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Mathematical Eigenfunctions analysis for 2nd Kind of Fredholm Integral equations

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Abstract. The current paper is a trial to investigate the Eigen functions appears through mathematical treatment of 2^{nd} kind of Fredholm integral equations by making use a new developed an Inverse Iterative Numerical Scheme (IINS). To test the applicability and accuracy of the proposed IINS, two numerical examples were solved and compared with previous results. The discussion of the results gave a good coincides up to some decimal places of results with the available ones.

2020 Mathematics Subject Classifications: 45Bxx, 45B05

Key Words and Phrases: Fredholm Integral equations, Eigen functions analysis, Iterative methods

1. Introduction

Integral equations are a very complicated and old branch of mathematics, and solutions for such equations need an appearing effort even after numerical methods and approaches became excellent tools for accurate results. Integrals equations spread among wide range of topics and applications exist in science, engineering and technology, e.g., continuum mechanics, potential theory, geo-physics, electricity, gas kinetic theory [1, 4, 5, 13, 16]. Also in biology, theory of renewal energy, quantum mechanics, theory of radiation, theory of optimization, economic mathematics, population, the theory queuing, some medical applications, transport phenomena, acoustic applications, phase change problems [2, 3, 9, 11, 14]. When the various sciences became complicated as a result of the interactions between them and developed greatly, scientists began to study natural phenomena, whether they were physical, and to explain these phenomena and find different solutions to them, whether analytical or numerical. Integrative equations have their own importance among the different types of mathematical sciences such as partial and differential equations, functional

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analysis, the theory of operators, transformers and special functions. Therefore, it can be said that there is no science from the different sciences except that the integrative equations play a prominent role in it. Therefore, we find that many researchers have been able to devise many different ways to solve the integrative equations, whether the nucleus of the integrative equation is connected or unconnected and other methods that are represented as ways Analytical or numerical. In one of the papers, a new analytical method was presented for solving systems of linear differential integrals, which is a new and powerful method through which effective recursive relationships were obtained to solve these systems, and it is an appropriate method to use it as an alternative to the mathematical methods used in such problems. Fredholm integral equation is one of the most important integral equations, and researchers look to the basis of integral equations as a transformation of some points over given vector space that have specific criteria by making use of certain specified operators to other points located in the same space [15]. There are wide ranges of numerical methods for implementing of Fredholm 2^{nd} type, some of these methods, the β-wavelet method [7], moments method based on β-wavelets [18], and variational iteration method [12]. Some of the numerical trials for solving Fredholm integral equation of 2^{nd} kind that proposed by Maleknejad et al., [6]. The Homotopy perturbation method is one of the iterative numerical methods that gave good approximate solutions for the problem underhand. In the present paper, certain type of integral equations is analyzed. The analysis is to approximate Eigenvalues of 2^{nd} type, Fredholm equation using a proposed inverse numerical iterative scheme. The current paper is a trial to investigate the Eigen functions appears through mathematical treatment of 2^{nd} kind of Fredholm integral equations by making use a new developed an (IINS). To test the applicability and accuracy of the proposed NIS, two numerical examples were solved and compared with previous results. The discussion of the results gave a good coincides up to some decimal places of results with the available ones.

2. Mathematical Formulation

Starting by the so called (CFI) defined as:

$$\int_{\Omega} \Im\left(\varepsilon_1, \varepsilon_2\right) \phi\left(\zeta_2\right) d\aleph_{\varpi_2} = \chi \phi\left(\varepsilon_1\right) \tag{1}$$

In which, $d\aleph_{x_2}$ is Stieltjes measure, defined as:

$$d\aleph_{\varepsilon_2} = d\varepsilon_2 + \sum_{i=1}^{i=k} m_k \delta\left(\upsilon_i\right) \tag{2}$$

$\Im\left(\varepsilon_1, \varepsilon_2\right)$: A positive symmetric kernel,

m_k: CPM,

$\upsilon_i, i = 1, 2, ...k$: Arbitrary points at which the massesm$_k$ are concentrated,

δ: DDF

The CFI has wide applications in the field of vibrations of a string [8, 10, 17, 19], the concepts of the so called INIS will be applied, and the general case of the string with different densities $\rho(\zeta_2)$ will be used herein in the current paper. The density will be charged by a finite number of cursors; therefore, its measurement using the following equation:

$$d\aleph_{\varepsilon_2} = \rho(\varepsilon_2)\, d\zeta_2 + \sum_{i=1}^{i=k} m_k \delta(\upsilon_i) \tag{3}$$

Therefore; $\ell^2_{d\aleph}$ will be (HS) equipped by the scalar product:

$$[\sigma_1, \sigma_2]_{dN} := (\sigma_1, \sigma_2)_{d\varepsilon 1_\rho} + (\sigma_1, \sigma_2)_{\Delta_r(\varepsilon_1)} \tag{4}$$

Where

$$(\sigma_1, \sigma_2)_{d\varpi 1_\rho} = \int_\Omega \sigma_1(\varepsilon_1)\, \sigma_2(\varepsilon_1)\, \rho(\varepsilon_1)\, d\varepsilon_1 \tag{5}$$

$$(\sigma_1, \sigma_2)_{\Delta_r(\varepsilon_1)} := \sum_{i=1}^{i=k} m_k \sigma_1(\upsilon_i)\, \sigma_2(\upsilon_i) \tag{6}$$

The $\ell^2_{d\aleph}$, has no singularity at the points υ_i and now considered the point is located inside the space, therefore; Eigenvalue problem appears as:

$$\tau\phi = \chi\phi \tag{7}$$

The term $\tau\zeta$ is called (CF), defined as:

$$\tau\phi = \int_\Omega \Im(., \varepsilon_2)\, \phi(\varepsilon_2)\, d\aleph_{\varepsilon_2} \tag{8}$$

By making use of equation (8) and equation (3), leads to:

$$(\tau\phi)(\varepsilon_2) = \int_\Omega \Im(\varepsilon_1, \varepsilon_2)\, \phi(\varepsilon_2)\, \rho(\varepsilon_2)\, d\varepsilon_2 + \sum_{i=1}^{i=k} n_k \Im(\varepsilon_1, \upsilon_i)\, \phi(\upsilon_i) \tag{9}$$

$$(\tau\phi)(\varepsilon_2) = (\Im(\varepsilon_1, \varepsilon_2), \phi(\varepsilon_2))_{dx_{\varpi\rho}} + (\Im(\varepsilon_1, \varepsilon_2), \phi(\varepsilon_2))_{\Delta_r(\varepsilon_2)} \tag{10}$$

And so;

$$[\tau\sigma_1, \sigma_2]\, d\aleph = ((\tau\sigma_1)(\varepsilon_1), \sigma_2(\varepsilon_1))\, d\varepsilon 1_\rho + ((\tau\sigma_1)(\varepsilon_1), \sigma_2(\varepsilon_1))_{\Delta_r(\varepsilon_1)} \tag{11}$$

$$[\tau\sigma_1, \sigma_2]\, d\aleph = \Big((\Im(\varepsilon_1, \varepsilon_2)\, \sigma_1(\varepsilon_2))\, d\varepsilon 2_\rho + \Big((\Im(\varepsilon_1, \varepsilon_2)\, \sigma_1(\varepsilon_2))_{\Delta_r(x_2)}, \sigma_2(\varepsilon_1)\Big)\Big)_{d\varepsilon 1_\rho}$$

$$+\left(\left(\Im\left(\varepsilon_1,\varepsilon_2\right)\sigma_1\left(\varepsilon_2\right)\right)d\varepsilon_{2\rho}+\left(\left(\Im\left(\varepsilon_1,\varepsilon_2\right)\sigma_1\left(\varepsilon_2\right)\right)_{\Delta_r(x_2)},\sigma_2\left(\varepsilon_1\right)\right)\right)_{\Delta_r(\varpi_1)} \tag{12}$$

By making use of the kernel symmetry, one can get:

$$\left(\left(\Im\left(\varepsilon_1,\varepsilon_2\right)\sigma_1\left(\varepsilon_2\right)\right)_{\Delta_r(x_2)},\sigma_2\left(\varepsilon_1\right)\right)_{\Delta_r(x_1)}=\sum_{i=1}^{i=k}\sum_{l=1}^{l=k}n_i n_l \Im\left(m_i,m_l\right)\sigma_1\left(m_i\right)\sigma_2\left(m_l\right) \tag{13}$$

3. Inverse Iterative Numerical Scheme (IINS)

The kernel $\Im\left(\varepsilon_1,\varepsilon_2\right)$ is assumed to be continuous function, over a square of unit length in the positive and first quadrant, real, symmetric, and regular Additional assumptions to Eigenvalues, they are real and positive.

Assuming Eigenvalues defined as:

$$0<.....\leq\gamma_3\leq\gamma_2\leq\gamma_1$$

Now let us consider some points inside the square in the 1st and +ve quadrant as follows:

$$0=:\upsilon_1<\upsilon_2<\upsilon_3<.....\upsilon_k<\upsilon_{k+1}:=1$$

Assume that in each interval $[\upsilon_i,\upsilon_{i+1}]\ni n-\ nodesof$ (MGLQ), as following:

$$\varepsilon_{11}^{(i)},\varepsilon_{12}^{(i)},\varepsilon_{13}^{(i)},.....,\varepsilon_{1n}^{(i)} \tag{14}$$

Make use of equation (14) over $[\upsilon_k,\upsilon_{k+1}]$, leads to:

$$\upsilon_k:=\varepsilon_{10}^{(i)}<\varepsilon_{11}^{(i)}<\varepsilon_{12}^{(i)}<.....<\varepsilon_{1n}^{(i)}<\varepsilon_{1n+1}^{(i)}:=\upsilon_{i+1},0\leq i\leq k \tag{15}$$

There exists corresponding points over vertical axis, defined as:

$$\upsilon_l:=y_{10}^{(l)}<y_{11}^{(l)}<y_{12}^{(l)}<.....<y_{1n}^{(l)}<y_{1n+1}^{(l)}:=\upsilon_{l+1},0\leq l\leq k \tag{16}$$

Assume subinterval $\mathfrak{R}_{i.j}^{(k,l)}$ as:

$$\mathfrak{R}_{i.j}^{(k,l)}:=\left\{\left(\varepsilon_1,\varepsilon_2\right)\middle|\varepsilon_{1_i}^{(k)}<\varepsilon_1<\varepsilon_{1_{i+1}}^{(k)};\middle|y_{1_j}^{(l)}<y_1<y_{1_{j+1}}^{(l)}\right\}i,j=1,2,...\&k,l=0,1,2,...., \tag{17}$$

Defining $\hat{\mathfrak{R}}_{i.j}^{(k,l)}$ within $\mathfrak{R}_{i.j}^{(k,l)}$ and $\mathfrak{R}_{i.j}^{-(k,l)}$ containing singular point for $\Im\left(\zeta_1,\zeta_2\right)$, therefore;

$$\overline{\Im}\left(\varepsilon_1,\varepsilon_2\right)=\begin{cases}\Im\left(\varepsilon_1,\varepsilon_{2j}\right)\ \text{when}\ \left(\varepsilon_1,\varepsilon_2\right)\in\mathfrak{R}_{i.j}^{(k,l)}\\ \aleph_{i,j}\ \text{when}\ \left(\varepsilon_1,\varepsilon_2\right)\in\hat{R}_{i.j}^{(k,l)}\end{cases} \tag{18}$$

The $\aleph_{i,j}$, are constants satisfying the next condition:

So as the norm defined by equation (19), to < epsilon, the next inequality should be verified:

$$\left\|\mathfrak{T}\left(\varepsilon_1, \varepsilon_{2j}\right) - \overline{\mathfrak{J}}\left(\varepsilon_1, \varepsilon_2\right)\right\|_{\ell^2_{d\aleph}} \geq \left|\lambda_\omega - \bar{\lambda}_\omega\right| \tag{19}$$

λ_ω Represents the exact Eigenvalues, while $\overline{\lambda}_\omega$ represents the approximate values.

4. Results & Discussion

Considering the following kernel:

$$\mathfrak{J}\left(\varepsilon_{1_i}, \varepsilon_{2_j}\right) = \begin{cases} \varepsilon_1\left(u - \varepsilon_2\right) 0 \leq \varepsilon_1 \leq 1 \\ 0 \leq \varepsilon_2 \leq 1 \\ \varepsilon_1 \leq \varepsilon_2 \\ \varepsilon_2\left(u - \varepsilon_1\right) 0 \leq \varepsilon_1 \leq 1 \\ 0 \leq \varepsilon_2 \leq 1 \\ \varepsilon_2 \leq \varepsilon_1 \end{cases}$$

And

$$u = u$$

or

$$u \neq u$$

Assuming that:

$$d\aleph_{\varepsilon_2} = \rho\left(\varepsilon_2\right) d\varepsilon_2$$

4.1. Problem-1

$$\begin{aligned} &\rho\left(\varpi_2\right) = u + \varpi_2 \\ &\& \\ &u = u = 1 \end{aligned}$$

By following up the procedure (IINS) described in the above section, and comparing the results with (RRM), the results are shown in table (1).

Table 1: Comparison between approximate Eigenvalues

Eigenvalue number	Eigenvalues		Absolute Error
	IINS	RRM	
1	0.15271	0.15265	0.00006
2	0.03779	0.03777	0.00002
3	0.01676	0.01672	0.00004
4	0.00942	0.00940	0.00002
5	0.00603	0.00601	0.00002
6	0.00418	0.00417	0.00001

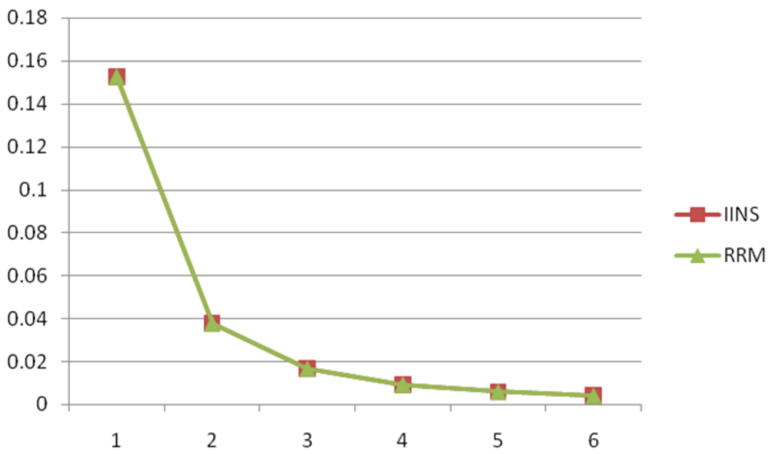

Figure 1: Comparison between IINS and RRM verses Eigenvalues for problem-1

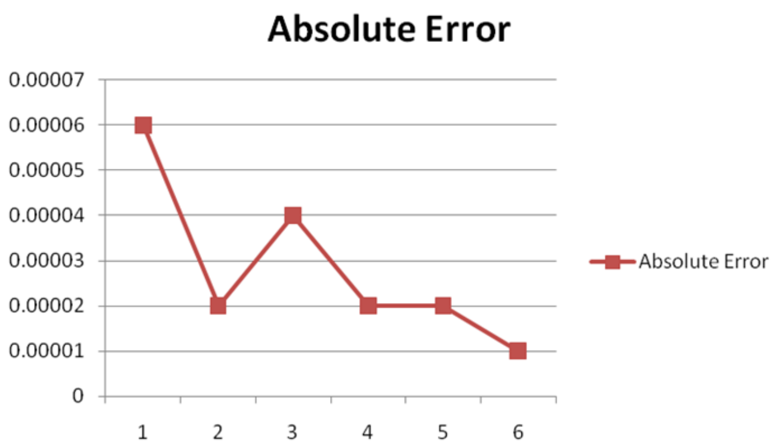

Figure 2: Absolute error between IINS and RRM for Problem-1

From table (1) that there is a very small errors and can be neglected.

4.2. Problem-2

$$d\aleph_{\varepsilon_2} = \rho(\varepsilon_2)\, d\varepsilon_2 + \sum_{i=1}^{4} n_i \delta(v_i)$$

$$\rho(\varepsilon_2) = \begin{cases} 16\varepsilon_2^2 - 4\varepsilon_2 + 0.25, 0.00 \leq \varepsilon_2 \leq 0.25 \\ 16\varepsilon_2^2 - 12\varepsilon_2 + 02.25, 0.25 \leq \varepsilon_2 \leq 0.50 \\ 16\varepsilon_2^2 - 20\varepsilon_2 + 06.25, 0.50 \leq \varepsilon_2 \leq 0.75 \\ 16\varepsilon_2^2 - 28\varepsilon_2 + 12.25, 0.75 \leq \varepsilon_2 \leq 1.00 \end{cases}$$

$$v_i = 0.25i - 0.125, i = 1, 2, 3, 4$$

&

$$m_1 = 0.25, m_2 = 0.5, m_3 = 0.5, m_4 = 0.25$$

Table 2: Comparison between approximate Eigenvalues

Eigenvalue number	Eigenvalues		Absolute Error
	IINS	RRM	
1	0.20683	0.20677	0.00006
2	0.05254	0.05252	0.00002
3	0.02004	0.01999	0.00005
4	0.01936	0.01931	0.00005
5	0.00086	0.00084	0.00002
6	0.00085	0.00084	0.00001

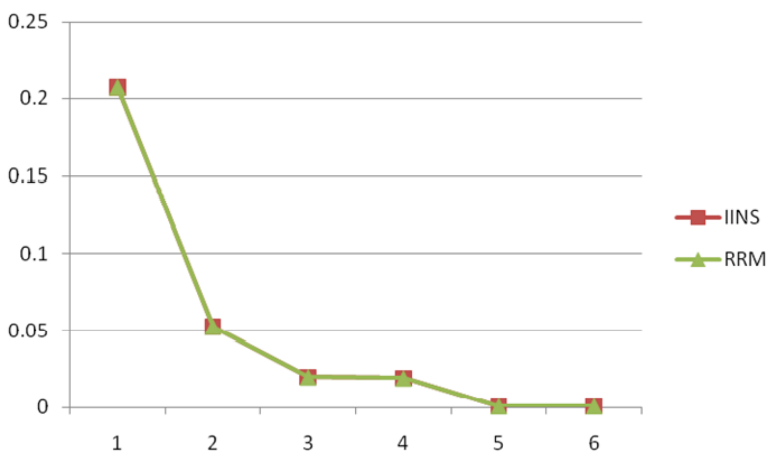

Figure 3: Comparison between IINS and RRM verses Eigenvalues for Problem-2

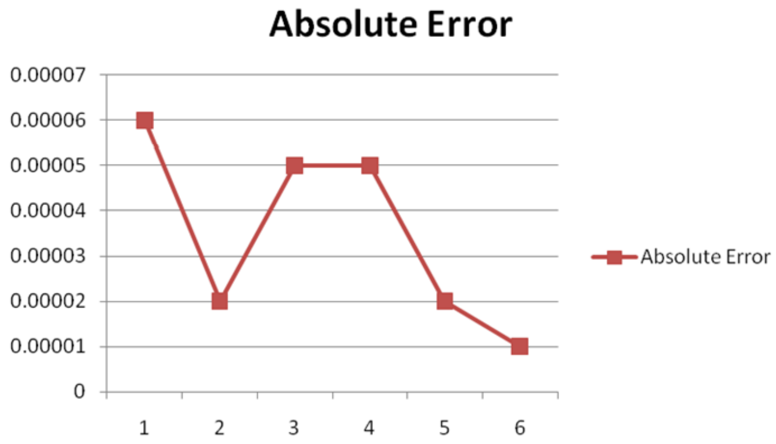

Figure 4: Absolute error between IINS and RRM for problem-2

5. Conclusion

The topics of the integral equations are so wide of their applications in science; engineering and technology, therefore the attention from researchers do not stop. The present paper, is just a step of these trials, in which, we tried to propose a technique based iterative numerical procedure. As seen from the computation, that the error is very small and can neglected.

Abbreviation

IINS	Inverse Iterative Numerical Scheme
HS	Hilbert space
CF	Compact Factor
DDF	Dirac-Delta Function
CPM	Combination of Positive Masses
MGLQ	Modified Gauss-Legendre Quadrature Rule
RRM	Rayleigh-Ritz Method

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References

[1] SG Ahmed. Boundary integral formulation and mushy zone model for phase change problem. *International Journal of Numerical Methods for Heat & Fluid Flow*, 2001. Publisher: MCB UP Ltd.

[2] SG Ahmed. An approximate method for evaporation problem with mushy zone. *Engineering analysis with boundary elements*, 29(2):161–165, 2005. Publisher: Elsevier.

[3] SG Ahmed. A numerical method for oxygen diffusion and absorption in a sike cell. *Applied mathematics and computation*, 173(1):668–682, 2006. Publisher: Elsevier.

[4] SG Ahmed, M Saif AlDien, and MY Youssef. A Numerical Method Based Boundary Integral Equation to Solve Multiphase Moving Boundary Problems. *British Journal of Mathematics & Computer Science*, 15(1):1, 2016. Publisher: SCIENCEDOMAIN International.

[5] Gemma Aiello. *Geophysics: Principles, applications and emerging technologies*. Nova Science Publishers, 2016.

[6] Aytac Arikoglu and Ibrahim Ozkol. Solutions of integral and integro-differential equation systems by using differential transform method. *Computers & Mathematics with Applications*, 56(9):2411–2417, 2008. Publisher: Elsevier.

[7] Hermann Brunner. *Collocation methods for Volterra integral and related functional differential equations*, volume 15. Cambridge university press, 2004.

[8] Ahmed A Mohammed Fawze, Borhan F Juma'a, and Waleed Al-Hayani. Modified of homotopy perturbation technique and its application to system of nonlinear fredholm integral equations of 2nd kind. In *Journal of Physics: Conference Series*, volume 1818, page 012057. IOP Publishing, 2021.

[9] G Fichera. Abstract and numerical aspects of eigenvalue theory. *Lecture Notes, The University of Alberta, Dept. of Math., Edmonton*, 1973.

[10] Khosrow Maleknejad, M Shahrezaee, and H Khatami. Numerical solution of integral equations system of the second kind by block–pulse functions. *Applied Mathematics and Computation*, 166(1):15–24, 2005. Publisher: Elsevier.

[11] Solomon Grigor'evich Mikhlin. *Integral equations and their applications*. Oxford, 2nd edition, 1964.

[12] Jafar Pour-Mahmoud, Mohammad Y Rahimi-Ardabili, and S21708451082 Shahmorad. Numerical solution of the system of Fredholm integro-differential equations by the Tau method. *Applied Mathematics and Computation*, 168(1):465–478, 2005. Publisher: Elsevier.

[13] K Prandecki. Theoretical Aspects of Sustainable Energy. energy and environmental engineering, vol. 2. 2014.

[14] Sean Stein Smith. How Cryptocurrencies Are Changing What CPAs Need to Know about Fraud Prevention. *Theoretical Economics Letters*, 8(14):3252–3266, 2018. Publisher: Scientific Research Publishing.

[15] Hari M Srivastava and Robert G Buschman. *Theory and applications of convolution integral equations*, volume 79. Springer Science & Business Media, 2013.

[16] P Tamilalagan and P Balasubramaniam. The solvability and optimal controls for fractional stochastic differential equations driven by Poisson jumps via resolvent operators. *Applied Mathematics & Optimization*, 77(3):443–462, 2018. Publisher: Springer.

[17] PJ Van der Houwen and BP Sommeijer. Euler-Chebyshev methods for integro-differential equations. *Applied Numerical Mathematics*, 24(2-3):203–218, 1997. Publisher: Elsevier.

[18] Elçin Yusufoğlu. Numerical solving initial value problem for Fredholm type linear integro-differential equation system. *Journal of the Franklin Institute*, 346(6):636–649, 2009. Publisher: Elsevier.

[19] Şuayip Yüzbaşı, Niyazi Şahin, and Mehmet Sezer. Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel polynomial bases. *Computers & Mathematics with Applications*, 61(10):3079–3096, 2011. Publisher: Elsevier.