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Strong Resolving Domination in the Lexicographic Product of Graphs

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Abstract. Let G be a connected graph. A subset $S \subseteq V(G)$ is a *strong resolving dominating set* of G if S is a dominating set and for every pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that $u \in I_G[v, w]$ or $I_G[u, w]$. The smallest cardinality of a strong resolving dominating set of G is called the *strong resolving domination number* of G. In this paper, we characterize the strong resolving dominating sets in the lexicographic product of graphs and determine the corresponding strong resolving domination number.

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Key Words and Phrases: Strong resolving dominating set, strong resolving domination number, lexicographic product

1. Introduction

All graphs considered in this study are finite, simple, and undirected connected graphs, that is, without loops and multiple edges. For some basic concepts in Graph Theory, we refer readers to [7].

Let $G = \big(V(G), E(G)\big)$ be a connected graph. The *open neighborhood* of $v \in V(G)$ is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any element u of $N_G(v)$ is called a *neighbor* of v. The *closed neighborhood* of $v \in V(G)$ is $N_G[v] = N_G(v) \cup \{v\}$. Thus, the degree of $v \in V(G)$ is given by $deg_G(v) = |N_G(v)|$. Customary, for $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$.

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A nonempty set $S \subseteq V(G)$ is a *dominating set* in graph G if $N_G[S] = V(G)$. Otherwise, we say S is a *non-dominating set* of G. The *domination number* of a graph G, denoted by $\gamma(G)$, is given by $\gamma(G) = \min|S| : S \textit{ is a dominating set of } G$. If S is a dominating set of G and if $|S| = \gamma(G)$, then S is called a *minimum dominating set* or a γ-set of G.

A vertex $w \in S$ *strongly resolves* two different vertices $u, v \in V(G)$ if $v \in I_G[u, w]$ or $u \in I_G[v, w]$. A set W of vertices in G is a *strong resolving set* of G if every two vertices of G are strongly resolved by some vertices of W. The smallest cardinality of a strong resolving set of G is called the *strong metric dimension* of G and is denoted by $sdim(G)$.

A subset $S \subseteq V(G)$ is a *strong resolving dominating set* of G if it is both strong resolving and dominating. The smallest cardinality of a strong resolving dominating set of G is called the *strong resolving domination number* of G and is denoted by $\gamma_{sR}(G)$. A strong resolving dominating set of cardinality $\gamma_{sR}(G)$ is called a γ_{sR}-set of G.

A *clique* in a graph G is a complete induced subgraph of G. A clique C in G is called a *superclique* if for every pair of distinct vertices $u, v \in C$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. A superclique C in G is called a *dominated superclique* if for every $u \in C$, there exists $v \in V(G) \setminus C$ such that $uv \in E(G)$.[5] A superclique (resp. dominated superclique) C is *maximum* in G if $|C| \geq |C^*|$ for all supercliques (resp. dominated supercliques) C^* in G. The *superclique (resp. dominated superclique) number*, $\omega_S(G)$ (resp. ω_{DS}) of G is the cardinality of a maximum superclique (resp. maximum dominated superclique) in G.

A vertex u of G is *maximally distant* from vertex v of G, $u \neq v$, if for every vertex $w \in N_G(u), d_G(v, w) \leq d_G(u, v)$. If u is maximally distant from v and v is maximally distant from u, then we say that u and v are *mutually maximally distant*, denoted by uMMDv.

In recent years, the concept of domination in graphs has been studied extensively and several research papers have been published on this topic. The said concept was not formally defined mathematically until the publications of the books by Claude Berge [3] in 1958 and Oystein Ore in 1962. In 1977, a survey paper by Cockayne and Hedetniemi [4] began to study the concept of domination.

On the other hand, the problem of uniquely recognizing the possible position of an intruder such as fault in a computer network and spoiled device was the principal motivation in introducing the concept of metric dimension in graphs.

Slater [10] brought in the notion of locating sets and its minimum cardinality as locating number. The same concept was also introduced by Harary and Melter [7] but using the terms resolving sets and metric dimension to refer to locating sets and locating number, respectively. However, in recent studies, locating sets and resolving sets are defined differently. In 2013, Canoy and Omega [8], defined a locating set as a subset S of $V(G)$ in a connected graph G satisfying that $N_G(u) \cap S \neq N_G(v) \cap S$ for all $u, v \in V(G) \setminus S$. Meanwhile, Bailey et al. [2] defined a resolving set as a set of vertices S in a graph G such that for any two vertices u, v, there exists $x \in S$ such that the distances $d(u, x) \neq d(v, x)$.

In 2007, Oellerman and Peter-Fransen [9] introduced the strong resolving graph G_{SR} of a connected graph G as a tool to study the strong metric dimension of G.

This study aims to define and characterize the strong resolving dominating sets in the lexicographic product of graphs and determine their corresponding strong resolving domination number.

2. Preliminary Results

Remark 1. [1] *Every strong resolving dominating set of a connected graph* G *is a dominating set of* G*. Hence,* $\gamma(G) \leq \gamma_{sR}(G)$*.*

Remark 2. [1] *Every strong resolving dominating set of a connected graph* G *is a strong resolving set of* G*. Hence,* $sdim(G) \leq \gamma_{sR}(G)$*.*

Remark 3. *For any connected graph* G *of order* $n \geq 2$*,*

$$1 \leq \gamma_{sR}(G) \leq n - 1.$$

Example 1. $\gamma_{sR}(P_2) = 1$ *and* $\gamma_{sR}(K_n) = n - 1$ *for* $n \geq 2$*.*

Remark 4. [1] *Any superset of a strong resolving dominating set is a strong resolving dominating set.*

Proposition 1. *Every strong resolving set of a connected graph* G *is a resolving set.*

Proof: Let $S \subseteq V(G)$ is a strong resolving set of G. Then for any pair of distinct vertices $u, v \in V(G)$, there exists $w \in S$ such that $u \in I_G[v, w]$ or $v \in I_G[u, w]$. If $u \in I_G[v, w]$, then $d_G(u, w) < d_G(v, w)$. Thus, $r_G(u/S) \neq r_G(v/S)$, showing that w resolves u and v. Hence, S is a resolving set of G. □

Proposition 2. *Every strong resolving dominating set of a connected graph* G *is a resolving dominating set. Hence,* $\gamma_R(G) \leq \gamma_{sR}(G)$*.*

Proof: Follows from Proposition 1. □

Remark 5. The converse of Proposition 2 is not true. To see this, consider the graph in Figure 1, the set $W = \{v_2, v_5\}$ is a resolving set since the representation of each vertex in G, with respect to W is unique. These representations are as follows: $r_G(v_1/W) = (1, 2), r_G(v_2/W) = (0, 1), r_G(v_3/W), r_G(v_4/W) = (2, 1)$ and $r_G(v_5/W) = (1, 0)$. However, none among the vertices in W strongly resolves the vertices v_1 and v_3. Thus, W is a resolving set but it is not a strong resolving set of G.

In the same graph, it is easy to verify that the set $\{v_1, v_3\}$ is a strong resolving set of G, hence a resolving set as well.

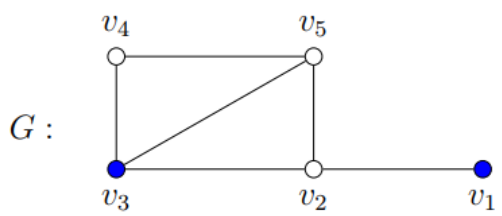

Figure 1. A strong resolving set $\{v_1, v_3\}$ *of* G

Proposition 3. [1] *Let G be a connected graph of order $n \geq 2$. Then,*

(i) $\gamma_{sR}(P_n) = \lceil \frac{n+1}{3} \rceil$

(ii) $\gamma_{sR}(K_n) = n - 1$

(iii) $\gamma_{sR}(C_n) = \begin{cases} 2 & , \text{ if } n = 3 \\ n-2 & , \text{ if } n > 3 \text{ and } n \text{ is odd} \\ \frac{n}{2} & , \text{ if } n > 3 \text{ and } n \text{ is even.} \end{cases}$

Remark 6. *Let G be a connected graph*

(i) *A set $\{u\} \subset V(G)$ induces a dominated superclique of G.*

(ii) *A clique $\langle C \rangle$ of G is a dominated superclique of G if $N_G[u] \neq N_G[v]$ for every pair of distinct vertices $u, v \in C$ and $V(G) \setminus C$ is a dominating set of G.*

(iii) *Every dominated superclique of G is a superclique of G.*

Example 2. Let n be a positive integer.

(i) $\omega_{DS}(K_n) = \omega_S(K_n) = 1$.

(ii) If $P_n = [v_1, v_2, \ldots, v_n]$ for $n \geq 4$, then the dominated supercliques of P_n are induced from the singletons $\{v_j\}$ for all $j = 1, 2, \ldots, n$ and $\{v_i, v_{i+1}\}$ for $i = 2, 3, \ldots, n-2$.

(iii) The dominated supercliques of a cycle C_n for $n \geq 4$, are $\langle \{u_j\} \rangle$ and induced from $\{u_i, u_j\} \subseteq V(C_n)$ where $u_i u_j \in E(C_n)$.

(iv) The dominated supercliques of a complete bipartite graph $K_{m,n}$ are the singleton sets $\{v\} \subset V(K_{m,n})$ and $n \neq 1,\ m \neq 1$.

(v) The maximum dominated supercliques of a complete bipartite graph $K_{m,n}$ are induced from the sets $\{x_i, x_j\} \subseteq V(K_{m,n})$ where $x_i x_j \in E(K_{m,n})$ and $n \neq 1$, $m \neq 1$.

Theorem 1. *Let G be a connected graph of order n. Then $\omega_{DS}(G) = 1$ if and only if $\gamma(G) = K_n$ or $G = K_{1,n-1}$.*

Proof: Suppose $\omega_{DS}(G) = 1$. If $n = 1$ or $n = 2$, then $G = K_n$. If $n = 3$, then $G = K_3$ or $G = K_{1,2}$. Suppose $n \geq 4$ and $G \neq K_n$. Then there exist distinct vertices a and b of G such that $d_G(a, b) = 2$. Let $v \in N_G(a) \cap N_G(b)$. Since $\{a, v\}$ is a superclique and $\omega_{DS}(G) = 1$, $|N_G(a)| = 1$. Similarly, $|N_G(b)| = 1$. Suppose there exists $y \in V(G) \setminus N_G(v)$. We may assume that $d_G(y, v) = 2$. Let $z \in N_G(y) \cap N_G(v)$. Then $\{z, v\}$ is a dominated superclique of G, contrary to the assumption that $\omega_{DS}(G) = 1$. Hence, $x \in N_G(v)$ for all $x \in V(G) \setminus \{v\}$. Also, for any distinct vertices $x, y \in V(G) \setminus \{v\}$, $xy \notin E(G)$. Therefore, $G = \langle v \rangle + \bigcup_{x \in V(G) \setminus \{v\}} \langle x \rangle = K_{1,n-1}$.

For the converse, suppose $G = K_n$ or $G = K_{1,n-1}$. Then, clearly, $\omega_{DS}(G) = 1$. □

3. Strong Resolving Domination in the Lexicographic Product of Graphs

Lemma 1. *Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph with $\gamma(H) \neq 1$. Then $A \times C \subseteq V(G[H])$ is a superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a superclique in H.*

Proof: Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph. Suppose $A \times C \subseteq V(G[H])$ is a superclique in $G[H]$. Then $A \subseteq V(G), C \subseteq V(H)$, $A \neq \varnothing$ and $C \neq \varnothing$. If $|C| = 1$, then we are done. Suppose $|C| \geq 2$. Let $x, y \in C, x \neq y$. Since $A \times C$ is a superclique in $G[H]$, $(v,x)(v,y) \in E(G[H])$ for all $v \in A$ and there exists $(w,z) \in V(G[H]) \setminus (A \times C)$ such that $(w,z) \in N_{G[H]}((v,x)) \setminus N_{G[H]}((v,y))$ or $(w,z) \in N_{G[H]}((v,y)) \setminus N_{G[H]}((v,x))$. Hence, $xy \in E(H)$. Since G is complete, $v = w$ and $z \in N_H(x) \setminus N_H(y)$ or $z \in N_H(y) \setminus N_H(x)$, where $z \in V(H) \setminus C$. Thus, C is a superclique in H. Conversely, suppose $A \subseteq V(G)$, $A \neq \varnothing$ and C is a superclique in H. Then $A \times C \subseteq V(G[H])$ and $A \times C \neq \varnothing$. If $|A \times C| = 1$, then we are done. Suppose $|A \times C| \geq 2$. Let $(u,x),(v,y) \in A \times C, (u,x) \neq (v,y)$. Consider the following cases:

Case 1. $u = v$

Then $x \neq y$. Since $x, y \in C$ and C is a superclique in H, $xy \in E(H)$ and there exists $z \in V(H) \setminus C$ such that $z \in N_H(x) \setminus N_H(y)$ or $z \in N_H(y) \setminus N_H(x)$. Thus, $(u,x)(v,y) \in E(G[H]), (u,z) \notin A \times C$ and $(u,z) \in N_{G[H]}((u,x)) \setminus N_{G[H]}((v,y))$ or $(u,z) \in N_{G[H]}((v,y)) \setminus N_{G[H]}((u,x))$.

Case 2. $u \neq v$

Subcase 2.1 $x = y$

Since $\gamma(H) \neq 1$, there exists $z \in V(H)$ such that $z \notin N_H(x)$. Also, since $x \in C$ and $z \notin C$, we have $(u,z) \notin A \times C$ and $(u,z) \in N_{G[H]}((v,x)) \setminus N_{G[H]}((u,x))$. Since G is complete and $u \neq v, (u,x)(v,x) \in E(G[H])$.

Subcase 2.2 $x \neq y$

Since $x, y \in C$ and C is a superclique in H, $xy \in E(H)$ and there exists $z \notin C$ such that $z \in N_H(x) \setminus N_H(y)$ or $z \in N_H(y) \setminus N_H(x)$. Hence, $(v,z) \in N_{G[H]}((u,x)) \setminus N_{G[H]}((v,y))$ or $(v,z) \in N_{G[H]}((v,y)) \setminus N_{G[H]}((u,x))$ for some $(v,z) \notin A \times C$.

In any case, $A \times C$ is a superclique in $G[H]$. □

Theorem 2. [6] *Let G and H be connected graphs. Then $C \subseteq V(G + H)$ is a dominating set in $G + H$ if and only if at least one of the following is true:*

(i) $C \cap V(G)$ is a dominating set in G.

(ii) $C \cap V(H)$ is a dominating set in H.

(iii) $C \cap V(G) \neq \varnothing$ and $C \cap V(H) \neq \varnothing$.

Theorem 3. [6] *Let G and H be connected graphs. Then $C \subseteq V(G + H)$ is a dominating set in $G[H]$ if and only if $C = \bigcup_{x \in S} \big(\{x\} \times T_x\big)$ and either*

(i) S is a total dominating set in G or

(ii) S is a dominating set in G and T_x is a dominating set in H for every $x \in S \setminus N_G(S)$.

Lemma 2. Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph with $\gamma(H) \neq 1$. Then $A \times C \subseteq V\big(G[H]\big)$ induces a dominated superclique in $G[H]$ if and only if one of the following hold:

(i) $A \subseteq V(G)$ with $1 \leq |A| \leq n-2$ and $\langle C \rangle$ is a superclique of H.

(ii) $A \subseteq V(G)$ with $|A| = n-1$ and $\langle V(H) \setminus C \rangle$ is a dominated superclique of H.

Proof: Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. Suppose $A \times C \subseteq V\big(G[H]\big)$ induces a dominated superclique of $G[H]$. By Remark 6(ii), $V\big(G[H]\big) \setminus (A \times C) = (V(G) \setminus A) \times \big(V(H) \setminus C\big)$ is a dominating set of $G[H]$. Thus, by Theorem 2, $V(G) \setminus A$ is a total dominating set of G or $V(G) \setminus A$ is a dominating set of G and $V(H) \setminus C$ is a dominating set of H. Since $G = K_n$, $1 \leq |A| \leq n-2$ or $|A| = n-1$ and $V(H) \setminus C$ is a dominating set of H. By Lemma 1, $\langle C \rangle$ is a superclique of H. Thus, (i) and (ii) hold.

The converse follows immediately from Theorem 3 and Lemma 1. □

Proposition 4. [1] *Let G be a non-trivial connected graph with $diam(G) \leq 2$. Then $W \subseteq V(G) \setminus C$ is a strong resolving set of G if and only if $C = \varnothing$ or C is a superclique in G. In particular, $sdim(G) = |V(G)| - \omega_S(G)$.*

Theorem 4. *Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph with $\gamma(H) \neq 1$. A subset S of $V\big(G[H]\big)$ is a strong resolving dominating set of $G[H]$ if and only if $S = V(G[H]) \setminus (A \times C)$ satisying either of the following:*

(i) $A \subseteq V(G)$ with $1 \leq |A| \leq n-2$ and $\langle C \rangle$ is a superclique of H.

(ii) $A \subseteq V(G)$ with $|A| = n-1$ and $\langle C \rangle$ is a dominated superclique of $G[H]$.

Proof: Let S be a strong resolving dominating set of $G[H]$. Since $diam(G[H]) = 2$, by Proposition 4, $S = V(G[H]) \setminus C_0$, where $C_0 = \varnothing$ or $\langle C \rangle$ is a superclique of $G[H]$. By Lemma 2, (i) and (ii) follow.

For the converse, suppose $S = V(G[H]) \setminus (A \times C)$ satisfying condition (i) or (ii). By Lemma 2, $\langle A \times C \rangle$ is a dominated superclique of $G[H]$. Since $diam(G[H]) = 2$, S is a strong resolving dominating set of $G[H]$ by Proposition 4. □

Lemma 3. *Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph with $\gamma(H) = 1$. Then $A \times C \subseteq V(G[H])$ is a superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a superclique in H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set C^* of H.*

Proof: Suppose $A \times C \subseteq V(G[H])$ is a superclique in $G[H]$. Then $A \subseteq V(G)$, $C \subseteq V(H)$, $A \neq \varnothing$ and $C \neq \varnothing$. If $|C| = 1$, then C is a superclique in H. Suppose $|C| \geq 2$ and $x, y \in C$ where $x \neq y$. Then $(u,x)(u,y) \in A \times C$ for all $u \in A$. Since $(u,x) \neq (u,y)$ and $A \times C$ is a superclique in $G[H]$, $(u,x)(u,y) \in E(G[H])$ and $(w,z) \in N_{G[H]}((u,x)) \setminus N_{G[H]}((u,y))$

or $(w,z) \in N_{G[H]}((u,y)) \setminus N_{G[H]}((u,x))$ for some $(w,z) \in V(G[H]) \setminus (A \times C)$. Thus, $xy \in E(H)$, $w = u$ and $z \in N_H(x) \setminus N_H(y)$ or $z \in N_H(y) \setminus N_H(x)$, where $z \notin C$. Hence, C is a superclique in H. Let $a \in C$ where $\{a\}$ is a γ-set of H. Suppose $|A| \geq 2$ and let $u, v \in A$, where $u \neq v$. Then $(u,a),(v,a) \in A \times C, (u,a) \neq (v,a)$. Since $A \times C$ is a superclique, there exists $(w,b) \in V(G[H]) \setminus (A \times C)$ such that $(w,b) \in N_{G[H]}((v,a)) \setminus N_{G[H]}((u,a))$ or $(w,b) \in N_{G[H]}((u,a)) \setminus N_{G[H]}((v,a))$. Since G is complete, $w = u$ $w = v$ and $b \notin N_H(a)$. This is a contrdiction since $\{a\}$ is a γ-set of H. Thus, $|A| \leq 1$. Since $A \neq \varnothing$, $|A| = 1$.

Coversely, suppose $A \subseteq V(G)$, $A \neq \varnothing$ and C is a superclique in H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set C^* of H. Then $A \times C \subseteq V(G[H])$ and $A \times C \neq \varnothing$. If $|A \times C| = 1$, then we are done. Suppose $|A \times C| \geq 2$. Let $(u,x),(v,y) \in A \times C$, $(u,x) \neq (v,y)$. Consider the following cases:

Case 1. $u = v$

Then $x \neq y$. Since $x, y \in C$ and C is a superclique in H, $xy \in E(H)$ and there exists $z \in N_H(x) \setminus N_H(y)$ or $z \in N_H(y) \setminus N_H(x)$ for some $z \in V(H) \setminus C$. Hence, $(u,x)(v,y) \in E(G[H])$ and $(u,z) \in N_{G[H]}((u,x)) \setminus N_{G[H]}((v,y))$ or $(u,z) \in N_{G[H]}((v,y)) \setminus N_{G[H]}((u,x))$, where $(u,z) \in V(G[H]) \setminus C$.

Case 2. $u \neq v$

Subcase 2.1. $x = y$

Since $u, v \in A, u \neq v, |A| \geq 2$. By assumption, if $x \in C$, then $\{x\}$ is not a γ-set of H. Thus, there exists $z \in V(H) \setminus N_H(x)$. Hence, $(u,z) \in V(G[H]) \setminus (A \times C)$ and $(u,z) \in N_{G[H]}((v,y)) \setminus N_{G[H]}((u,x))$. Since G is complete, $(u,x)(v,y) \in E(G[H])$.

Subcase 2.2. $x \neq y$

Since $x, y \in C$ and C is a superclique in H, $xy \in E(H)$ and $z \in N_H(x) \setminus N_H(y)$ or $z \in N_H(y) \setminus N_H(x)$ for some $z \in V(H) \setminus C$. Thus, $(u,z) \in N_{G[H]}((v,y)) \setminus N_{G[H]}((u,x))$ where $(u,z) \in V(G[H]) \setminus (A \times C)$. Since G is complete and $u \neq v$, $(u,x)(v,y) \in E(G[H])$.

In any case, $A \times C$ is a superclique in $G[H]$. □

The next result follows immediately from Lemma 3 and Theorem 2.

Lemma 4. *Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph with $\gamma(H) = 1$. Then $A \times C \subseteq V(G[H])$ induces a dominated superclique of $G[H]$ if and only if the following hold:*

(i) $A \subseteq V(G)$ with $1 \leq |A| \leq n-2$ and $\langle C \rangle$ is a superclique of H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set C^* of H.

(ii) $A \subseteq V(G)$ with $|A| = n-1$ and $\langle C \rangle$ is a dominated superclique of $G[H]$.

Theorem 5. *Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph with $\gamma(H) = 1$. A subset S of $V\big(G[H]\big)$ is a strong resolving dominating set of $G[H]$ if and only if the following hold:*

(i) $A \subseteq V(G)$ with $1 \leq |A| \leq n-2$ and $\langle C \rangle$ is a superclique of H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set C^* of H.

(ii) $A \subseteq V(G)$ with $|A| = n-1$ and $\langle C \rangle$ is a dominated superclique of $G[H]$.

Proof: The proof is similar to the proof of Theorem 4 and by using Lemma 4 □

Corollary 1. Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph of order m. Then

$$\gamma_{sR}\big(G[H]\big) = nm - \omega_S\big(G[H]\big).$$

Proof: Let S be a strong metric basis of $G[H]$. Then S is a strong resolving set of $G[H]$. By Lemma 1, $S = V(G[H]) \setminus (A \times C)$, where $A \times C$ is a superclique of $G[H]$. Since S is a strong resolving dominating set, $A \times C$ is a maximum dominated superclique. Hence,

$$\gamma_{SR}(G[H]) = |S| = |V(G[H])| - |A \times C| = nm - \omega_{DS}(G[H]).$$

□

The next result follows immediately from Lemma 1 and Corollary 1.

Corollary 2. Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph of order m. Then

$$\gamma_{sR}\big(G[H]\big) = nm - \omega_{DS}\big(G[H]\big).$$

Theorem 6. Let G be a non-trivial connected graph and H be non-trivial complete graph. A subset

$$C = \left(\bigcup_{x\in S}\Big\{\,\{x\} \times T_x\Big\}\right) \bigcup \left(\bigcup_{x\in W_G}\Big\{\,\{x\} \times (V(H) \setminus T_x)\Big\}\right)$$

of $V\big(G[H]\big)$ where $W_G \subseteq S$ and $T_x \subseteq V(H),\ \forall x \in S$, is a strong resolving dominating set of $G[H]$ if and only if

(*i*) $S = V(G)$.

(*ii*) $V(H) \setminus T_x$ is a superclique of H.

(*iii*) W_G is a strong resolving set of G.

Proof: Let C be a strong resolving dominating set of $G[H]$. Suppose there exists $x \in V(G) \setminus S$. Let $p, q \in V(H)$ with $d_H(p,q) = diam(H)$. Then (x,p) MMD (x,p) implying that (x,p) and (x,q) cannot be resolved by C since $(x,p)(x,q) \notin C$. Hence, $S = V(G)$ and (*i*) holds.

Let $u, v \in V(H) \setminus T_x, u \neq v$. Then $(x,u),(x,v) \notin C$. Since C is a strong resolving dominating set of $G[H], (x,u)$ and (x,v) can be strongly resolved by $(y,w) \in C$. If $(x,u) \in I_{G[H]}[(x,v),(y,w)]$, then

$$d_{G[H]}((x,v),(x,u)) + d_{G[H]}((x,u),(y,w)) = d_{G[H]}((x,v),(y,w))$$

implying that $x = y$ and $w \in N_H(u) \setminus N_H(v)$. Similarly, if $(x,v) \in I_{G[H]}[(x,u),(y,v)]$, then $x = y$ and $w \in N_H(v) \setminus N_H(u)$. Hence, $V(H) \setminus T_x$ is a superclique of H. Thus, (*ii*) holds.

Let $p, q \in V(G) \setminus W_G$. Then $(p,r),(q,r) \notin C$, where $r \notin T_p, T_q$. Hence, there exists $(s,t) \in C$ that resolves (p,r) and (q,r). If $(p,r) \in I_{G[H]}[(q,r),(s,t)]$, then

$$d_{G[H]}((q,r),(p,r)) + d_{G[H]}((p,r),(s,t)) = d_{G[H]}((q,r),(s,t))$$

implying that $r = t$ and $p \in I_G[q,s]$. Similarly, if $(q,r) \in I_{G[H]}[(p,r),(s,t)]$, then $r = t$ and $q \in I_G[p,s]$. Hence, s strongly resolves p and q. Therefore, W_G is a strongly resolving dominating set of G and (iii) holds.

For the converse, suppose C satisfies the given property. Let $x = (x_1, x_2), y = (y_1, y_2) \notin C, x \neq y$. Then consider the following cases:

Case 1. $x_1 \in V(G) \setminus W_G$ and $y_1 \in V(G) \setminus W_G, x_1 \neq y_1$.

By (iii), there exists $z_1 \in V(G) \cap W_G$ that resolves x_1 and y_1. If $x_1 \in I_G[y_1, z_1]$, then

$$d_G(y_1, x_1) + d_G(x_1, z_1) = d_G(y_1, z_1).(1)$$

Choose $z_2 \in V(H) \setminus T_{z_1}$. Clearly, $(z_1, z_2) \in C$. We claim that (z_1, z_2) strongly resolves $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $G[H]$. Using equation 1, we have

$$d_{G[H]}((y_1,y_2),(x_1,x_2)) + d_{G[H]}((x_1,x_2),(z_1,z_2)) = d_{G[H]}((y_1,y_2),(z_1,z_2))$$

implying that $(x_1, x_2) \in I_{G[H]}[(y_1,y_2),(z_1,z_2)]$. Similarly, if $y_1 \in I_G[x_1, z_1]$, then (z_1, z_2) strongly resolves $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Case 2. $x_1 = y_1$

Then $x_2 \neq y_2$ since $x \neq y$. It follows that $T_{x_1} = T_{y_1}$. Note that $x = (x_1, x_2)$ and $y = (y_1, y_2) \notin C$, then $x_2 \notin T_{x_1}$ and $y_2 \notin T_{y_1}$. By (ii), there exists $z \in T_{x_1}$ such that $z \in N_G(y_2) \setminus N_G(x_2)$ or $z \in N_G(x_2) \setminus N_G(y_2)$. Clearly, $(x_1, z) \in C$ and either $(y_1, y_2) \in I_{G[H]}[(x_1,x_2),(x_1,z)]$. Thus, (x_1, z) strongly resolves (x_1, x_2) and (y_1, y_2). □

The next result follows immediately from Theorem 6.

Theorem 7. Let G be a non-trivial connected graph and H be non-trivial complete graph. A subset

$$C = \left(\bigcup_{x\in S}\Big\{\,\{x\} \times T_x\Big\}\right) \bigcup \left(\bigcup_{x\in W_G}\Big\{\,\{x\} \times (V(H) \setminus T_x)\Big\}\right)$$

of $V\big(G[H]\big)$ where $W_G \subseteq S$ and $T_x \subseteq V(H),\ \forall x \in S$, is a strong resolving dominating set of $G[H]$ if and only if

(i) $S = V(G)$.

(ii) $V(H) \setminus T_x$ is a dominated superclique of H.

(iii) W_G is a strong resolving dominating set of G.

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