_{uv} = \varnothing$. Then $N_{G \diamond H}(x) \cap C = \{v\}$. Let $w \in N_G(v) \setminus \{u\}$ and let $y \in V(H^{wv}) \setminus S_{wv}$. Suppose $w \notin A$. Then $N_{G \diamond H}(y) \cap C = [N_{H^{wv}}(y) \cap S_{wv}] \cup \{v\}$. Since C is a locating set of $G \diamond H$, $N_{G \diamond H}(x) \cap C \neq N_{G \diamond H}(y) \cap C$. This implies that $N_{H^{wv}}(y) \cap S_{wv} \neq \varnothing$, showing that (*iii*) holds.

For the converse, suppose that C has the form described and satisfies (*i*), (*ii*), and (*iii*). Let $z \in V(G \diamond H) \setminus C$ and let $uv \in E(G)$ such that $z \in \langle \{u, v\} \rangle + H^{uv}$. If $z = u$ or $z = v$, then there exists $t \in S_{uv} \subset C$ such that $z \in N_{G \diamond H}(t)$ by (*ii*)(*a*). Suppose $z \in V(H^{uv}) \setminus S_{uv}$. If $u \in A$ or $v \in A$, then $uz \in E(G \diamond H)$ or $vz \in E(G \diamond H)$. If $u, v \notin A$, then there exists $s \in S_{uv} \cap N_{G \diamond H}(z)$ by (*ii*)(*b*). Hence, C is a dominating set of $G \diamond H$.

Next, let $p, q \in V(G \diamond H) \setminus C$ with $p \neq q$ and let $uv, xy \in E(G)$ such that $p \in \langle\{u,v\}\rangle + H^{uv}$ and $q \in \langle\{x,y\}\rangle + H^{xy}$. Consider the following cases:

Case 1. The edges uv and xy are non-adjacent (i.e., they do not share a common vertex).

Suppose that $p \in \{u,v\}$ or $q \in \{x,y\}$. Since $S_{uv} \subseteq N_{G\diamond H}(p)$ and $S_{xy} \subseteq N_{G\diamond H}(q)$, $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$. Suppose that $p \notin \{u,v\}$ and $q \notin \{x,y\}$. Then $p \in V(H^{uv}) \setminus S_{uv}$ and $q \in V(H^{xy}) \setminus S_{xy}$. Since $N_{G\diamond H}(p) \cap C \subseteq V(\langle\{u,v\}\rangle + H^{uv})$ and $N_{G\diamond H}(q) \cap C \subseteq V(\langle\{x,y\}\rangle + H^{xy})$, it follows that $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$.

Case 2. The edges uv and xy are distinct and adjacent.

We may assume that $x = u$. If $p \in \{u,v\}$ or $q \in \{x,y\}$, then $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$ (as in Case 1). So suppose that $p \notin \{u,v\}$ and $q \notin \{x,y\}$. If $N_{H^{uv}}(p) \cap S_{uv} \neq \varnothing$ or $N_{H^{xy}}(y) \cap S_{xy} \neq \varnothing$, then $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$. Suppose that $N_{H^{uv}}(p) \cap S_{uv} = \varnothing$ or $N_{H^{xy}}(y) \cap S_{xy} = \varnothing$. If $u \in A$, then $y \in A$ or $v \in A$ by (iii). Suppose that $u \notin A$. Then by $(ii)(b)$, $y, v \in A$. Since $v \in N_{G\diamond H}(p) \cap C$, $y \in N_{G\diamond H}(y) \cap C$, and $y \neq v$, it follows that $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$.

Case 3. The edges uv and xy are the same.

We may assume that $x = u$ and $y = v$. Suppose first that $p = u$ and $q = v$. Since G is connected and $G \neq K_2$, we may assume that there exists $w \in V(G) \setminus \{u,v\}$ such that $vw \in E(G)$. Because $\varnothing \neq S_{vw} \subseteq (N_{G\diamond H}(q) \cap C \setminus (N_{G\diamond H}(p) \cap C)$, we have $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$. Suppose that $p, q \in V(H^{uv}) \setminus S_{uv}$. By $(ii)(a)$, $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$. Finally, suppose that $p \in V(H^{uv}) \setminus S_{uv}$ (or $q \in V(H^{uv}) \setminus S_{uv}$) and $q \in \{u,v\}$ (resp. $p \in \{u,v\}$). We may assume without loss of generality that $q = u$. Consider the following subcases:

Subcase 1. $u, v \notin \mathcal{L}(G)$.

Then there exist $a, b \in V(G)$ such that $au, bv \in E(G)$. Since $S_{au} \subseteq N_{G\diamond H}(q) \cap C$ and $S_{au} \cap N_{G\diamond H}(p) = \varnothing$, $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$.

Subcase 2. $u \in \mathcal{L}(G)$ or $v \in \mathcal{L}(G)$.

By $(ii)(c)$ and $(ii)(d)$, S_{uv} is a strictly locating set of $V(H^{uv})$. It follows that $N_{H^{uv}}(p) \cap S_{uv} \neq S_{uv}$. Since $S_{uv} \subseteq N_{G\diamond H}(q) \cap C$, $N_{G\diamond H}(p) \cap C \neq N_{G\diamond H}(q) \cap C$.

Accordingly, C is locating-dominating set of $G \diamond H$. □

A set $S \subseteq V(G)$ is a *vertex cover* of G if for every $uv \in E(G)$, $u \in S$ or $v \in S$. A vertex cover S is a *perfect vertex cover* of G if for each $v \in S$ and for each pair of distinct edges uv and wv of G, $u \in S$ or $w \in S$. The smallest size of a perfect vertex cover of

G, denoted by $\beta_p(G)$, is called the the *perfect vertex covering number* of G. Any perfect vertex cover of G of size $\beta_p(G)$ is called a *β_p-set* or a *minimum perfect vertex cover* of G.

Example 1. $\beta_p(K_n) = n - 1$ *for each* $n \geq 2$.

Corollary 1. Let G be a connected graph of order $m \geq 3$ and let H be any non-trivial connected graph.

(i) If $L(G) = \varnothing$, then

$$\gamma_L(G \diamond H) \leq \min\{\beta_p(G) + |E(G)|ln(H), |E(G)|\gamma_L(H)\}.$$

(ii) If $L(G) \neq \varnothing$, then

$$\gamma_L(G \diamond H) \leq \min\{\beta_p(G) + |E(G)|sln(H), |E(G)|\gamma_{SL}(H)\}.$$

Proof. (i) Suppose that $L(G) = \varnothing$. Let S_1 be a β_p-set of G and let S_{uv} be a minimum locating set of H^{uv} for each $uv \in E(G)$. Then $C_1 = S \cup [\cup_{uv\in E(G)} S_{uv}]$ is a locating dominating set of $G \diamond H$ by Theorem 2. Hence, $\gamma_L(G \diamond H) \leq |C_1| = \beta_p(G) + |E(G)|ln(H)$. Now, let L_{uv} be a γ_L-set of H^{uv} for each $uv \in E(G)$. Then $C_2 = \cup_{uv\in E(G)} L_{uv}$ is a locating dominating set of $G \diamond H$ by Theorem 2. This implies that $\gamma_L(G \diamond H) \leq |C_2| = |E(G)|\gamma_L(H)$. Therefore, (i) holds.

(ii) Suppose that $L(G) \neq \varnothing$. Let S be a β_p-set of G and let S'_{uv} be a minimum strictly locating set of H^{uv} for each $uv \in E(G)$. Then $C_3 = S \cup [\cup_{uv\in E(G)} S'_{uv}$ is a locating dominating set of $G \diamond H$ by Theorem 2. Hence, $\gamma_L(G \diamond H) \leq |C_3| = \beta_p(G) + |E(G)|sln(H)$. Let R_{uv} be a γ_{SL}-set of H^{uv} for each $uv \in E(G)$. Then $C_4 = \cup_{uv\in E(G)} R_{uv}$ is a locating dominating set of $G \diamond H$ by Theorem 2. This implies that $\gamma_L(G \diamond H) \leq |C_4| = |E(G)|\gamma_{SL}(H)$, showing that (ii) holds. □

Remark 1. *The bounds in Corollary 2 are sharp.*

Indeed, it can be verified that

$$\gamma_L(K_3 \diamond P_5) = |E(K_3)|\gamma_L(P_5) = 6 < 8 = \beta_p(K_3) + |E(K_3)|ln(P_5),$$

$$\gamma_L(K_3 \diamond P_3) = \beta_p(K_3) + |E(K_3)|ln(P_3) = 5 < 6 = |E(K_3)|\gamma_L(P_3),$$

$$\gamma_L(P_3 \diamond P_3) = |E(P_3)|\gamma_{SL}(P_3) = 4 < 6 = \beta_p(P_3) + |E(P_3)|sln(P_3), \text{ and}$$

$$\gamma_L(P_4 \diamond P_5) = \beta_p(P_4) + |E(P_4)|sln(P_5) = 8 < 9 = |E(P_4)|\gamma_{SL}(P_5).$$

Theorem 3. Let G be a connected graph of order $m \geq 3$ and let H be any non-trivial connected graph. Then C is a stable locating-dominating set of $G \diamond H$ if and only if $C = A \cup [\cup_{uv\in E(G)} S_{uv}]$ and satisfies the following conditions:

(i) $A \subseteq V(G)$.

(ii) For each $uv \in E(G)$,

- (a) S_{uv} is a stable locating set of H^{uv};
- (b) S_{uv} is a stable locating-dominating set of H^{uv} whenever $u, v \notin A$;
- (c) S_{uv} is a stable strictly locating set of H^{uv} for each $v \in L(G)$ with $v \notin A$; and
- (d) S_{uv} is a stable strictly locating-dominating set of H^{uv} whenever $u, v \notin A$ and $\{u, v\} \cap L(G) \neq \varnothing$.

(iii) For each $w \in A$ and for each $z \in N_G(w)$, we have:

- (a) S_{zw} is a strictly locating set of H^{zw} whenever $w \in L(G)$ and
- (b) S_{zw} is a strictly locating-dominating set of H^{zw} whenever $z \notin A$ and $\{z, w\} \cap L(G) \neq \varnothing$.

(iv) For each $zw \in E(G)$ with $z \in A$ and $w \notin A$, if $x \in V(H^{zw}) \setminus [S_{zw} \setminus \{p\}]$ for $p \in S_{zw}$ and $N_{H^{zw}}(x) \cap (S_{zw} \setminus \{p\}) = \varnothing$, then for each $y \in N_G(z) \setminus \{w\}$ and for each $q \in V(H^{yz}) \setminus S_{yz}$, it holds that $y \in A$ or $N_{H^{yz}}(q) \cap S_{yz} \neq \varnothing$.

Proof. Suppose C is a stable locating-dominating set of $G \diamond H$. Let $A = C \cap V(G)$ and $S_{uv} = C \cap V(H^{uv})$ for each $uv \in E(G)$. Then $C = A \cup [\cup_{uv \in E(G)} S_{uv}]$ and (i) holds. Let $xy \in E(G)$. By Theorem 2$(ii)(a)$, S_{xy} is a locating set of H^{xy}. Let $p \in S_{xy}$. Then by assumption, $C \setminus \{p\} = A \cup [\cup_{uv \in [E(G) \setminus \{xy\}]} S_{uv}] \cup (S_{xy} \setminus \{p\})$ is a locating-dominating set of $G \diamond H$. It follows from Theorem 2$(ii)(a)$ that $S_{xy} \setminus \{p\}$ is a locating set of H^{xy}. If $x, y \notin A$, then $S_{xy} \setminus \{p\}$ is a locating-dominating set of H^{xy} by Theorem 2$(ii)(b)$. If one of x and y, say $x \in L(G) \setminus A$, then $S_{xy} \setminus \{p\}$ is a strictly locating set of H^{xy} by Theorem 2$(ii)(c)$. Morover, if $x, y \notin A$ and $x \in L(G)$ or $y \in L(G)$, then S_{xy} is a strictly locating-dominating set of H^{xy} by Theorem 2$(ii)(d)$. Therefore, $(a), (b), (c)$, and (d) hold.

Next, let $w \in A$ and let $z \in N_G(w)$. Since C is a stable locating-dominating set of $G \diamond H$,

$$C \setminus \{w\} = (A \setminus \{w\}) \cup [\cup_{uv \in E(G)} S_{uv}]$$

is a locating-dominating set of $G \diamond H$. It follows from (c) and (d) of Theorem 2 that S_{zw} is a strictly locating set of H^{zw} whenever $w \in L(G)$ and S_{zw} is a strictly locating-dominating set of H^{zw} whenever $z \notin A$ (hence, $z \notin A \setminus \{w\}$) and $\{z, w\} \cap L(G) \neq \varnothing$. This shows that (iii) holds.

Finally, let $zw \in E(G)$ with $z \in A$ and $w \notin A$. Let $p \in S_{zw}$. Then, again,

$$C \setminus \{p\} = A \cup [\cup_{uv \in [E(G) \setminus \{zw\}]} S_{uv}] \cup (S_{zw} \setminus \{p\})$$

is a locating-dominating set of $G \diamond H$. Hence, by Theorem 2(iii), statement (iv) holds.

For the converse, suppose that C has the given form and satisfies (i), (ii), (iii) and (iv). By (i) and (ii), it follows that (i) and (ii) of Theorem 2 are satisfied by C. Let $zw \in E(G)$ with $z \in A$ and $w \notin A$. Let $x \in V(H^{zw}) \setminus S_{zw}$. Then $x \in V(H^{zw}) \setminus [S_{zw} \setminus \{p\}]$ for $p \in S_{zw}$. Suppose $N_{H^{zw}}(x) \cap S_{zw} = \varnothing$. Then $N_{H^{zw}}(x) \cap (S_{zw} \setminus \{p\}) = \varnothing$. Hence, by (iv), for each $y \in N_G(z) \setminus \{w\}$ and for each $q \in V(H^{yz}) \setminus S_{yz}$, it holds that $y \in A$ or

$N_{H^{yz}}(q) \cap S_{yz} \neq \varnothing$. Thus, (iii) of Theorem 2 also holds for C. Therefore, C is a locating dominating set of $G \diamond H$. Let $q \in C$ and let $uv \in E(G)$ such that $q \in V(\langle\{u,v\}\rangle + H^{uv})$. Suppose first that $p \in \{u,v\}$. Then

$$C^* = C \setminus \{p\} = (A \setminus \{p\}) \cup [\cup_{uv \in E(G)} S_{uv}].$$

Accordingly, C is a stable locating-dominating set of $G \diamond H$. □

The next two results follow from Theorem 3.

Corollary 2. Let G be a connected graph of order $m \geq 3$ with $L(G) = \varnothing$ and let H be any non-trivial connected graph. If $C = \cup_{uv \in E(G)} S_{uv}$ and S_{uv} is a stable locating-dominating set of H^{uv} for each $uv \in E(G)$, then C is a stable locating-dominating set of $G \diamond H$. In particular,

$$\gamma_{sL}(G \diamond H) \leq |E(G)|\gamma_{sL}(H).$$

Proof. Since $L(G) = \varnothing$ and S_{uv} is a stable locating-dominating set of H^{uv} for each $uv \in E(G)$, $C = \cup_{uv \in E(G)} S_{uv}$ satisfies the conditions in Theorem 3. Thus, C is a stable locating-dominating set of $G \diamond H$ and $\gamma_{sL}(G \diamond H) \leq |C| = |E(G)|\gamma_{sL}(H)$. □

Corollary 3. Let G be a connected graph of order $m \geq 3$ with $L(G) \neq \varnothing$ and let H be any non-trivial connected graph with $\gamma(H) \neq 1$. If $C = \cup_{uv \in E(G)} S_{uv}$ and S_{uv} is a stable strictly locating-dominating set of H^{uv} for each $uv \in E(G)$, then C is a stable locating-dominating set of $G \diamond H$. Moreover,

$$\gamma_{sL}(G \diamond H) \leq |C| = |E(G)|\gamma_{sSL}(H).$$

Proof. By Theorem 1 and the assumption that $\gamma(H) \neq 1$, H admits a stable strictly locating-dominating set. Since S_{uv} is a stable strictly locating-dominating set of H^{uv} for each $uv \in E(G)$ and $L(G) \neq \varnothing$, $C = \cup_{uv \in E(G)} S_{uv}$ satisfies the conditions in Theorem 3. Hence, C is a stable locating-dominating set of $G \diamond H$ and $\gamma_{sL}(G \diamond H) \leq |C| = |E(G)|\gamma_{sSL}(H)$. □

The next result is found in [15].

Theorem 4. Let G and H be non-trivial connected graphs such that $\Delta(H) \leq |V(H)| - 2$. Then $C = \bigcup_{x \in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a locating-dominating set of $G[H]$ if and only if the following hold.

(i) $S = V(G)$.

(ii) T_x is a locating set in H for every $x \in V(G)$.

(iii) T_x or T_y is strictly locating in H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$.

(*iv*) T_x or T_y is a dominating set in H whenever x and y are distinct non-adjacent vertices of G with $N_G(x) = N_G(y)$.

Theorem 5. Let G and H be non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$. Then $C = \bigcup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a stable locating-dominating set of $G[H]$ if and only if each of the following conditions hold.

(*i*) $S = V(G)$.

(*ii*) T_x is a stable locating set in H for every $x \in V(G)$.

(*iii*) If x and y are adjacent vertices of G with $N_G[x] = N_G[y]$ and one, say T_x is not strictly locating, then T_y is a stable strictly locating set of H.

(*iv*) If x and y are distinct non-adjacent vertices of G with $N_G(x) = N_G(y)$ and one, say T_x is not a dominating set, then T_y is a stable dominating set of H.

Proof. Suppose C is a stable locating-dominating set of $G[H]$. By Theorem 4, $S = V(G)$ and T_x is a locating set of H for each $x \in V(G)$. Let $z \in S$ and let $a \in T_z$. By assumption, $C \setminus \{(z, a)\} = [\bigcup_{x\in S\setminus\{z\}}(\{x\} \times T_x)] \cup [\{z\} \times (T_z \setminus \{a\})]$ is a locating-dominating set of $G[H]$. By Theorem 4(*ii*), $T_z \setminus \{a\}$ is a locating set of H. This implies that T_z is a stable locating set of H. Thus, (*ii*) holds.

Suppose now that x and y are adjacent vertices of G with $N_G[x] = N_G[y]$. Suppose that one, say T_x is not a strictly locating set of H. By Theorem 4(*iii*), T_y is a strictly locating set of H. Let $p \in T_y$. Since

$$C \setminus \{(y, p)\} = [\bigcup_{z\in S\setminus\{y\}}(\{z\} \times T_z)] \cup [\{y\} \times (T_y \setminus \{p\})]$$

is a locating-dominating set of $G[H]$ and T_x is not strictly locating, it follows from Theorem 4(*iii*) that $T_y \setminus \{p\}$ is strictly locating. Therefore, T_y is a stable strictly locating set of H, showing that (*iii*) holds.

Next, suppose that x and y are distinct non-adjacent vertices of G with $N_G(x) = N_G(y)$. Suppose that one, say T_x is not a dominating set of H. Then T_y is a dominating set of G by Theorem 4(*iv*). Let $q \in T_y$. Since

$$C \setminus \{(y, q)\} = [\bigcup_{z\in S\setminus\{y\}}(\{z\} \times T_z)] \cup [\{y\} \times (T_y \setminus \{q\})]$$

is a locating-dominating set of $G[H]$ and T_x is not a dominating set of G, $T_y \setminus \{q\}$ is a dominating set of H by Theorem 4(*iv*). This shows that T_y is a stable dominating set of H. Thus, (*iv*) holds.

For the converse, suppose that C satisfies (*i*), (*ii*), (*iii*) and (*iv*). Then C satisfies the conditions (*i*), (*ii*), (*iii*) and (*iv*) of Theorem 4. Hence, C is a locating-dominating set of

$G[H]$. Let $(y,a) \in C$. Then

$$C^* = C \setminus \{(y,a)\} = [\bigcup_{x \in S \setminus \{y\}} (\{x\} \times T_x)] \cup [\{y\} \times (T_y \setminus \{a\})].$$

By (ii), $T_y \setminus \{a\}$ is a locating set and T_x are stable locating sets of H for each $x \in S \setminus \{y\}$. This would also imply that $C_G^* = S^* = S = V(G)$. Let x and z be adjacent vertices of G with $N_G[x] = N_G[z]$. If T_x is strictly locating, then we are done. So suppose that T_x is not strictly locating. Then by (iii), T_z is a stable strictly locating set. Hence, if $z \neq y$, then T_y is strictly locating and, if $z = y$, then $T_y \setminus \{a\}$ is strictly locating. Finally, let u and w be distinct non-adjacent vertices of G with $N_G(u) = N_G(w)$. If T_u is dominating is a dominating set, then we are done. Suppose T_u is not a dominating set in H. Then by (iv), T_w is a stable dominating set of H. This implies that T_w is a dominating set if $w \neq y$ and $T_y \setminus \{a\}$ is a dominating set if $w = y$. Therefore, C^* is a locating-dominating set of $G[H]$. Accordingly, C is a stable locating-dominating set of $G[H]$. □

Given a non-trivial connected graph H, we denote by $\gamma_L^s(H)$ the smallest size of a dominating stable locating set of H, i.e.,

$$\gamma_L^s(H) = \min\{|S| : S \text{ is a dominating stable locating set of } H\}.$$

Any dominating stable locating set of H of size $\gamma_L^s(H)$ is called a γ_L^s-set of H. Note that since $V(H)$ is a dominating stable locating set, it follows that H admits a dominating stable locating set.

Consider the graph H in Figure 1 below. Clearly, $S = \{c, d, e\}$ is a dominating stable locating set of H and $\gamma_L^s(H) = |S| = 3$.

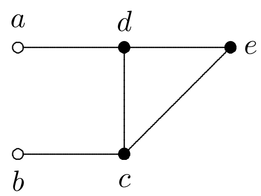

Figure 1

Corollary 4. Let G and H be non-trivial connected graphs. If G is point determining, then

$$\gamma_{sL}(G[H]) \leq |V(G)|\gamma_L^s(H).$$

Proof. Let D be a γ_L^s-set of H and let $T_x = D$ for each $x \in V(G)$. Then $C = \bigcup_{x \in V(G)} (\{x\} \times T_x)$ is a stable locating-dominating set of $G[H]$ by Theorem 5. Thus, $\gamma_{sL}(G[H]) \leq |C| = |V(G)|\gamma_L^s(H)$. □

Note that the bound in Corollary 4 is tight. To see this, consider the graph H in Figure 1. It can easily be verified that $\gamma_{sL}(P_3[H]) = 9 = 3.3 = |V(P_3)|\gamma_L^s(H)$.

The next result follows from Theorem 5.

Corollary 5. Let G be a connected totally point determining graph and let H be any non-trivial connected graph. Then $C = \bigcup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a γ_{sL}-set of $G\,[H]$ if and only if $S = V(G)$ and T_x is an *sbln*-set of H for every $x \in V(G)$. In particular,

$$\gamma_{sL}(G[H]) = |V(G)|sbln(H).$$

Proof. By Theorem 5, $C = \bigcup_{x\in S}(\{x\} \times T_x)$, where is a γ_{sL}-set of $G\,[H]$ if and only if $S = V(G)$ and T_x is an *sbln*-set of H for every $x \in V(G)$. Now, let D be an *sbln*-set of H and let $T_x = D$ for each $x \in V(G)$. Then $C_0 = \bigcup_{x\in V(G)}(\{x\} \times T_x)$ is a γ_{sL}-set of $G\,[H]$. Therefore, $\gamma_{sL}(G[H]) = |C| = |V(G)|sbln(H)$. □

3. Conclusion

The locating dominating sets in the edge corona of graphs were characterized and bounds for its locating-domination number were obtained. The stable locating-dominating sets in the edge corona and lexicographic products of graphs were also characterized. Tight bounds for their stable locating-domination numbers were determined. A further study of stable locating-domination in other graphs is highly recommended. It is not yet known if the stable locating dominating set problem is NP-complete.

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