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Cliques and Supercliques in a Graph

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Abstract. A set $S \subseteq V(G)$ of an undirected graph G is a clique if every two distinct vertices in S are adjacent. A clique is a superclique if for every pair of distinct vertices $v, w \in S$, there exists $u \in V(G) \setminus S$ such that $u \in N_G(v) \setminus N_G(w)$ or $u \in N_G(w) \setminus N_G(v)$. The maximum cardinality of a clique (resp. superclique) in G is called the clique (resp. superclique) number of G. In this paper, we determine the clique and superclique numbers of some graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Clique, clique number, superclique, superclique number

1. Introduction

Recently, Dela Cerna and Canoy (see [3]) initiated the study of the concept of superclique in a graph. It is known that the superclique number of a graph is at most equal to the clique number of the graph. Moreover, it was shown that any two positive integers a and b with $2 \leq a \leq b$ are, respectively, realizable as the superclique number and clique number of a connected graph. This result also implies that the difference of the clique number and the superclique number can be made arbitrarily large. As pointed out in an earlier study, superclique and superclique number were introduced and first used in the study of Acal, Monsanto, Sumaoy, and Rara in [1], [14], and [21] when they investigated some variations of resolving domination for graphs under some binary operations. Their study was motivated by the concepts of strong resolving set, strong metric dimension, and resolving domination which were introduced and studied in [2], [16], and [19]. These latter studies, in turn, came after Slater in [20] introduced the concepts of resolving set and metric dimension. The same concepts were also independently investigated by Harary and Melter in [9]. Chartrand et al. (see [4]) also studied resolving set and metric dimension of a graph.

Domination and some variations of domination are found in [10]. Other studies on domination are in [11], [12], [17], and [22]. Some studies involving cliques can be found in [5], [6], [7], [8], [13], [15], [18], and [23].

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2. Terminologies and Notations

Let $G = (V(G), E(G))$ be a simple undirected graph. The distance between two vertices u and v of G, denoted by $d_G(u, v)$, is equal to the length of a shortest path connecting u and v. Any path connecting u and v of length $d_G(u, v)$ is called a *u-v geodesic*. The *open neighbourhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighbourhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighbourhood* of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v\in S} N_G(v)$ and its *closed neighbourhood* is the set $N_G[S] = N_G(S) \cup S$. The *degree* of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The smallest cardinality of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set of G with with cardinality $\gamma(G)$ is called a *γ-set* of G.

A set $S \subseteq V(G)$ is a *clique* in a graph G if the graph $G[S] = \langle S\rangle$ induced by S is a complete subgraph of G. A clique C in G is called a *superclique* if for every pair of distinct vertices $u, v \in C$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. The *clique number* (resp. *superclique number*) of G, denoted by $\omega(G)$ (resp. $\omega_s(G)$), is the largest cardinality of a clique (resp. superclique) in G. Any clique (resp. superclique) in G with cardinality $\omega(G)$ (resp. $\omega_s(G)$) is called a *maximum clique* or ω-set (resp. *maximum superclique* or ω_s-set).

Let G and H be graphs. The *edge corona* $G\diamond H$ of graphs G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of every edge uv in G to every vertex of the copy H^{uv} of H. (that is forming the join $\langle\{u, v\}\rangle + H^{uv}$ for each $uv \in E(G)$). The *Tensor product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and (u, v) is adjacent with (u', v') whenever $uu' \in E(G)$ and $uv' \in E(H)$. The *strong product* $G \otimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u, v) is adjacent with (u', v') whenever $[uu' \in E(G)$ and $v = v']$ or $[vv' \in E(H)$ and $u = u']$ or $[uu' \in E(G)$ and $vv' \in E(H)]$. We note that every non-empty subset C of $V(G) \times V(H)$ can be expressed as $C = \cup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x = \{a \in V(H) : (x, a) \in C\}$ for each $x \in S$.

3. Results

The first result is found in [3]. Recall that two adjacent vertices v and w of a graph G are *true twins* if $N_G[v] = N_G[w]$.

Theorem 1. *Let G be any graph. Then each of the following statements holds:*

(*i*) *G admits a superclique and $1 \leq \omega_s(G) \leq \omega(G)$.*

(*ii*) *$\omega_s(G) = 1$ if and only if every component of G is complete.*

(*iii*) *$\omega_s(G) = \omega(G)$ if and only if G has a maximum clique containing no true twin vertices.*

Theorem 2. *Let G be a nontrivial connected graph and H be any graph. Then S is clique in $G \diamond H$ if and only if one of the following holds:*

(i) S is a clique in G.

(ii) S is clique in H^{uv} for some $uv \in E(G)$.

(iii) $S = S_{uv} \cup D$, where S_{uv} is a clique in H^{uv} and $\varnothing \neq D \subseteq \{u, v\}$ for some $uv \in E(G)$.

Proof. Suppose S is a clique in $G \diamond H$. If $S \subseteq V(G)$, then S is a clique in G and (i) holds. Suppose that $S \subseteq V(H^{uv})$ for some $uv \in E(G)$. Since S is a clique in $G \diamond H$, S_{uv} is a clique in H^{uv}. Hence, (ii) holds. Suppose now that $D = S \cap \{u, v\} \neq \varnothing$ and $S_{uv} = S \cap V(H^{uv}) \neq \varnothing$ for some $uv \in E(G)$. Then clearly, S_{uv} is a clique in H^{uv} and $S = S_{uv} \cup D$. Thus, (iii) holds.

The converse is clear.

The next result is immediate from Theorem 2.

Corollary 1. *Let G be a nontrivial connected graph and let H be any graph. Then*

$$\omega(G \diamond H) = max\{\omega(G),\ \omega(H) + 2\}.$$

Theorem 3. *Let G be a nontrivial connected graph such that $G \neq K_2$ and let H be any graph. Then S is superclique in $G \diamond H$ if and only if one of the following holds:*

(i) S is a clique in G.

(ii) S is superclique in H^{uv} for some $uv \in E(G)$.

(iii) $S = S_{uv} \cup D$ for some $uv \in E(G)$, where S_{uv} is a superclique in H^{uv} and D is a nonempty subset of $\{u, v\}$ such that $D = \{u\}$ if $deg_G(v) = 1$ and $D = \{v\}$ if $deg_G(u) = 1$.

Proof. Suppose S is a superclique in $G \diamond H$. Then S is a clique in $G \diamond H$. If $S \subseteq V(G)$ or $S \subseteq V(H^{uv})$ for some $uv \in E(G)$, then S is a clique in G or H^{uv}, respectively, by (i) and (ii) of Theorem 2. Suppose $S = S_{uv}$ for some $uv \in E(G)$ and let $a, b \in S_{uv}$. Since S is a superclique in $G \diamond H$, there exists $c \in V(G \diamond H) \setminus S$ such that $c \in N_{G\diamond H}(a) \setminus N_{G\diamond H}(b)$ or $c \in N_{G\diamond H}(b) \setminus N_{G\diamond H}(a)$. This implies that $c \in V(H^{uv}) \setminus S_{uv}$ and $c \in N_{H^{uv}}(a) \setminus N_{H^{uv}}(b)$ or $c \in N_{H^{uv}}(b) \setminus N_{H^{uv}}(a)$. Hence, S_{uv} is a superclique in H^{uv}, showing that (i) or (ii) holds.

Next, suppose that $S \cap \{u, v\} \neq \varnothing$ and $S \cap V(H^{uv}) \neq \varnothing$ for some $uv \in V(G)$. Then $S = S_{uv} \cup D$, where S_{uv} is a clique in H^{uv} and $\varnothing \neq D \subseteq \{u, v\}$ for some $uv \in E(G)$, by Theorem 2(iii). Again, since S is a superclique in $G \diamond H$, S_{uv} is a superclique in H^{uv}. Suppose now that $deg_G(u) = 1$ or $deg_G(v) = 1$, say $deg_G(v) = 1$. Pick any $x \in S_{uv}$. Then $N_{G\diamond H}(x) \cap [V(G \diamond H) \setminus S] = N_{G\diamond H}(u) \cap [V(G \diamond H) \setminus S]$. Thus, $u \notin D$. Therefore, $|D| = 1$. In particular, $D = \{u\}$ showing that (iii) holds.

For the converse, suppose first that (i) holds. Let $u, v \in S$ with $u \neq v$. Since G is connected and $G \neq K_2$, $deg_G(u) \geq 2$ or $deg_G(v) \geq 2$. Assume that $deg_G(u) \geq 2$. Let $w \in N_G(u) \setminus \{v\}$ and pick any $q \in V(H^{uw})$. Then $q \in V(G \diamond H) \setminus S$ and $q \in N_{G\diamond H}(u) \setminus N_{G\diamond H}(v)$. Therefore, S is a superclique in $G \diamond H$. Next, suppose that (ii) holds.

Since S is a superclique in H^{uv}, it is a superclique in $G \diamond H$. Finally, suppose that (iii) holds. By Theorem 2(iii), S is a clique in $G \diamond H$. Let $x, y \in S$ with $x \neq y$. If $x, y \in S_{uv}$, then there exists $z \in V(H^{uv} \setminus S_{uv}) \subseteq V(G \diamond H) \setminus S$ such that $z \in N_{H^{uv}}(x) \setminus N_{H^{uv}}(y)$ or $z \in N_{H^{uv}}(y) \setminus N_{H^{uv}}(x)$ because S_{uv} is a superclique in H^{uv}. Therefore, $z \in V(G \diamond H) \setminus S$ and $z \in N_{G \diamond H}(x) \setminus N_{G \diamond H}(y)$ or $z \in N_{G \diamond H}(y) \setminus N_{G \diamond H}(x)$. Suppose $x \in D$ and $y \in S_{uv}$. Assume, without lost of generality, that $x = u$. Then $deg_G(u) \geq 2$. Let $w \in N_G(u) \setminus \{v\}$. Then $w \in V(G \diamond H) \setminus S$ and $w \in N_{G \diamond H}(x) \setminus N_{G \diamond H}(y)$. Lastly, suppose that $x, y \in D$. In particular, let $x = u$ and $y = v$. Then, by assumption, $deg_G(u) \geq 2$ and $deg_G(v) \geq 2$. Let $z \in N_G(u) \setminus \{v\}$ and choose any $p \in V(H^{uz})$. Then $p \in V(G \diamond H) \setminus S$ and $p \in N_{G \diamond H}(x) \setminus N_{G \diamond H}(y)$. Therefore, in either case, S is a superclique in $G \diamond H$. □

Corollary 2. *Let G be a nontrivial connected graph such that G is not a star and let H be any graph. Then*

$$\omega_s(G \diamond H) = \max\{\omega(G),\ \omega_s(H) + 2\}.$$

Proof. Let S be a maximum clique in G and let $u, v \in V(G)$ with $deg_G(u) \geq 2$ and $deg_G(v) \geq 2$ (these vertices exist because G is not a star). Let S_{uv} be a maximum superclique in H^{uv}. Then S and $S^* = S_{uv} \cup \{u, v\}$ are supercliques in $G \diamond H$ by Theorem 3. This implies that

$$\omega_s(G \diamond H) \geq \max\{|S|,\ |S^*|\} = \max\{\omega(G),\ \omega_s(H) + 2\}.$$

On the other hand, if S_0 is a maximum superclique in $G \diamond H$, then S_0 is a clique in G or $S_0 = S_{uv} \cup D$ for some $uv \in E(G)$ satisfying the conditions in Theorem 3(iii). Hence,

$$\omega_s(G \diamond H) = |S_0| \leq \max\{\omega(G),\ \omega_s(H) + 2\},$$

establishing the desired equality. □

Theorem 4. *Let $G = K_{1,m} = \langle v_0 \rangle + \overline{K}_m$, where $m \geq 2$, and let H be any graph. Then S is a superclique in $G \diamond H$ if and only if one of the following holds:*

(i) *S is a clique in G.*

(ii) *S is a superclique in H^{uv_0} for some $u \in V(G) \setminus \{v_0\}$.*

(iii) *$S = S_{uv_0} \cup \{v_0\}$ for some $u \in V(G) \setminus \{v_0\}$, where S_{uv_0} is a superclique in H^{uv_0}.*

Proof. Suppose S is a superclique in $G \diamond H$. Then (i), (ii), or (iii) holds by Theorem 3.

The converse also follows from Theorem 3. □

Corollary 3. *Let $G = K_{1,m}$, where $m \geq 2$, and let H be any graph. Then*

$$\omega_s(G \diamond H) = \omega_s(H) + 1.$$

Proof. Clearly, $\omega(G) = 2$. Since $\omega_s(H) \geq 1$, it follows that $\omega(G) \leq \omega_s(H) + 1$. The desired equality now follows from Theorem 4. □

Theorem 5. *Let G and H be nontrivial connected graphs. Then $C = \cup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a clique in $G \boxtimes H$ if and only if the following statements hold:*

(i) S is a clique in G.

(ii) $|T_x| = 1$ for each $x \in S$, $T_x \neq T_y$ if $x \neq y$, and $\cup_{x\in S}T_x$ is a clique in H.

Proof. Suppose C is a clique in $G\boxtimes H$. Let $x, y \in S$ such that $x \neq y$ and let $a \in T_x$ and $b \in T_y$. Then $(x, a)(y, b) \in C$ and $(x, a) \neq (y, b)$. It follows that $(x, a)(y, b) \in E(G \boxtimes H)$. Hence, $xy \in E(G)$, showing that S is a clique in G. Thus, (i) holds.

Next, let $x \in S$ and suppose that $|T_x| \geq 2$. Let $p, q \in T_x$ such that $p \neq q$. Then $(x, p)(x, q) \in C$ and $(x, p) \neq (x, q)$. Since C is a clique in $G \boxtimes H$, $(x, p)(x, q) \in E(G \boxtimes H)$ which is not possible. It follows that $|T_x| = 1$ for each $x \in S$. Now, let $s, t \in \cup_{x\in S}T_x$ such that $s \neq t$. Then $s \in T_v$ and $t \in T_w$ for some $v, w \in S$ with $v \neq w$. Since $(v, s), (w, t) \in C$ and C is a clique in $G \boxtimes H$, $(v, s)(w, t) \in E(G \boxtimes H)$. This implies that $st \in E(H)$. Therefore, $\cup_{x\in S}T_x$ is a clique in H, showing that (ii) holds.

Conversely, suppose that C satisfies (i) and (ii). Let $(v, p), (w, q) \in C$ such that $(v, p) \neq (w, q)$. If $v = w$, then $p \neq q$ and $p, q \in T_v$ contrary to the assumption that $|T_x| = 1$ for all $x \in S$. Thus, $v \neq w$ and $p \neq q$. Since S and $\cup_{x\in S}T_x$ are cliques in G and H, respectively, $vw \in E(G)$ and $pq \in E(H)$. Thus, $(v, p), (w, q) \in E(G \boxtimes H)$. Therefore, C is a clique in $(G \boxtimes H)$. □

Corollary 4. *Let G and H be nontrivial connected graphs. Then*

$$\omega(G \boxtimes H) = min\{\omega(G), \omega(H)\}.$$

Proof. Let S and D be maximum cliques in G and H. Suppose first that $\omega(G) = |S| \leq |D| = \omega(H)$. Let $D' \subseteq D$ such that $|S| = |D'|$. Let $S = \{v_1, v_2, \ldots, v_k\}$ and $D' = \{a_1, a_2, \ldots, a_k\}$. Set $T_{v_j} = \{a_j\}$ for each $j \in [k]$. Then $C = \cup_{j=1}^{k}(\{v_j\} \times T_{v_j})$ is a clique in $G \boxtimes H$ by Theorem 5. Consequently, $\omega(G \boxtimes H) \geq |C| = |S| = \omega(G)$. A similar argument can be used to show that $\omega(G \boxtimes H) \geq |D| = \omega(G)$ if $|D| \leq |S|$.

Suppose now that $C_0 = \cup_{x\in S_0}(\{x\} \times R_x)$ is a maximum clique in $G \boxtimes H$. Then S is a clique in G, $|R_x| = 1$ for each $x \in S$, $R_x \neq R_y$ if $x \neq y$, and $\cup_{x\in S}R_x$ is a clique in H by Theorem 5. It follows that

$$\omega(G \boxtimes H) = |C_0| = \sum_{x\in S} |R_x| = |S| = | \cup_{x\in S} R_x| \leq min\{\omega(G), \omega(H)\}.$$

This establishes the desired equality. □

Theorem 6. *Let G and H be nontrivial connected graphs such that $G \neq K_2$ or $H \neq K_2$. Then $C = \cup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a superclique in $G \boxtimes H$ if and only if it is a clique.*

Proof. Since every superclique is a clique, it remains to show that the converse is true. To this end, suppose that C is a clique in $G \boxtimes H$. Then C satisfies (i) and (ii) of Theorem 5. Let $(v, p), (w, q) \in C$ such that $(v, p) \neq (w, q)$. Then $(v, p), (w, q) \in C$. Hence, $vw \in E(G)$ and $pq \in E(H)$. Suppose $G \neq K_2$. Since G is connected, $deg_G(v) \geq 2$ or $deg_G(w) \geq 2$. Assume that $deg_G(v) \geq 2$ and let $z \in N_G(v) \setminus \{w\}$. Since C is a clique and $(z, q)(w, q) \notin E(G \boxtimes H)$, it follows that $(z, q) \notin C$. Hence, $(z, q) \in N_{G\boxtimes H}((v, p)) \setminus N_{G\boxtimes H}((w, q))$. Next, suppose that $H \neq K_2$. Then $deg_H(p) \geq 2$ or $deg_H(q) \geq 2$. Assume that $deg_H(p) \geq 2$ and let $t \in N_H(p) \setminus \{q\}$. Since C is a clique and $(w, t)(w, q) \notin E(G \boxtimes H)$, it follows that $(w, t) \notin C$. Hence, $(w, t) \in N_{G\boxtimes H}((v, p)) \setminus N_{G\boxtimes H}((w, q))$. In either case, C is a superclique in $G \boxtimes H$. $\square$

It is clear that $\omega_s(K_2 \boxtimes K_2) = \omega_s(K_2 \cup K_2) = 1$.

Corollary 5. *Let G and H be nontrivial connected graphs such that $G \neq K_2$ or $H \neq K_2$. Then*

$$\omega_s(G \boxtimes H) = \omega(G \boxtimes H) = \min\{\omega(G), \omega(H)\}.$$

Theorem 7. *Let G and H be nontrivial connected graphs. Then $C = \cup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a clique in $G \otimes H$ if and only if the following statements hold:*

(i) *S is a clique in G.*

(ii) *T_x is a clique in H for each $x \in S$.*

(iii) *$\cup_{x\in S} T_x$ is a clique in H.*

Proof. Suppose C is a clique in $G \otimes H$. Let $x, y \in S$ such that $x \neq y$ and let $a \in T_x$ and $b \in T_y$. Then $(x, a)(y, b) \in C$ and $(x, a) \neq (y, b)$. By assumption, $(x, a)(y, b) \in E(G \otimes H)$. This implies that $xy \in E(G)$, showing that (i) is true.

Let $x \in S$ and let $p, q \in T_x$ such that $p \neq q$. Then $(x, p)(x, q) \in C$ and $(x, p) \neq (x, q)$. Since C is a clique in $G \otimes H$, $(x, p)(x, q) \in E(G \otimes H)$. Adjacency in $G \otimes H$ would imply that $pq \in E(H)$. This shows that T_x is a clique in H as asserted in (ii). Next, let $s, t \in \cup_{x\in S} T_x$ such that $s \neq t$. If $s, t \in T_x$ for $x \in S$, then $st \in E(H)$ because T_x is a clique. Suppose $s \in T_v$ and $t \in T_w$ for some $v, w \in S$ with $v \neq w$. Since S is a clique in G, $vw \in E(G)$. Also, since C is a clique in $G \otimes H$, $(v, s)(w, t) \in E(G \otimes H)$. Consequently, $st \in E(H)$ by the definition of the adjacency in $G \otimes H$. This proves (iii).

For the converse, suppose that C satisfies (i), (ii), and (iii). Let $(x, p), (y, q) \in C$ such that $(x, p) \neq (y, q)$. If $x = y$, then $p \neq q$ and $p, q \in T_x$. It follows from (ii) that $pq \in E(H)$. Hence, $(x, p)(y, q) \in E(G \otimes H)$. Suppose now that $x \neq y$. Then $xy \in E(G)$ by (i). If $p = q$, then $(x, p)(y, q) \in E(G \otimes H)$ by the definition of $G \otimes H$. Suppose $p \neq q$. The assumption that (iii) holds would imply that $T_x \cup T_y$ is a clique in H. Thus, $pq \in E(G)$ and $(x, p)(y, q) \in E(G \otimes H)$. This proves that C is a clique in $G \otimes H$. $\square$

Corollary 6. *Let G and H be nontrivial connected graphs. Then*

$$\omega(G \otimes H) = \omega(G)\omega(H).$$

Proof. Let S and D be maximum cliques in G and H, respectively. Set $T_x = D$ for each $x \in S$. Then $C = \cup_{x\in S}(\{x\} \times T_x) = S \times D$ is a clique in $G \otimes H$ by Theorem 7. Therefore, $\omega(G \otimes H) \geq |C| = |S||D| = \omega(G)\omega(H)$.

Next, suppose that $C_0 = \cup_{x\in S_0}(\{x\} \times R_x)$ is a maximum clique in $G \otimes H$. Then C_0 satisfies properties (i), (ii), and (iii) of Theorem 7. It follows that

$$\omega(G \otimes H) = |C_0| = \sum_{x\in S_0} |R_x| \leq |S_0|\omega(H) \leq \omega(G)\omega(H).$$

This establishes the desired equality. □

Theorem 8. *Let G and H be non-trivial connected graphs. Then $C = \cup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a superclique in $G \otimes H$ if and only if it satisfies the following conditions:*

(i) *S is a clique in G.*

(ii) *T_x is a superclique in H for each $x \in S$.*

(iii) *$\cup_{x\in S}T_x$ is a clique in H.*

(iv) *For each distinct pairs of vertices $v, w \in S$ such that $T_w \cap T_v \neq \varnothing$, there exists $u \in V(G) \setminus S$ such that $u \in N_G(v) \setminus N_G(w)$ or $u \in N_G(w) \setminus N_G(v)$.*

(v) *For each pair of distinct vertices $v, w \in S$ such that $N_G[v] = N_G[w]$, and for each distinct vertices a and b such that $a \in T_v$ and $b \in T_w$, there exists $c \in V(H)$ such that $[c \notin T_v$ and $c \in N_H(a) \setminus N_H(b)]$ or $[c \notin T_w$ and $c \in N_H(b) \setminus N_H(a)]$.*

Proof. Suppose C is superclique in $G \otimes H$. Since C is a clique, each T_x is clique in H, and (i) and (iii) hold by Theorem 7. Let $x \in S$ and $p, q \in T_x$ such that $p \neq q$. Since T_x is a clique, $pq \in E(H)$. Also, since C is a superclique in $\otimes H$, we may assume that there exists $(u, t) \in V(G \otimes H) \setminus C$ such that $(u, t) \in N_{G\otimes H}((x, p)) \setminus N_{G\otimes H}((x, q))$. Suppose first that $u \neq x$. Then $ux \in E(G)$. If $p = t$, then $(u, t)(x, q) \in E(G \otimes H)$, a contradiction. Thus, $pt \in E(H)$ because $(u, t) \in N_{G\otimes H}((x, p))$. Since $(u, t) \notin N_{G\otimes H}((x, q))$, $t \neq q$ and $tq \notin E(H)$. This implies that $t \in V(H) \setminus T_x$ and $t \in N_H(p) \setminus N_H(q)$. Next, suppose that $u = x$. Then $pt \in E(H)$ and $qt \notin E(H)$. It follows that $t \in V(H) \setminus T_x$ and $t \in N_H(p) \setminus N_H(q)$. In either case, T_x is a superclique in H, showing that (ii) holds. Next, suppose that $v, w \in S$ such that $v \neq w$ and $T_v \cap T_w \neq \varnothing$, say $p \in T_v \cap T_w$. Again, since C is a superclique and $(v, p), (w, p) \in C$, we may assume that there exists $(z, s) \in V(G\otimes H) \setminus C$ such that $(z, s) \in N_{G\otimes H}((v, p)) \setminus N_{G\otimes H}((w, p))$. Suppose $z = v$. Then $ps \in E(H)$ because $(z, s) \in N_{G\otimes H}((v, p))$. Further, since $vw \in E(G)$, $(z, s) \in N_{G\otimes H}((w, p))$, a contradiction. Thus, $z \neq v$. This implies that $zv \in E(G)$. Suppose $zw \in E(G)$. If $p = s$, then

$(z,s) \in N_{G\otimes H}((w,p))$. If $p \neq s$, then $ps \in E(H)$ since $(z,s) \in N_{G\otimes H}((v,p))$. Hence, $(z,s) \in N_{G\otimes H}((w,p))$. In either case, we get a contradiction. Therefore, $zw \notin E(G)$, that is, $z \in V(G) \setminus S$ and $z \in N_G(v) \setminus N_G(w)$, showing that (iv) holds.

For the converse, suppose that C satisfies (i), (ii), (iii), and (iv). Then C is a clique by Theorem 7. Let $(x,a),(z,b) \in C$ such that $(x,a) \neq (z,b)$. Consider the following cases:

Case 1. $x = z$.
Then $a, b \in T_x$ and $a \neq b$ Since T_x is a clique, $ab \in E(H)$. Moreover, since T_x is a superclique, we may assume that there exists $p \in V(H) \setminus T_x$ such that $p \in N_H(a) \setminus N_H(b)$. Hence, $(x,p) \in V(G \otimes H) \setminus C$ and $(x,p) \in N_{G\otimes H}((x,a)) \setminus N_{G\otimes H}((x,b))$.

Case 2. $x \neq z$.
Then $xz \in E(G)$ because S is a clique in G. Suppose $a = b$. Then $a \in T_x \cap T_z$. By (iv), we may assume that there exists $y \in V(G) \setminus S$ such that $y \in N_G(x) \setminus N_G(z)$. It follows that $(y,a) \in V(G \otimes H) \setminus C$ and $(y,a) \in N_{G\otimes H}(x,a) \setminus N_{G\otimes H}(z,b)$. Finally, suppose that $a \neq b$. Then $ab \in E(G)$ by (iii). Suppose $N_G[x] \neq N_G[z]$. We may assume that there exists $v \in N_G(x) \setminus N_G(z)$. Clearly, $v \in V(G) \setminus S$. Hence, $(v,a) \in V(G \otimes H) \setminus C$ and $(v,a) \in N_{G\otimes H}(x,a) \setminus N_{G\otimes H}(z,b)$. If $N_G[x] = N_G[z]$, then there exists $c \in V(H)$ such that $[c \in V(H) \setminus T_x$ and $c \in N_H(a) \setminus N_H(b)]$ or $[c \in V(H) \setminus T_z$ and $c \in N_H(b) \setminus N_H(a)]$ by property (v). Assume that $c \in V(H) \setminus T_x$ and $c \in N_H(a) \setminus N_H(b)$. Then $(x,c) \in V(G \otimes H) \setminus C$ and $(x,c) \in N_{G\otimes H}(x,a) \setminus N_{G\otimes H}(z,b)$.

Accordingly, C is a superclique in $G \otimes H$. □

Corollary 7. *Let G and H be non-trivial connected graphs. Then*

$$\omega_s(G)\omega_s(H) \leq \omega_s(G \otimes H) \leq \omega(G)\omega_s(H).$$

Moreover, if $\omega_s(G) = \omega(G)$, then $\omega_s(G \otimes H) = \omega_s(G)\omega_s(H)$.

Proof. Let S and D be ω_s-sets in G and H, repectively. Set $T_x = D$ for each $x \in S$. Then $C = \cup_{x\in S}(\{x\} \times T_x) = S \times D$ is a superclique in $G \otimes H$ by Theorem 8. Therefore, $\omega_s(G \otimes H) \geq |C| = |S||D| = \omega_s(G)\omega_s(H)$.

On the other hand, suppose that $C' = \cup_{x\in S'}(\{x\} \times D_x)$ is an ω_s-set in $G \otimes H$. Then S' is a clique in G and each D_x is a superclique in H by Theorem 8. It follows that $\omega_s(G \otimes H) = |C'| \leq \omega(G)\omega_s(H)$. This proves the assertion. □

Example 1. *For any two positive integers $m \geq 3$ and $n \geq 3$,*

$$\omega_s(P_m \otimes P_n) = 4 = \omega_s(P_m)\omega_s(P_n) = \omega(P_m)\omega_s(P_n).$$

Note that

$$\omega_s(P_2 \otimes P_3) = \omega_s(P_2)\omega_s(P_3) = 2 \neq 4 = \omega(P_2)\omega_s(P_3).$$

4. Conclusion

Cliques and supercliques in the edge corona, Tensor product, and strong product of two graphs have been characterized and the corresponding clique and supercliques numbers have been described. These concepts can be studied further for other graphs. Moreover, it may be interesting to investigate the complexity of the superclique problem.

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