EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 5-17
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Geodetic Hop Dominating Sets in a Graph

Chrisley Jade C. Saromines [1,*], Sergio R. Canoy, Jr.[1]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Algebra and Analysis-PRISM, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let G be an undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A subset S of vertices of G is a geodetic hop dominating set if it is both a geodetic and a hop dominating set. The geodetic hop domination number of G, $\gamma_{hg}(G)$, is the minimum cardinality among all geodetic hop dominating sets in G. Geodetic hop dominating sets in a graph resulting from some binary operations have been characterized. These characterizations have been used to determine some tight bounds for the geodetic hop domination number of each of the graphs considered.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Geodetic domination, hop domination, corona, lexicographic

---

1. Introduction

Frank Harary et al. in [10] introduced a graph theoretical parameter called geodetic number of a graph. Geodetic sets and geodetic numbers are studied further in Chartrand [7]. In 2011, H. Escuadro et al. (see [8]) introduced the concept of geodetic domination in graphs. After their introduction, more studies have been done on the concepts. Some of the studies dealing with geodetic sets, geodetic number, and geodetic dominating sets can be found in [4], [5], [6], [7], [8], [9], [10], [14], and [24].

The concept of hop domination in graphs was introduced and initially investigated by Natarajan and S. K. Ayyaswamy [19]. The study was then followed by numerous studies on the topic. In particular, a lot of variations of the concept have been introduced and studied (see [2], [3], [11], [12], [13], [15], [16], [18], [20], [21], [17], [22], and [23]). Recently, Anusha and Robin [1] introduced the concept of geodetic hop domination and studied it for join and corona of graphs. In this present paper, we revisit the concept of geodetic hop domination and give further results of the new parameter.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4646

Email addresses: `chrisleyjade.saromines@g.msuiit.edu.ph` (C.J. Saromines), `sergio.canoy@g.msuiit.edu.ph` (S. Canoy, Jr.)

2. Terminology and Notation

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u,v)$ is called a *u-v geodesic*. The interval $I_G[u,v]$ consists u,v and all vertices lying on a u-v geodesic. The interval $I_G(u,v) = I_G[u,v] \setminus \{u,v\}$. The *open neighborhood* of a vertex u is the set $N_G(u)$ consisting of all vertices v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup_{v\in A} N_G(v)$ is called the *open neighborhood of A* and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of A*. The *open hop neighborhood* of a vertex u is the set $N_G^2(u) = \{v \in V(G) : d_G(v,u) = 2\}$. The *closed hop neighborhood* of u is $N_G^2[u] = N_G^2(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G^2(A) = \bigcup_{v\in A} N_G^2(v)$ is called the *open hop neighborhood of A* and $N_G^2[A] = N_G^2(A) \cup A$ is called the *closed hop neighborhood of A*.

A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. The smallest cardinality of a dominating set in G, denoted by $\gamma(G)$ is called the *domination number* of G. The *geodetic closure* of a set $S \subseteq V(G)$, denoted by $I_G[S]$, is the union of the intervals $I_G[u,v]$, where $u,v \in S$. Set S is *geodetic set* in G if $I_G[S] = V(G)$. The smallest cardinality among all geodetic sets in G, denoted by $g(G)$, is called the *geodetic number* of G. A geodetic set of cardinality $g(G)$ is called a g-set of G. A set $S \subseteq V(G)$ is a *geodetic dominating set* in G if it is both a dominating and a geodetic set.

A set $S \subseteq V(G)$ is a *hop dominating* set if $N_G^2[S] = V(G)$. The minimum cardinality of a hop dominating set of a graph G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. A subset S of $V(G)$ is a *total hop dominating* set of G if for every $v \in V(G)$, there exists $u \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of a total hop dominating set of G, denoted by $\gamma_{th}(G)$ is called the *total hop domination number* of G. Any total hop dominating set of G with cardinality $\gamma_{th}(G)$ is called a γ_{th}-set.

A subset S of vertices of G is a *geodetic hop dominating* set if it is both a geodetic and a hop dominating set. The *geodetic hop domination number* $\gamma_{hg}(G)$ of G is the minimum cardinality among all geodetic hop dominating sets in G. Any geodetic hop dominating set of G with cardinality $\gamma_{hg}(G)$ is called a γ_{hg}-set.

A set $S \subseteq V(G)$ of a graph G is called a 2-*path closure absorbing* if for each $x \in V(G) \setminus S$ there exist $u,v \in S$ such that $d_G(u,v) = 2$ and $x \in I_G(u,v)$. The minimum cardinality of a 2-path closure absorbing set in G is denoted by $\rho_2(G)$. Any 2-path closure absorbing set of G with cardinality $\rho_2(G)$ is called a ρ_2-set.

A set $D \subseteq V(G)$ is a *pointwise non-dominating* set of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a pointwise non-dominating set of G, denoted by $pnd(G)$, is called the *pointwise non-domination number* of G. A pointwise non-dominating set $S \subseteq V(G)$ of a graph G is called a 2-*path closure absorbing pointwise non-dominating set* if it is a 2-path closure absorbing set. The minimum cardinality of a 2-path closure absorbing pointwise non-dominating set in G is denoted by $\rho_{2pnd}(G)$. Any 2-path closure absorbing pointwise non-dominating set of G

with cardinality $\rho_{2pnd}(G)$ is called a ρ_{2pnd}-set.

Let K_n be the complete graph of order $n \geq 3$ and Ω a family of complete proper subgraphs of K_n. We say that Ω is an *independent set* if no two distinct subgraphs in Ω have common vertex. The graph G of order n obtained from K_n by deleting the edges in Ω is denoted $K_n \setminus E(\Omega)$. Hence, $xy \in E(G)$ if and only if xy is not an edge in any subgraph in Ω.

Let G and H be two graphs. The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in V(G)$ and write $v + H^v$ for $\langle v \rangle + H^v$. The *lexicographic product* $G[H]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. Note that any non-empty set $C \subseteq V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

3. Results

Remark 1. *Let G be a connected graph of order n. If S is a geodetic hop dominating set then S is a hop dominating set. In particular, $\gamma_h(G) \leq \gamma_{hg}(G)$.*

Theorem 1. *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{hg}(G) = n$ if and only if one of the following holds:*

(*i*) $G = K_n$

(*ii*) *$G \neq K_n$ and there exist dominating vertices $v_1, v_2, ..., v_k$ such that $H_1 = G \setminus v_1$, $H_2 = H_1 \setminus v_2$, . . . , $H_{k-1} = H_{k-2} \setminus v_{k-1}$ are connected graphs and $H_k = H_{k-1} \setminus v_k$ is the union of atleast 2 complete components.*

Proof. Suppose $\gamma_{hg}(G) = n$ and suppose that $G \neq K_n$. Then there exist $x, y \in V(G)$ such that $d_G(x,y) = 2$. Let $v_1 \in N_G(x) \cap N_G(y)$. Suppose there exists $z \in V(G) \setminus \{v_1\}$ such that $v_1 z \notin E(G)$. We may pick z so that $d_G(v_1, z) = 2$. Then $V(G) \setminus \{v_1\}$ is a geodetic hop dominating set of G, contrary to our assumption that $\gamma_{hg}(G) = n$. Thus, $N_G[v_1] = V(G)$.

Next, let $H_1 = G \setminus v_1$. Suppose that H_1 is disconnected and suppose that H_1 has a component $H_1^{'}$ that is not complete. Then there exists $s, t \in V(H_1^{'})$ such that $d_{H_1^{'}}(s,t) = d_{H_1}(s,t) = d_G(s,t) = 2$. Let $r \in N_G(s) \cap N_G(t)$. Then $V(G) \setminus \{r\}$ is a geodetic hop dominating set of G, a contradiction. Therefore, all components of H_1 are complete.

Suppose H_1 is connected. Suppose further that $d_{H_1}(x,y) \geq 3$, say $[x_1, x_2, ..., x_k]$, where $x_1 = x$ and $x_k = y$, be an x-y geodesic in H. Then $V(G) \setminus \{x_2\}$ is a geodetic hop dominating set, a contradiction. Thus, $d_H(x,y) = 2$. Let $v_2 \in N_{H_1}(x) \cap N_{H_2}(y)$. Suppose there exists $p \in V(H_1)$ such that $d_{H_1}(v_2, p) = 2$. Then $V(G) \setminus \{v_2\}$ is geodetic hop dominating set of G, a contradiction. Thus, $N_{H_1}(v_2) = V(H_1)$ and $N_G[v_2] = V(G)$.

Continuing in this manner, there exists a finite sequence of dominating vertices $v_1, v_2, ..., v_k$ such that $H_1 = G \setminus v_1$, $H_2 = H_1 \setminus v_2$, . . . , $H_{k-1} = H_{k-2} \setminus v_{k-1}$ are connected graphs and $H_k = H_{k-1} \setminus v_k$ is the union of at least 2 complete components.

For the converse, suppose that $G = K_n$. Then, clearly, $\gamma_{gh}(G) = n$. Let $v_1, v_2, ..., v_k$ be dominating vertices such that $H_1 = G \setminus v_1, ..., H_{k-1} = H_{k-2} \setminus v_{k-1}$ are connected and $H_k = H_{k-1} \setminus v_k$ is the union of at least two complete graphs. Let S be a γ_{hg}-set of G. Then $v_1, v_2, ..., v_k \in S$. Let $v \in V(G) \setminus \{v_1, v_2, ..., v_k\}$. Suppose $v \in S$. Since S is a hop dominating set, there exists $w \in S \cap N_G^2(v)$. Also, since S is a geodetic set, there exists $p, q \in S$ such that $[p, v, q]$ is a p-q geodesic. Since p, v and q are not dominating vertices, $p, v, q \in V(H_k \setminus v_k)$. It follows that the component of $H_k \setminus v_k$ containing p, v and q is not complete, contrary to our assumption. Therefore, $v \in S$. Accordingly, $S = V(G)$ and $\gamma_{hg}(G) = n$. □

Proposition 1. *Let n be a positive integer.*

(i) For a path P_n on n vertices , $\gamma_{hg}(P_n) = \begin{cases} n, & \text{if } n = 1, 2. \\ \frac{n+6}{3}, & \text{if } n \equiv 0(\text{mod}3), \\ \frac{n+2}{3}, & \text{if } n \equiv 1(\text{mod}3), \\ \frac{n+4}{3}, & \text{if } n \equiv 2(\text{mod}3), \end{cases}$

(ii) For a cycle C_n on n vertices, $\gamma_{hg}(C_n) = \begin{cases} 3, & \text{if } n = 3, 4, 5 \\ \frac{n}{3}, & \text{if } n \equiv 0(\text{mod}3), \\ \frac{n+2}{3}, & \text{if } n \equiv 1(\text{mod}3), \\ \frac{n+4}{3}, & \text{if } n \equiv 2(\text{mod}3), \end{cases}$

Proof.

(i) Let $P_n = [v_1, v_2, ..., v_n]$ and S be γ_{hg}-set of P_n. Since S is a geodetic set, $v_1, v_n \in S$. Consider the following cases:

Case 1. $n \equiv 0(\text{mod}3)$

Let $n = 3r$, for some positive integer r. Then $S_1 = \{v_1, v_4, ..., v_{3r-2}, v_{3r-1}, v_{3r}\}$ and $S_2 = \{v_{3r}, v_{3r-3}, ..., v_3, v_2, v_1\}$ are the only γ_{hg}-sets of P_n. Hence, $\gamma_{hg}(P_n) = |S_1| = \frac{n+6}{3}$.

Case 2. $n \equiv 1(\text{mod}3)$

Let $n = 3t + 1$, for some non-negative integer t. Then $S_3 = \{v_1, v_4, ..., v_{3t+1}\}$ is the unique γ_{hg}-set of P_n. Hence, $\gamma_{hg}(P_n) = |S_3| = \frac{n+2}{3}$.

Case 3. $n \equiv 2(\text{mod}3)$

Let $n = 3s + 2$, for some non-negative integer s. Then $S_4 = \{v_1, v_4, ..., v_{3s+1}, v_{3s+2}\}$ and $S_5 = \{v_{3s+2}, v_{3s-1}, ..., v_2, v_1\}$ are the only γ_{hg}-sets of P_n. Hence, $\gamma_{hg}(P_n) = |S_4| = \frac{n+4}{3}$.

(ii) Let $C_n = [v_1, v_2, ..., v_n, v_1]$ and let D be γ_{hg}-set of C_n. By circularity property of C_n, we may assume that $v_1 \in D$. Consider the following cases:

Case 1. $n \equiv 0(\text{mod}3)$

Let $n = 3r$ for some positive integer r. Then $D = \{v_1, v_4, ..., v_{3r-2}\}$. Hence, $\gamma_{hg}(P_n) = |D| = \frac{n}{3}$.

Case 2. $n \equiv 1(\text{mod}3)$

Let $n = 3t + 1$ for some non-negative integer t. Then $D = \{v_1, v_4, ..., v_{3t+1}\}$. Hence, $\gamma_{hg}(C_n) = |D| = \frac{n+2}{3}$.

Case 3. $n \equiv 2(\text{mod}3)$

Let $n = 3s + 1$ for some non-negative integer s. Then $D = \{v_1, v_4, ..., v_{3s+1}, v_{3s+2}\}$. Hence, $\gamma_{hg}(C_n) = |D| = \frac{n+4}{3}$.

Remark 2. *If $P_n = [v_1, v_2, ..., v_n]$ and S is a geodetic set of P_n, then $\{v_1, v_n\} \subseteq S$.*

Theorem 2. *Let K_n be the complete graph of order $n \geq 3$ and Ω an independent family of complete proper subgraph of K_n, each of order at least 2. If $G = K_n \setminus E(\Omega)$, then*

$$\gamma_{hg}(G) = \begin{cases} n, & \text{if } |\Omega| = 1 \\ |\Omega| - 2 + n - \sum_{K_q \in \Omega} q + min\{4, p+1\}, & \text{if } |\Omega| \geq 2 \end{cases}$$

where $p = \min\{q : K_q \in \Omega\}$.

Proof. Suppose that S is a γ_{hg}-set of G and let $\Omega = \{K_p\}$. If $K_p = K_n$, then we are done. If $K_p \neq K_n$, then there exist dominating vertices $v_1, v_2, ..., v_k$ such that $H_1 = G \setminus v_1$, $H_2 = H_1 \setminus v_2$, . . . , $H_{k-1} = H_{k-2} \setminus v_{k-1}$ are connected graphs and $H_k = H_{k-1} \setminus v_k$ is the union of at least 2 complete components. By Theorem 2, $\gamma_{hg}(G) = n$.

Suppose $|\Omega| \geq 2$. Let $D_1 = V(G) \setminus \bigcup_{K_q \in \Omega} V(K_q)$ and let D_2 be a smallest subset of S such that $V(G) \setminus S \subseteq I_G(D_2)$. Since G is non-complete, there exist $u, v \in S$ such that $d_G(u, v) = 2$. Since Ω is an independent set, $u, v \in V(K_q)$ for a unique $K_q \in \Omega$. We may assume that $u, v \in D_2$. Consider the following cases:

Case 1. $p < 4$.

Then $V(G) \setminus S \subseteq I_G(V(K_p))$. If there exists $w \in V(K_p) \setminus S$, then there exist $x, y \in S$ such that $[x, w, y]$ is an x-y geodesic. It follows that $x, y \in V(K_r)$ for some $K_r \in \Omega \setminus \{K_p\}$. Since $u, v, x, y \in D_2$, $|D_2| \geq 4 > p$, a contradiction. Thus, $V(K_p) \subseteq S$ and $D_2 = V(K_p)$.

Next, let $K_q \in \Omega \setminus \{K_p\}$. Since S is a hop dominating set of G, $S \cap V(K_q) \neq \varnothing$. Moreover, $|S \cap V(K_q)| = 1$ since S is a γ_{hg}-set of G. Let $S \cap V(K_q) = \{x_q\}$ for each $K_q \in \Omega \setminus \{K_p\}$. Note that if $D_1 \neq \varnothing$, then D_1 contains all the dominating vertices of G. Hence $D_1 \subseteq S$. Therefore,

$$S = D_1 \cup D_2 \cup \left(\bigcup_{q \neq p} \{x_q\} \right)$$

and

$$\begin{aligned}\gamma_{hg}(G) &= |S| = n - \sum_{K_q \in \Omega} q + p + |\Omega| - 1 \\ &= \left(n - \sum_{K_q \in \Omega} q\right) + p + 1 + |\Omega| - 2.\end{aligned}$$

Case 2. $p \geq 4$.

Suppose $D_2 = V(K_p)$. Then $|S| = n - \sum_{K_q \in \Omega} q + p + |\Omega| - 1$. Let $K_t \in \Omega \setminus \{K_p\}$. Pick $a, b \in V(K_t)$ with $a \neq b$. Let $D^{'} = \{u, v, a, b\}$. Then $V(G) \setminus S \subseteq I(D^{'})$. For each $K_q \in \Omega \setminus \{K_p, K_t\}$, pick $y_q \in V(K_q)$. Let $S^{'} = D_1 \cup D^{'} \cup \left(\bigcup_{q \neq p,t} \{y_q\}\right)$. Then $S^{'}$ is a geodetic hop dominating set of G and

$$\begin{aligned}\left|S^{'}\right| &= \left(n - \sum_{K_q \in \Omega} q\right) + 4 + |\Omega| - 2 \\ &< \left(n - \sum_{K_q \in \Omega} q\right) + p + |\Omega| - 1.\end{aligned}$$

This is a contradiction to the above assumption. Thus, $D \neq V(K_p)$.

Using the preceding arguments, we have

$$\gamma_{hg}(G) = |S| = \left(n - \sum_{K_q \in \Omega} q\right) + 4 + |\Omega| - 2.$$

This proves the assertion. □

Corollary 1. *Let K_n be the complete graph of order $n \geq 4$ and Ω an independent family of complete proper subgraphs of K_n, each of order at least 2. If $K_2 \in \Omega$ and $G = K_n \setminus E(\Omega)$, then*

$$\gamma_{hg}(G) = \begin{cases} n, & \text{if } |\Omega| = 1 \\ |\Omega| + 1 + n - \sum_{K_q \in \Omega}, & \text{if } |\Omega| \geq 2. \end{cases}$$

Corollary 2. *Let K_n be the complete graph of order $n \geq 4$. If G is a graph of order n obtained from K_n by deleting an edge, then $\gamma_{hg}(G) = n$*

The next result is a restatement of the one obtained in [15].

Theorem 3. *Let G and H be any two graphs. A set $C \subseteq V(G \circ H)$ is a hop dominating set of $G \circ H$ if and only if*

$$C = A \cup \left(\cup_{v \in V(G)} S_v\right),$$

where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:

(i) For each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $S_y \neq \varnothing$.

(ii) $S_v \subseteq V(H^v)$ is a pointwise non-dominating set of H^v for each $v \in V(G) \setminus N_G(A)$.

Theorem 4. *Let G and H be any two graphs. A set $C \subseteq V(G \circ H)$ is a geodetic hop dominating set of $G \circ H$ if and only if*

$$C = A \cup \left(\cup_{v \in V(G)} S_v\right),$$

where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:

(i) $S_v \subseteq V(H^v)$ is a pointwise non-dominating set of H^v for each $v \in V(G) \setminus N_G(A)$.

(ii) For each $w \in V(G) \setminus A$, one of the following condition holds:

(1) *$\exists\ a, b \in S_w$ with $d_{H_w}(a, b) \neq 1$.*

(2) *$\exists\ x, y \in V(G)$ with $w \in I_G(x, y)$.*

(3) *$\exists\ s \in S_w$ and $t \in A$.*

(iii) S_v is a 2-path closure absorbing set in H^v $\forall v \in V(G)$.

Proof. Suppose C is a geodetic hop dominating set of $G \circ H$. Let $A = C \cap V(G)$ and $S_v = C \cap V(H^v)$ for each $v \in V(G)$. Since C is a geodetic set, $S_v \neq \varnothing$ for each $v \in V(G)$. By Theorem 3, (i) holds. Let $w \in V(G) \setminus A$. Since C is a geodetic set of $G \circ H$, at least one of the three statements in (ii) holds. Let $v \in V(G)$. Let $p \in V(H^v) \setminus S_v$. Since C is a geodetic set, there exist $s, t \in C$ such that $p \in I_{G \circ H}(s, t)$. It follows that $s, t \in S_v$ and $d_{H^v}(s, t) = 2$ and $[s, p, t]$ is an s-t geodesic in H^v. Thus, S_v is a 2-path closure absorbing set in H^v, showing that (iii) holds.

Conversely, suppose C satisfies the given conditions. Let $v \in V(G) \setminus A$ and choose any $y \in N_G(v)$. By assumption, $S_y \neq \varnothing$. Hence, by Theorem 3, C is hop dominating set of $G \circ H$. Let $z \in V(G \circ H) \setminus C$ and let $w \in V(G)$ such that $z \in V(w + H^w)$. Consider the following cases:
Case 1. $z = w$.

Then $w \in V(G) \setminus A$. Suppose condition (1) of (ii) holds. Then $a, b \in C$ and $z \in I_{G \circ H}(a, b)$. Suppose (2) holds. Let $p \in S_x$ and $q \in S_y$. Then $p, q \in C$ and $z \in I_{G \circ H}(p, q)$. Next, suppose that (3) holds. Then $s, t \in C$ and $z \in I_{G \circ H}(s, t)$.
Case 2. $z \neq w$.

Then $z \in V(H^w) \setminus S_w$. By (iii), S_w is a 2-path closure absorbing set in H^w; hence, there exists $a, b \in S_w$ such that $[a, z, b]$ is an a-b geodesic in H^w. Therefore, $a, b \in C$ and $[a, z, b]$ is an a-b geodesic in $G \circ H$.

Accordingly, C is a geodetic hop dominating set of $G \circ H$. □

Corollary 3. *Let G be a connected non-trivial graph on n vertices and let H be any noncomplete graph. Then*

$$\gamma_{hg}(G \circ H) = \min \{n\rho_{2pnd}(H), \gamma(H) + n\rho_2(H)\}.$$

Proof. For each $v \in V(G)$, let S_v be a ρ_{2pnd}-set of H^v. Then $C = \cup_{v \in V(G)} S_v$ is a geodetic hop dominating set of $G \circ H$ by Theorem 4. Next, let A be a γ-set of G. For each $v \in V(G)$, let T_v be a ρ_2 set of H^v. By Theorem 4, $C' = A \cup \left(\cup_{v \in V(G)} T_v\right)$ is a geodetic hop dominating set of $G \circ H$. Thus,

$$\gamma_{hg}(G \circ H) \leq min \left\{|C|, |C'|\right\} = min \{n\rho_{2pnd}(H), \gamma(H) + n\rho_2(H)\}.$$

Let R_v be a ρ_{2pnd}-set of H^v. Let $A_1 = V(G) \setminus N_G(A_0)$ and $A_2 = N_G(A_0)$. Then $C_0 = A_0 \cup \left(\cup_{v \in V(G)} R_v\right)$ is a geodetic hop dominating set of $G \circ H$ by Theorem 4. Thus

$$|C_0| = |A_0| + \sum_{v \in A_1} |R_v| + \sum_{v \in A_2} |R_v|$$

$$\geq |A_0| + |A_1| \rho_{2pnd}(H) + |A_2| \rho_2(H).$$

Suppose $\gamma(G) + n\rho_2(H) \leq n\rho_{2pnd}(H)$. Then $\rho_2(H) < \rho_{2pnd}(H)$, that is, $\rho_2(H) + 1 \leq \rho_{2pnd}(H)$. It follows that

$$\begin{aligned} |C_0| &\geq |A_0| + |A_1| (\rho_2(H) + 1) + |A_2| \rho_2(H) \\ &= |A_0| + |A_1| + |A_1| + |A_2| \rho_2(H) \\ &= |A_0| + |A_1| + n\rho_2(H) \\ &\geq \gamma(G) + n\rho_2(H) \end{aligned}$$

since $A_0 \cup A_1$ is a dominating set of G and $|A_0 + A_1| \leq |A_0| + |A_1|$.

Suppose $n\rho_{2pnd}(H) < \gamma(G) + n\rho_2(H)$. Then $\rho_{2pnd}(H) = \rho_2(H)$. Thus,

$$\begin{aligned} |C_0| &\geq |A_0| + |A_1| \rho_{2pnd}(H) + |A_2| \rho_2(H) \\ &= |A_0| + (|A_1| + |A_2|) \rho_{2pnd}(H) \\ &= |A_0| + n\rho_{2pnd}(H) \\ &\geq n\rho_{2pnd}(H). \end{aligned}$$

Therefore,

$$\gamma_{hg}(G \circ H) = |C_0| \geq min \{n\rho_{2pnd}(H), \gamma(G) + n\rho_{2pnd}(H)\}$$

Accordingly,

$$\gamma_{hg}(G \circ H) = min\left\{n\rho_{2pnd}(H), \gamma(G) + n\rho_{2pnd}(H)\right\}.$$

□

Canoy et al. in [15] obtained the next result.

Theorem 5. *Let G and H be connected non-trivial graphs. A subset $C = \bigcup_{x\in S}[x \times T_x]$ of $V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a hop dominating set of $G[H]$ if and only if the following conditions hold:*

(i) S is a hop dominating set of G;

(ii) T_x is a pointwise non-dominating set of H for each $x \in S \setminus N_G^2(S)$.

Theorem 6. *Let G and H be connected non-trivial graphs. A subset $C = \bigcup_{x\in S}[x \times T_x]$ of $V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a geodetic hop dominating set of $G[H]$ if and only if the following conditions hold:*

(i) S is a geodetic hop dominating set of G,

(ii) T_x is a pointwise non-dominating set of H for each $x \in S \setminus N_G^2(S)$.

(iii) T_x is a 2-path closure absorbing set of H for each $x \in S \setminus I_G(S)$.

Proof. Suppose C is a geodetic hop dominating set of $G[H]$. By Theorem 5, S is a hop dominating set of G and (ii) holds. Suppose $v \in V(G) \setminus S$. Let $a \in V(H)$. Since C is a geodetic set, there exist $(x, p), (y, q) \in C$ such that $(v, a) \in I_{G[H]}((x, p), (y, q))$. Then $x, y \in S$ and $v \in I_G(x, y)$. This shows that S is a geodetic set of G, showing that (i) holds. Next, let $x \in S \setminus I_G(S)$. If $T_x = V(H)$, then it is a 2-path closure absorbing set of H. Suppose $T_x \neq V(H)$ and let $b \in V(H) \setminus T_x$. Since $(x, b) \notin C$ and C is a geodetic set, there exist $(u, k), (w, t) \in C$ such that $(x, b) \in I_{G[H]}((u, k), (w, t))$. Since $x \in S \setminus I_G(S)$, $u = w = x$ and $d_{G[H]}((u, k)(w, t)) = 2$. Because $(x, b) \in I_{G[H]}((u, k), (w, t))$, this would imply that $d_H(k, t) = 2$ and $x \in I_H(k, t)$. This shows that T_x is a 2-path closure absorbing set of H, that is, (iii) holds.

Conversely, suppose C satisfies (i), (ii) and (iii). By Theorem 5, S is a hop dominating set of G. Let $(v, p) \in V(G[H]) \setminus C$. Consider the following cases:

Case 1. $v \notin S$.

Since S is a geodetic set of G, there exist $u, w \in S$ such that $v \in I_G(u, w)$. Let $[v_1, v_2, ..., v_k]$, where $v_1 = u$ and $v_k = w$, a u-w geodesic in G. Let $v = v_j$ where $1 < j < k$. Let $s \in T_u$ and $t \in T_w$. Then $[(v_1, s), (v_2, p), ..., (v_j, p), ..., (v_{k-1}, p), (v_k, t)]$ is (u, s)-(w, t) geodesic in $G[H]$ containing (v, p).

Case 2. $v \in S$.

Then $p \notin T_v$. If $v \in I_G(S)$, then following the arguments of Case 1, there exist $(x, a), (y, b) \in C$ such that $(v, p) \in I_{G[H]}((x, a), (y, b))$. Suppose $v \notin I_G(S)$. By (iii),

T_v is a 2-path closure absorbing set of H. This implies that there exists $c, d \in T_v$ such that $d_H(c,d) = 2$ and $p \in I_H(c,d)$. Hence, $(v,c),(v,d) \in C$ and $[(v,c),(v,p),(v,d)]$ is a (v,c)-(v,d) geodesic in $G[H]$.

Therefore, C is a geodetic hop dominating set of $G[H]$. □

Corollary 4. *Let G and H be connected non-trivial graphs. Then*

$$\gamma_{hg}(G[H]) \leq \gamma_{hg}(G)\ \rho_{2pnd}(H)$$

Proof. Let S be a γ_{hg}-set of G and let D be a ρ_{2pnd}-set of H. For each $x \in S$, let $T_x = D$. Then

$$C = \bigcup_{x \in S} (\{x\} \times T_x) = S \times D$$

is a geodetic hop dominating set by Theorem 6. Therefore,

$$\gamma_{hg}(G\,[H]) \leq |C| = |S|\,|D| = \gamma_{hg}(G)\rho_{2pnd}(H).$$

□

Remark 3. *Strict inequality in Corollary 4 can be attained.*

Example 1. *Consider the graph $P_3\,[P_3]$. It can be verified that*

$$\gamma_{hg}(P_3\,[P_3]) = 7 < 9 = \gamma_{hg}(P_3)\rho_{2pnd}(P_3)$$

.

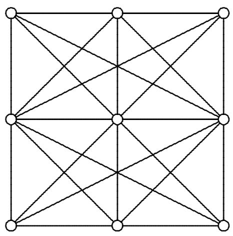

Figure 1: The lexicographic product $P_3[P_3]$

Corollary 5. *Let $n \geq 2$ be a positive integer and let H be any connected non-trivial graph. Then*

$$\gamma_{hg}\left(K_n\,[H]\right) = n\rho_{2pnd}(H)$$

.

Proof. Let $C = \bigcup_{x \in S} (\{x\} \times T_x)$ be a γ_{hg}-set of $K_n\,[H]$. Then $S = V(K_n)$ by Theorem 6(i). Also, by (ii) and (iii) of Theorem 6, T_x is a pointwise non-dominating and 2-path closure absorbing set of H for every $x \in S$. Thus,

$$\begin{aligned}\gamma_{hg}\left(K_n\left[H\right]\right) = |C| &= \sum_{x\in S} |T_x| \\ &\geq |S|\,\rho_{2pnd}(H) \\ &= n\rho_{2pnd}(H)\end{aligned}$$

By Corollary 4, $\gamma_{hg}\left(K_n\left[H\right]\right) = n\rho_{2pnd}(H)$. □

Example 2. *Consider the graph $K_3\left[P_4\right]$. It can be verified that*

$$\gamma_{hg}(K_3\left[P_4\right]) = 9 = \gamma_{hg}(K_3)\rho_{2pnd}(P_4)$$

.

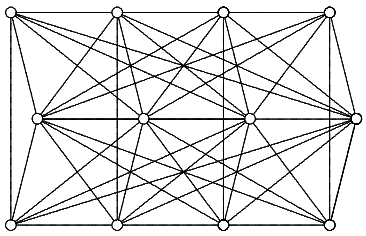

Figure 2: The lexicographic product $K_3[P_4]$

4. Conclusion

This paper investigated the concept of geodetic hop domination, a variant of hop domination, which was definedand studied previously by some authors. Some bounds of the parameter are determined and graphs attaining these bounds are also characterized. Characterizations of geodetic hop dominating sets in the corona and lexicographic product of two graphs are given. These characterizations were used to obtain exact or tight bounds for the geodetic hop domination number of the corresponding graphs. It is recommended that some other bounds for the geodetic hop domination be determined and that the parameter be studied for other interesting graphs.

Acknowledgements

The authors are very much grateful to the referees for the corrections and suggestions they made in the initial manuscript. The authors would like to thank the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP)-Philippines, and MSU-Iligan Institute of Technology for funding this research.

References

[1] D. Anusha and S. Joseph Robin. Geodetic hop domination in join and corona of graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 21(3):1117–1127, 2011.

[2] S. Arriola and S. Canoy Jr. $(1,2)^*$-domination in graphs. *Advances and Applications in Discrete Mathematics.*, 18(2):179–190, 2017.

[3] S. Ayyaswamy, B. Krishnakumari, B. Natarjan, and Y. Venkatakrishnan. Bounds on the hop domination number of a tree. *Proceedings-Mathematical Sciences*, 125(4):449–455, 2015.

[4] G. Cagaanan and S. Canoy Jr. On the geodesic and hull numbers of the sum of graphs. *Congresus Numerantium*, 161:97–104, 2003.

[5] G. Cagaanan and S. Canoy Jr. On the geodetic bases of the composition $g[k_m]$. *Ars Combinatoria*, 79:33–45, 2006.

[6] G. Cagaanan and S. Canoy Jr. Bounds for the geodetic number of the cartesian product of graphs. *Utilitas Matematica*, 79:91–98, 2009.

[7] G. Chartrand, F. Harary, and P. Zhang. The geodetic number of a graph. *Networks: An International Journal*, 39(1):1–6, 2002.

[8] H. Escuardo, R. Gera, A. Hansberg, N. Jafari Rad, and L. Volkmann. Geodetic domination in graphs. *J. Combin. Math. Combin. Comput.*, 77(1):89–101, 2022.

[9] A. Hansberg and L. Volkmann. On the geodetic and geodetic domination numbers of a graph,. *Discrete Math*, 310(1):2140–2146, 2010.

[10] F. Harary. The geodetic number of a graph. *Mathl. Comput. Modelling*, 17(11):89–95, 1993.

[11] J. Hassan and S. Canoy Jr. Grundy hop domination in graphs. *European Journal of Pure and Applied Mathematics*, 15(4):1623–1636, 2022.

[12] J. Hassan and S. Canoy Jr. Hop independent hop domination in graphs. *European Journal of Pure and Applied Mathematics*, 15(4):1783–1796, 2022.

[13] M. Henning and N. Rad. On 2-step and hop dominating sets in graphs. *Graphs and Combinatorics.*, 33(4):913–927, 2017.

[14] S. Canoy Jr., G. Cagaanan, and S. Gervacio. Convexity, geodetic and hull numbers of the join of graphs. *Utilitas Matematica*, 71:143–159, 2006.

[15] S. Canoy Jr., R. Mollejon, and J. G. Canoy. Hop dominating sets in graphs under binary operations. *European Journal of Pure and Applied Mathematics*, 12(4):1455–1463, 2019.

[16] S. Canoy Jr. and G. Salasalan. Global hop domination numbers of graphs. *European Journal of Pure and Applied Mathematics*, 14(1):112–125, 2021.

[17] S. Canoy Jr. and G. Salasalan. Locating-hop domination in graphs. *Kyungpook Mathematical Journal*, 62:193–204, 2022.

[18] S. Canoy Jr. and G. Salasalan. A variant of hop domination in a graph. *European Journal of Pure and Applied Mathematics*, 15(2):342–353, 2022.

[19] C. Natarajan and S. Ayyaswamy. Hop domination in graphs ii. *Versita*, 23(2):187–199, 2015.

[20] R. Rakim and H. Rara. Total perfect hop domination in graphs under some binary operations. *European Journal of Pure and Applied Mathematics*, 14(3):803–815, 2021.

[21] R. Rakim, H. Rara, and C.J. Saromines. Perfect hop domination in graphs. *Applied Mathematical Sciences*, 12(13):635–649, 2018.

[22] G. Salasalan and S. Canoy Jr. Revisiting domination, hop domination, and global hop domination in graphs. *European Journal of Pure and Applied Mathematics*, 14(4):1415–1428, 2021.

[23] C. J. Saromines and S. Canoy Jr. Outer-connected hop dominating sets in graphs. *European Journal of Pure and Applied Mathematics*, 15(4):1966–1981, 2022.

[24] T. Tacbobo, F. Jamil, and S. Canoy Jr. Monophonic and geodetic domination in the join, corona and composition of graphs. *Ars Combin*, 112(1):13–32, 2013.