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On Double Roman Dominating Functions in Graphs

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Abstract. Let G be a connected graph. A function $f : V(G) \to \{0, 1, 2, 3\}$ is a double Roman dominating function of G if for each $v \in V(G)$ with $f(v) = 0$, v has two adjacent vertices u and w for which $f(u) = f(w) = 2$ or v has an adjacent vertex u for which $f(u) = 3$, and for each $v \in V(G)$ with $f(v) = 1$, v is adjacent to a vertex u for which either $f(u) = 2$ or $f(u) = 3$. The minimum weight $\omega_G(f) = \sum_{v\in V(G)} f(v)$ of a double Roman dominating function f of G is the double Roman domination number of G. In this paper, we continue the study of double Roman domination introduced and studied by R.A. Beeler et al. in [2]. First, we characterize some double Roman domination numbers with small values in terms of the domination numbers and 2-domination numbers. Then we determine the double Roman domination numbers of the join, corona, complementary prism and lexicographic product of graphs.

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Key Words and Phrases: Domination number, 2-domination number, double Roman dominating function, double Roman domination number

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1. Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with $V(G)$ and $E(G)$ being the *vertex set* and *edge set* of G, respectively. For $S \subseteq V(G)$, the symbol $|S|$ refers to the cardinality of S. In particular, $|V(G)|$ is the *order* of G. For other basic concepts not presented but are used here are adopted from ([4, 11]).

For a vertex v of a graph G, the *open neighborhood* of v refers to the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ while its *closed neighborhood* is the set $N_G[v] = \{v\} \cup N_G(v)$. Vertex v is an *isolated vertex* if $N_G(v) = \varnothing$. For $S \subseteq V(G)$, the *open neighborhood* and *closed neighborhood* of S are the sets $N_G(S) = \cup_{v\in S} N_G(v)$ and $N_G[S] = \cup_{v\in S} N_G[u]$, respectively.

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A set $S \subseteq V(G)$ is said to be a *dominating set* of G if $N_G[S] = V(G)$. The minimum cardinality of a dominating set is called the *domination number* of G, and is denoted by $\gamma(G)$. Any dominating set of cardinality $\gamma(G)$ is referred to as a γ-*set* of G. We refer to [1, 3, 5, 7, 8, 12, 14] for the introduction, fundamental concepts and some studies on domination in graphs.

A set $S \subseteq V(G)$ is called a 2-*dominating set* if each $v \in V(G) \setminus S$, $|S \cap N_G(v)| \geq 2$. The 2-*domination number* of G, denoted $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G. References [7, 10] provide a good study on 2-domination.

A *Roman dominating function* on G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that for each $u \in V(G)$ for which $f(u) = 0$, there exists $v \in V(G)$ such that $f(v) = 2$ and $uv \in E(G)$. The *weight* of f is the value $\omega_G(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function of G. The history, introduction and some of the recent studies in Roman domination have been provided in [6, 13, 15–17].

A function $f : V(G) \to \{0, 1, 2, 3\}$ is a double Roman dominating function of G, written $f \in DRD(G)$, if each of the following holds:

(1) for each $v \in V(G)$ with $f(v) = 0$ at least one of the following holds:

  (a) v has two adjacent vertices u and w for which $f(u) = f(w) = 2$; or

  (b) v has an adjacent vertex u for which $f(u) = 3$, and

(2) for each $v \in V(G)$ with $f(v) = 1$, v is adjacent to a vertex u for which either $f(u) = 2$ or $f(u) = 3$.

The *double Roman domination number* of G, denoted by $\gamma_{dR}(G)$, is the minimum weight $\omega_G(f) = \sum_{v \in V(G)} f(v)$ of a double Roman dominating function f of G. Any $f \in DRD(G)$ of weight equal to $\gamma_{dR}(G)$ is referred to as γ_{dR}-*function* of G.

The concept of double domination in graphs was proposed by Beeler, Haynes and Hedetniemi [2] in 2016. It is a stronger version of Roman domination. If in Roman domination only one legion is required to defend an attacked city, in double Roman domination any attack can be defended by at least two legions. Double Roman domination is further studied in [9, 18, 19].

In this paper, the double Roman domination in graphs is revisited. The main interest is particularly on the double Roman dominating function of the join, corona, complementary prism and lexicographic product of graphs.

The following results established in the referred articles are useful in this paper.

Proposition 1. [2] In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value 1.

Proposition 2. [9] For $n \geq 1$,

$$\gamma_{dR}(P_n) = \begin{cases} n, & \textit{if } n \equiv 0 \pmod 3, \\ n+1, & \textit{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

Proposition 3. [9] For $n \geq 3$,

$$\gamma_{dR}(C_n) = \begin{cases} n, & \textit{if } n \equiv 0, 2, 3, 4 \pmod 6, \\ n+1, & \textit{if } n \equiv 1, 5 \pmod 6. \end{cases}$$

Proposition 4. [2] For any graph G, $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$.

2. Results

For a function $f : V(G) \to \{0, 1, 2, 3\}$, we write $f = (V_0, V_1, V_2, V_3)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for all $i \in \{0, 1, 2, 3\}$. Hence, $f \in DRD(G)$ if and only if each of the following holds:

(1) for each $v \in V_0$, $|V_2 \cap N_G(v)| \geq 2$ or $|V_3 \cap N_G(v)| \geq 1$; and

(2) for each $v \in V_1$, either $|V_2 \cap N_G(v)| \geq 1$ or $|V_3 \cap N_G(v)| \geq 1$.

In view of Proposition 1, we may always assume that a γ_{dR}-function of G is of the form $f = (V_0, \varnothing, V_2, V_3)$. Thus, $\gamma_{dR}(G) \geq 2$ for all graphs G. More precisely, $\gamma_{dR}(G) = 2$ if and only if $G = K_1$.

Proposition 5. *Let G be a nontrivial connected graph. Then*

(i) $\gamma_{dR}(G) = 3$ *if and only if* $\gamma(G) = 1$; *and*

(ii) $\gamma_{dR}(G) = 4$ *if and only if* $\gamma(G) = 2 = \gamma_2(G)$.

Proof. If $\gamma_{dR}(G) = 3$ and $f = (V_0, \varnothing, V_2, V_3)$ is a γ_{dR}-function of G, then $V_2 = \varnothing$, $|V_3| = 1$ and $V_0 = V(G) \setminus V_3$. If $V_3 = \{v\}$, then $N_G[v] = V_0 \cup \{v\} = V(G)$. This means that $\gamma(G) = 1$.

Conversely, if $\gamma(G) = 1$ and $\{v\}$ is a dominating set of G, then $f = (V(G)\setminus\{v\}, \varnothing, \varnothing, \{v\}) \in DRD(G)$ so that $\gamma_{dR}(G) \leq \omega_G(f) = 3$. Since G is nontrivial, $\gamma_{dR}(G) = 3$. This proves (i).

Assume that $\gamma_{dR}(G) = 4$, and let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of G. Then $|V_2| = 2$ (say $V_2 = \{u, v\}$), $V_3 = \varnothing$ and $V_0 = V(G) \setminus \{u, v\}$. Thus, V_2 is a γ_2-set so that $\gamma_2(G) = 2$. Moreover, being a 2-dominating set, V_2 is a dominating set of G so that $\gamma(G) \leq 2$. By (i), $\gamma(G) = 2$.

Conversely, let $S = \{u, v\}$ be a γ_2-set of G. Since $f = (V(G) \setminus S, \varnothing, S, \varnothing) \in DRD(G)$, $\gamma_{dR}(G) \leq \omega_G(f) = 4$. Because G is nontrivial and $\gamma(G) \neq 1$, $\gamma_{dR}(G) \geq 4$ by (i). Hence, $\gamma_{dR}(G) = 4$. This proves (ii). □

Proposition 6. *For a nontrivial connected graph G, $\gamma_{dR}(G) = 5$ if and only if $\gamma_2(G) \geq 3$ and there exist $u, v \in V(G)$ for which the following holds:*

(i) $uv \notin E(G)$;

(ii) $V(G) \setminus N_G[v] = \{u\}$ *and* $V(G) \setminus N_G[u] \neq \{v\}$.

Proof. Assume that $\gamma_{dR}(G) = 5$. In view of Proposition 5, $\gamma_2(G) \geq 3$. Let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of G. Then $|V_2| = |V_3| = 1$. Let $V_2 = \{u\}$ and $V_3 = \{v\}$. Since f is a γ_{dR}-function of G, $N_G(v) = V_0$. Because $\gamma(G) \geq 2$, $uv \notin E(G)$. Hence, $V(G) \setminus N_G[v] = \{u\}$. Moreover, since $\gamma_2(G) \geq 3$, $uw \notin E(G)$ for some $w \in V(G) \setminus \{u, v\}$. Therefore, $V(G) \setminus N_G[u] \neq \{v\}$.

Conversely, suppose that $\gamma_2(G) \geq 3$, and let $u, v \in V(G)$ such that $uv \notin E(G)$, $V(G) \setminus N_G[v] = \{u\}$ and $V(G) \setminus N_G[u] \neq \{v\}$. In view of Proposition 5, $\{u, v\}$ is a γ-set of G. Since $f = (V(G) \setminus \{u, v\}, \varnothing, \{u\}, \{v\}) \in DRD(G)$, $\gamma_{dR}(G) \leq \omega_G(f) = 5$. Since $\gamma(G) = 2$ and $\gamma_2(G) \neq 2$, $\gamma_{dR}(G) = 5$ by Proposition 5. □

Corollary 1. *For a nontrivial connected graph G,* $\gamma_{dR}(G) = 5$ *if and only if* $\gamma(G) = 2$, $\gamma_2(G) \geq 3$ *and* $\gamma(G - v) = 1$ *for some* $v \in V(G)$*, where* $G - v$ *is the resulting graph after removing the vertex* v.

Proof. Assume that $\gamma_{dR}(G) = 5$. By Proposition 6, $\gamma_2(G) \geq 3$ and there exist vertices $u, v \in V(G)$ for which $uv \notin E(G)$, $V(G) \setminus N_G[u] = \{v\}$ and $V(G) \setminus N_G[v] \neq \{u\}$. This means that $\{u, v\}$ and $\{u\}$ are γ-sets of G and $G - v$, respectively. Thus, $\gamma(G) = 2$ and $\gamma(G - v) = 1$.

Conversely, suppose that $\gamma(G) = 2$, $\gamma_2(G) \geq 3$ and let $u, v \in V(G)$ such that $N_{G-v}[u] = V(G - v)$. Since $\gamma_2(G) \neq 2$, there exists $w \in V(G) \setminus \{u, v\}$ such that $vw \notin E(G)$. Thus, $w \in V(G) \setminus N_G[u]$ so that $V(G) \setminus N_G[v] \neq \{u\}$. Moreover, since $\gamma(G) = 2$, $uv \notin E(G)$ so that $V(G) \setminus N_G[u] = \{v\}$. By Proposition 6, $\gamma_{dR}(G) = 5$. □

Proposition 7. *For nontrivial connected graph G,* $\gamma_{dR}(G) = 6$ *if and only if one of the following holds:*

(i) $\gamma(G) = 2$, $\gamma_2(G) \geq 3$ *and* $\gamma(G - v) \geq 2$ *for all* $v \in V(G)$.

(ii) $\gamma(G) \geq 2$ *and* $\gamma_2(G) = 3$ *and* $\gamma(G - v) \geq 2$ *for all* $v \in V(G)$.

Proof. Let $\gamma_{dR}(G) = 6$. Then $\gamma(G) \geq 2$ by Proposition 5. Let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of G. Consider the following cases:

Case 1. Suppose that $V_2 = \varnothing$ and $|V_3| = 2$. Then V_3 is a dominating set of G and so $\gamma(G) = 2$. Since $\gamma_{dR}(G) \neq 4$, $\gamma_2(G) \geq 3$ by Proposition 5. Moreover, by Proposition 1, $\gamma(G - v) \geq 2$ for all $v \in V(G)$. This proves (i).

Case 2. Suppose that $|V_2| = 3$ and $V_3 = \varnothing$. Then V_2 is a 2-dominating set of G so that $\gamma_2(G) \leq 3$. Since $\gamma_{dR}(G) \neq 4$, $\gamma_2(G) = 3$ by Proposition 5. Hence, (ii) holds.

Conversely, by Proposition 5, Proposition 6 and Corollary 1, $\gamma_{dR}(G) \geq 6$. If $u, v \in V(G)$ such that $\{u, v\}$ dominates $V(G)$, then $f = (V(G) \setminus \{u, v\}, \varnothing, \varnothing, \{u, v\}) \in DRD(G)$ so that $\gamma_{dR}(G) \leq \omega_G(f) = 6$. On the other hand, if $\{u, v, w\}$ is a γ_2-set of G, then $f = (V(G) \setminus \{u, v, w\}, \varnothing, \{u, v, w\}, \varnothing) \in DRD(G)$ so that $\gamma_{dR}(G) \leq \omega_G(f) = 6$. Therefore, each of (i) and (ii) implies that $\gamma_{dR}(G) = 6$. □

Proposition 8. *For a nontrivial connected graph G, if $\gamma_{dR}(G) = 7$, then $\gamma(G) = 3$ and $\gamma_2(G) \geq 4$.*

Proof. Let $\gamma_{dR}(G) = 7$. By Proposition 5, Corollary 1, and Proposition 7, $\gamma(G) \geq 3$ and $\gamma_2(G) \geq 4$. Let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of G. Then $|V_2| = 2$ and $|V_3| = 1$. Write $V_2 = \{u, v\}$ and $V_3 = \{w\}$. Then $\{u, v, w\}$ is a dominating set of G, and so, $\gamma(G) \leq 3$. Hence, $\gamma(G) = 3$. □

The converse of Proposition 8 need not be true. Consider for example, the graph G in Figure 1 obtained from $P_9 = [x_1, x_2, x_3, \ldots, x_9]$ by adding the edges x_3x_5 and x_7x_5. Observe that $\gamma(G) = 3$, $\gamma_2(G) \geq 4$ but $\gamma_{dR}(G) = 9 > 7$.

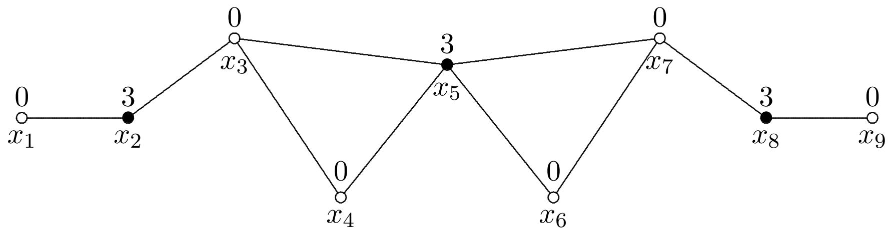

Figure 1: A graph G with $\gamma_{dR}(G) > 7$

Proposition 9. *Let G be a disconnected graph with components C_1, C_2, $\ldots$, C_k. Then $\gamma_{dR}(G) = \sum_{j=1}^{k} \gamma_{dR}(C_j)$. In particular, if $G = K_n$, then $\gamma_{dR}(\overline{G}) = 2n$.*

Proof. If $f_1, f_2, \ldots, f_k$ are γ_{dR}-functions of C_1, C_2, $\ldots$, C_k, respectively, then the function $f : V(G) \to \{0, 1, 2, 3\}$ given by $f(x) = f_k(x)$ for all $x \in V(C_k)$ is a γ_{dR}-function of G. Thus, $\gamma_{dR}(G) \leq \sum_{j=1}^{k} \gamma_{dR}(C_j)$.

Conversely, if f be a γ_{dR}-function of G, then the restriction $f|_{C_j}$ of f to C_j, for any $j = 1, 2, \ldots, k$, is a γ_{dR}-function of C_j. Thus, $\gamma_{dR}(C_j) \leq \omega_{C_j}(f|_{C_j})$ for all $j = 1, 2, \ldots, k$. Hence, $\sum_{j=1}^{k} \gamma_{dR}(C_j) \leq \gamma_{dR}(G)$. □

Proposition 10. (i) *For any path P_n of order n,*

$$\gamma_{dR}(\overline{P}_n) = \begin{cases} 2, & n = 1; \\ 4, & n = 2; \\ 5, & n \geq 3. \end{cases}$$

(ii) *For any cycle C_n of order $n \geq 3$, $\gamma_{dR}(\overline{C}_n) = 6$.*

Proof. **For** (i)**:** The cases where $n = 1, 2, 3, 4$ are clear. Suppose that $n \geq 5$. Let $V(P_n) = [v_1, v_2, \ldots, v_n]$. Then the sets $\{v_1, v_2\}$ and $\{v_1, v_2, v_3\}$ are γ-set and γ_2-set of $\overline{P}_n$, respectively. Moreover, $\gamma(\overline{P}_n - v_2) = 1$. By Proposition 6, $\gamma_{dR}(\overline{P}_n) = 5$.

For (ii)**:** The case where $n = 3, 4$ is clear. Suppose that $n \geq 5$. Let $u, w, v \in V(C_n)$ such that $u, w \in N_{C_n}(v)$. Then $\{u, v\}$ and $\{u, v, w\}$ are γ-set and γ_2-set of $\overline{C}_n$, respectively. Moreover, $\gamma(\overline{C}_n - v) = 2$ for all $v \in V(C_n)$. By Proposition 7, $\gamma_{dR}(\overline{C}_n) = 6$. □

The (n, m)-*tadpole graph* $T_{n,m}$ is obtained by joining a cycle graph C_n and a path P_m with a bridge. The graph in Figure 2 is the tadpole $T_{6,3}$.

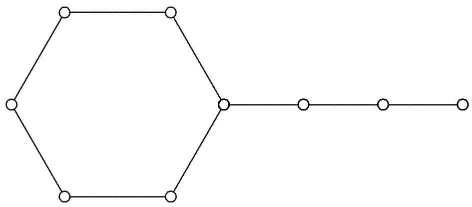

Figure 2: The tadpole $T_{6,3}$

Proposition 11. *For $T_{n,1}$ with $n \geq 3$, $n \leq \gamma_{dR}(T_{n,1}) \leq n + 1$. More precisely,*

$$\gamma_{dR}(T_{n,1}) = \begin{cases} n, & n \equiv 0, 3 \pmod 6; \\ n + 1, & n \equiv 1, 2, 4, 5 \pmod 6. \end{cases}$$

Proof. Let $v \in V(C_n)$ be the vertex that connects C_n to $P_1 = \{u\}$ and let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of C_n. We may assume that $v \notin V_0$. If $v \in V_3$, then $g = (V_0 \cup \{u\}, \varnothing, V_2, V_3) \in DRD(T_{n,1})$. If $v \in V_2$, then $g = (V_0, \{u\}, V_2, V_3) \in DRD(T_{n,1})$. In any case,

$$\gamma_{dR}(T_{n,1}) \leq \omega_{T_{n,1}}(g) \leq 1 + \gamma_{dR}(C_n).$$

Now, let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of $T_{n,1}$. If $v \notin V_0$, then $v \in V_3$ and $u \in V_0$ so that $g = (V_0 \setminus \{u\}, \varnothing, V_2, V_3) \in DRD(C_n)$. If $v \in V_0$ and $u \in V_2$, then $g = (V_0 \setminus \{v\}, \varnothing, (V_2 \cap V(C_n)) \cup \{v\}, V_3) \in DRD(C_n)$. And, if $v \in V_0$ and $u \in V_3$, then $g = (V_0 \setminus \{v\}, \varnothing, V_2, (V_3 \cap V(C_n)) \cup \{v\}) \in DRD(C_n)$. In any case,

$$\gamma_{dR}(C_n) \leq \omega_{C_n}(g) = \gamma_{dR}(T_{n,1}).$$

Hence, the desired inequalities hold.

Suppose that $n \equiv 0, 3 \pmod 6$. Then $\gamma_{dR}(C_n) = n$ (by Proposition 2.2.3) and C_n has a γ_{dR}-function $f = (V_0, \varnothing, V_2, V_3)$ with $V_3 \neq \varnothing$. By symmetry, we assume that $v \in V_3$. Thus, $g = (V_0 \cup \{u\}, \varnothing, V_2, V_3) \in DRD(T_{n,1})$. Together with the inequality,

$$n \leq \gamma_{dR}(T_{n,1}) \leq \omega_{T_{n,1}}(g) = \gamma_{dR}(C_n) = n.$$

Therefore, $\gamma_{dR}(T_{n,1}) = n$.

Suppose that $n \equiv 2, 4 \pmod 6$. Then $\gamma_{dR}(C_n) = n$ and $V_3 = \varnothing$ for all γ_{dR}-functions $f = (V_0, \varnothing, V_2, V_3)$ of C_n. With $v \in V_2$, $g = (V_0, \{u\}, V_2, V_3)$ is a γ_{dR}-function of $T_{n,1}$. Thus, $\gamma_{dR}(T_{n,1}) = n + 1$.

Finally, suppose that $n \equiv 1, 5 \pmod 3$. Then $\gamma_{dR}(C_n) = n + 1$ and C_n has a γ_{dR}-function $f = (V_0, \varnothing, V_2, V_3)$ with $V_3 \neq \varnothing$. With $v \in V_3$, $g = (V_0 \cup \{u\}, \varnothing, V_2, V_3) \in DRD(T_{n,1})$. Thus, $n+1 = \gamma_{dR}(C_n) \leq \gamma_{dR}(T_{n,1}) \leq \omega_{T_{n,1}}(g) = \omega_{C_n}(f) = n + 1$. □

Proposition 12. *Let* $n \geq 3$ *and* $m \geq 2$.

(*i*) *If* $n \equiv 0, 3 \pmod 6$, *then*

$$\gamma_{dR}(T_{n,m}) = \begin{cases} n+2, & \text{if } m = 2 \\ n + \gamma_{dR}(P_{m-1}), & \text{if } m \geq 3. \end{cases}$$

(*ii*) *If* $n \equiv 2, 4 \pmod 6$, *then*

$$\gamma_{dR}(T_{n,m}) = \begin{cases} n+2, & \text{if } m = 2 \\ n + \gamma_{dR}(P_m), & \text{if } m \geq 3;\ m \equiv 0 \pmod 3), \\ n + \gamma_{dR}(P_m) - 1, & \text{if } m \geq 3;\ m \equiv 1,2 \pmod 3). \end{cases}$$

(*iii*) *If* $n \equiv 1, 5 \pmod 6$, *then*

$$\gamma_{dR}(T_{n,m}) = \begin{cases} n+3, & \text{if } m = 2 \\ n + 1 + \gamma_{dR}(P_{m-1}), & \text{if } m \geq 3. \end{cases}$$

Proof. Write $P_m = [v_1, v_2, \ldots, v_m]$. Let $v \in V(C_n)$ be the vertex that connects C_n to P_m through the edge vv_1. We consider the following cases:

Case 1: Suppose that $n \equiv 0, 3 \pmod 6$. Let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of C_n with $V_3 \neq \varnothing$. Assume $v \in V_3$. If $m = 2$, then $g = (V_0 \cup \{v_1\}, \varnothing, V_2 \cup \{v_2\}, V_3)$ is a γ_{dR}-function of $T_{n,2}$. Thus, $\gamma_{dR}(T_{n,2}) = \gamma_{dR}(C_n) + 2 = n+2$. Assume $m \geq 3$, and let $m = 3k+r$, where $0 \leq r \leq 2$. Put $V_3^* = \{v_{3j} : j \in \{1, 2, \ldots, k\}\}$. If $0 \leq r \leq 1$, put $V_0^* = V(P_m) \setminus V_3^*$ and $V_2^* = \varnothing$. On the other hand, if $r = 2$, put $V_0^* = V(P_m) \setminus (V_3^* \cup \{v_{3k+2}\})$ and $V_2^* = \{v_{3k+2}\}$. Since $(V_0^* \setminus \{v_1\}, \varnothing, V_2^*, V_3^*)$ is a γ_{dR}-function of $P_m - v_1 \cong P_{m-1}$, $g = (V_0 \cup V_0^*, \varnothing, V_2 \cup V_2^*, V_3 \cup V_3^*)$ is a γ_{dR}-function of $T_{n,m}$. Thus, $\gamma_{dR}(T_{n,m}) = n + \gamma_{dR}(P_{m-1})$.

Case 2: Suppose that $n \equiv 2, 4 \pmod 6$. Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{dR}-function of C_n. Accordingly, $V_3 = \varnothing$ and we may assume that $v \in V_2$. If $m = 2$, then $g = (V_0 \cup \{v_1\}, \varnothing, V_2 \cup \{v_2\}, \varnothing)$ is a γ_{dR}-function of $T_{n,2}$. Thus, $\gamma_{dR}(T_{n,2}) = n + 2$.

Suppose that $m \geq 3$, and let $m = 3k + r$, where $0 \leq r \leq 2$. Whenever $r = 0$, put $V_3^* = \{v_{3j-1} : j \in \{1, 2, \ldots, k\}\}$, $V_0^* = V(P_m) \setminus V_3^*$ and $V_2^* = \varnothing$. Then $(V_0^*, \varnothing, V_2^*, V_3^*)$ is a γ_{dR}-function of P_m. Thus $g = (V_0 \cup V_0^*, \varnothing, V_2 \cup V_2^*, V_3 \cup V_3^*)$ is a γ_{dR}-function of $T_{n,m}$. Consequently, $\gamma_{dR}(T_{n,m}) = n + \gamma_{dR}(P_m)$.

Suppose that $r = 1$. Let j be the largest positive integer for which $2j \leq 3k$. If $2j = 3k$, put $V_2^* = \{v_{2i} : i \in \{1, 2, \ldots, j-1\}\}$, $V_0^* = V(P_m) \setminus (V_2^* \cup \{v_{3k}\})$ and $V_3^* = \{v_{3k}\}$. On the other hand, if $2j < 3k$, put $V_2^* = \{v_{2i} : i \in \{1, 2, \ldots, j\}\} \cup \{v_{3k+1}\}$, $V_0^* = V(P_m) \setminus V_2^*$

and $V_3^* = \varnothing$. In either case, $g = (V_0 \cup V_0^*, \varnothing, V_2 \cup V_2^*, V_3^* \cup V_3^*)$ is a γ_{dR}-function of $T_{n,m}$. Thus, $\gamma_{dR}(T_{n,m}) = n + \gamma_{dR}(P_m) - 1$.

Suppose that $r = 2$. Let j be the largest positive integer for which $2j \leq 3k$. If $2j = 3k$, put $V_2^* = \{v_{2i} : i \in \{1, 2, \ldots, j+1\}\}$, $V_0^* = V(P_m) \setminus V_2^*$ and $V_3^* = \varnothing$. On the other hand, if $2j < 3k$, put $V_2^* = \{v_{2i} : i \in \{1, 2, \ldots, j\}\}$, $V_0^* = V(P_m) \setminus (V_2^* \cup \{v_{3k+1}\})$ and $V_3^* = \{v_{3k+1}\}$. In either case, $g = (V_0 \cup V_0^*, \varnothing, V_2 \cup V_2^*, V_3^* \cup V_3^*)$ is a γ_{dR}-function of $T_{n,m}$. Thus, $\gamma_{dR}(T_{n,m}) = n + \gamma_{dR}(P_m) - 1$.

Case 3: Suppose that $n \equiv 1, 5 \pmod 6$. Let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of C_n with $V_3 \neq \varnothing$ and $v \in V_3$. If $m = 2$, then $g = (V_0 \cup \{v_1\}, \varnothing, V_2 \cup \{v_2\}, V_3)$ is a γ_{dR}-function of $T_{n,2}$. Thus,

$$\gamma_{dR}(T_{n,2}) = \gamma_{dR}(C_n) + 2 = n + 1 + 2.$$

Suppose that $m \geq 3$, and let $m = 3k + r$, where $0 \leq r \leq 2$. If $r = 0$, put $V_3^* = \{v_{3j} : j \in \{1, 2, \ldots, k\}\}$, $V_0^* = V(P_m) \setminus V_3^*$ and $V_2^* = \varnothing$. If $r = 1$, put $V_3^* = \{v_{3j} : j \in \{1, 2, \ldots, k\}\}$, $V_0^* = V(P_m) \setminus V_3^*$ and $V_2^* = \varnothing$. And if $r = 2$, put $V_3^* = \{v_{3j} : j \in \{1, 2, \ldots, k\}\}$, $V_0^* = V(P_m) \setminus (V_3^* \cup \{v_{3k+2}\})$ and $V_2^* = \{v_{3k+2}\}$. Since $(V_0^* \setminus \{v_1\}, \varnothing, V_2^*, V_3^*)$ is a γ_{dR}-function of $P_m - v_1 \equiv P_{m-1}$, $g = (V_0 \cup V_0^*, \varnothing, V_2 \cup V_2^*, V_3 \cup V_3^*)$ is a γ_{dR}-function of $T_{n,m}$. Thus, $\gamma_{dR}(T_{n,m}) = n + 1 + \gamma_{dR}(P_{m-1})$. □

Let G and H be graphs with disjoint vertex sets. The *join* of G and H is the graph $G + H$ with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Proposition 13. (join of graphs) *Let G and H be nontrivial graphs. Then*

$$3 \leq \gamma_{dR}(G + H) \leq 6. \tag{1}$$

More precisely,

(*i*) $\gamma_{dR}(G + H) = 3$ *if and only if* $\gamma_{dR}(G) = 3$ *or* $\gamma_{dR}(H) = 3$.

(*ii*) $\gamma_{dR}(G + H) = 4$ *if and only if* $\min\{\gamma_{dR}(G), \gamma_{dR}(H)\} = 4$.

(*iii*) $\gamma_{dR}(G + H) = 5$ *if and only if* $\min\{\gamma_{dR}(G), \gamma_{dR}(H)\} = 5$.

(*iv*) $\gamma_{dR}(G + H) = 6$ *if and only if* $\gamma_{dR}(G) \geq 6$ *and* $\gamma_{dR}(H) \geq 6$.

Proof. Since $G + H$ is nontrivial, $\gamma_{dR}(G + H) \geq 3$. Now, let $u \in V(G)$ and $v \in V(G)$. Then $f = (V(G + H) \setminus \{u, v\}, \varnothing, \varnothing, \{u, v\}) \in DRD(G + H)$. Thus, $\gamma_{dR}(G + H) \leq \omega_{G+H}(f) = 6$.

To prove (*i*), we have from Proposition 5,

$$\begin{aligned} \gamma_{dR}(G + H) = 3 \quad &\Longleftrightarrow \quad \gamma(G + H) = 1 \\ &\Longleftrightarrow \quad \gamma(G) = 1 \text{ or } \gamma(H) = 1 \end{aligned}$$

$$\iff \quad \gamma_{dR}(G) = 3 \text{ or } \gamma_{dR}(H) = 3.$$

For (ii)-(iii), put $\alpha = \min\{\gamma_{dR}(G), \gamma_{dR}(H)\}$. Suppose that $\gamma_{dR}(G+H) = 4$. By Proposition 5, $\gamma(G+H) = \gamma_2(G+H) = 2$. Let $S = \{u, v\}$ be a γ_2-set of $G+H$. If $u \in V(G)$ and $v \in V(H)$, then $\gamma(G) = 1$ and $\gamma(H) = 1$, a contradiction. Thus, $S \subseteq V(G)$ or $S \subseteq V(H)$. Consequently, $\gamma(G) = \gamma_2(G) = 2$ or $\gamma(H) = \gamma_2(H) = 2$. By Proposition 5, $\gamma_{dR}(G) = 4$ or $\gamma_{dR}(H) = 4$. By (i), $\alpha = 4$. Conversely, suppose that $\alpha = 4$, and let f be a γ_{dR}-function of G. Then $g = (V_0 \cup V(H), V_1, V_2, V_3) \in DRD(G+H)$ with $\omega_{G+H}(g) = 4$. Hence, $\gamma_{dR}(G+H) \leq \omega_{G+H}(g) = \omega_G(f) = 4$. By (i), $\gamma_{dR}(G+H) = 4$.

Suppose that the $\gamma_{dR}(G+H) = 5$. By (i) and (ii), $\alpha \geq 5$. It follows from Proposition 6 that $\gamma(G+H) = 2$, $\gamma_2(G+H) \geq 3$ and there exists $v \in V(G+H)$ for which $\gamma((G+H)-v) = 1$. WLOG, assume that $v \in V(G)$. Then $\gamma(G-v) = 1$ and, consequently, $\gamma(G) = 2$. If $\gamma_2(G) = 2$, then $\gamma_{dR}(G) = 4$ by Proposition 5, a contradiction by (ii). Thus, $\gamma_2(G) \geq 3$ so that $\gamma_{dR}(G) = 5$. Thus, $\alpha \leq 5$. Conversely, assume $\alpha = \gamma_{dR}(G) = 5$, and let $f = (V_0, V_1, V_2, V_3)$ be a γ_{dR}-function of G. Then $g = (V_0 \cup V(H), V_1, V_2, V_3) \in DRD(G+H)$. Hence, $\gamma_{dR}(G+H) \leq \omega_{G+H}(g) = \omega_G(f) = 5 = \alpha$. But by (i) and (ii), $\gamma_{dR}(G+H) \geq 5$. Therefore, $\gamma_{dR}(G+H) = 5$.

Finally, (iv) follows immediately from Equation 1 and statements (i), (ii) and (iii). □

The *complementary prism* is the graph $G\overline{G}$ formed from G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. If for each $v \in V(G)$, $\overline{v}$ is the vertex in $\overline{G}$ corresponding to v, then $G\overline{G}$ is formed by adding the edge $v\overline{v}$ for every $v \in V(G)$.

Remark 1. (i) *For any path P_n of order $n \geq 3$,*

$$\gamma_{dR}(P_n\overline{P_n}) = \begin{cases} 3+n, & \text{if } n \equiv 0 \pmod 3, \\ 3+(n+1), & \text{if } n \equiv 1,2 \pmod 3. \end{cases}$$

(ii) *For any cycle C_n of order $n \geq 3$,*

$$\gamma_{dR}(C_n\overline{C_n}) = \begin{cases} 4+n, & \text{if } n \equiv 1,2,3,5 \pmod 6; \\ 5+n, & \text{if } n \equiv 0,4 \pmod 6. \end{cases}$$

The following lemma is obvious.

Lemma 1. *For any graph G, $\gamma(G\overline{G}) = 1$ if and only if $G = K_1$.*

Proposition 14. *Let G be a nontrivial graph. Then*

(i) $\gamma_{dR}(G) \neq 4$*;*

(ii) $\gamma_{dR}(G\overline{G}) = 3$ *if and only if $G = K_1$; and* (*iii*) $\gamma_{dR}(G\overline{G}) = 5$ *if and only if* $G = \{K_2, \overline{K_2}\}$.

Proof. To prove (*i*), we claim that $\gamma_2(G\overline{G}) \neq 2$. Suppose, in the contrary, that there exist $u, v \in V(G\overline{G})$ such that $S = \{u, v\}$ is a γ_2-set of $G\overline{G}$. If $u, v \in V(G)$, then $\overline{u}v \notin E(G\overline{G})$, a contradiction. Similar contradiction is attained if $u, v \in V(\overline{G})$. Assume $v \in V(G)$ and $u \in V(\overline{G})$. If $u = \overline{v}$, then for each $w \in V(G) \setminus \{v\}$ either $wv \notin E(G\overline{G})$ or $u\overline{w} \notin E(G\overline{G})$, a contradiction. Suppose that $u \neq \overline{v}$. A contradiction is already attained if $\overline{u}v \notin E(G)$. However, if $\overline{u}v \in E(G)$, then $u\overline{v} \notin E(G)$, a contradiction. Therefore, $\gamma_2(G\overline{G}) \neq 2$. By Proposition 5, $\gamma_{dR}(G\overline{G}) \neq 4$.

Clearly, if $G = K_1$, then $\gamma_{dR}(G\overline{G}) = 3$. Suppose that $\gamma_{dR}(G\overline{G}) = 3$. Then $\gamma(G\overline{G}) = 1$, by Proposition 5. Thus, by Lemma 1, $G = K_1$. This proves (*ii*).

Now, we prove (*iii*). If $G \in \{K_2, \overline{K_2}\}$, then $G\overline{G} \cong P_4$ so that $\gamma_{dR}(G\overline{G}) = 5$. Conversely, assume $\gamma_{dR}(G\overline{G}) = 5$. By (*ii*), $G \neq K_1$. Suppose that $G \notin \{K_2, \overline{K_2}\}$. Let u, v and w be distinct vertices of G. Then $\overline{u}, \overline{w} \in V(G\overline{G}) \setminus N_{G\overline{G}}[v]$. This means that $|V(G\overline{G}) \setminus N_{G\overline{G}}[v]| \geq 2$ for all $v \in V(G)$. Similarly, $|V(G\overline{G}) \setminus N_{G\overline{G}}[v]| \geq 2$ for all $v \in V(\overline{G})$. Therefore, $|V(G\overline{G}) \setminus N_{G\overline{G}}[v]| \geq 2$ for all $v \in V(G\overline{G})$. This is a contradiction to Proposition 6. Therefore, $G \in \{K_2, \overline{K_2}\}$. □

Theorem 1. (complementary prism) *Let G be a graph of order $n \geq 3$. Assume $\gamma_{dR}(\overline{G}) \leq \gamma_{dR}(G)$. Then*

$$1 + \gamma_{dR}(G) \leq \gamma_{dR}(G\overline{G}) \leq \rho,$$

where

$$\rho = \min\{\omega_G(f) + 2\,(n - |V_3|) - |V_2| : f = (V_0, V_1, V_2, V_3) \in DRD(G) \cup DRD(\overline{G})\}.$$

Moreover, these bounds are sharp.

Proof. Let $f = (V_0, V_1, V_2, V_3) \in DRD(G)$. Extend f to a function on $V(G\overline{G})$ by defining

$$f(\overline{v}) = \begin{cases} 0, & \text{if } v \in V_3; \\ 1, & \text{if } v \in V_2; \\ 2, & \text{if } v \in V_0 \cup V_1. \end{cases}$$

Then $f \in DRD(G\overline{G})$ so that $\gamma_{dR}(G\overline{G}) \leq \omega_G(f) + 2\,(n - |V_3|) - |V_2|$. Thus, $\gamma_{dR}(G\overline{G}) \leq \rho$.

Now, we show the left-hand inequality. Let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of $G\overline{G}$. If $V(\overline{G}) \subseteq V_0$, then $V_3 = V(G)$ so that $\gamma_{dR}(G\overline{G}) = 3|V_3| = 3n \geq 1 + \gamma_{dR}(G)$. Suppose that $V(\overline{G}) \cap (V_2 \cup V_3) \neq \varnothing$. Let $A = \{v \in V_0 \cap V(G) : V_3 \cap N_{G\overline{G}}(v) = \{\overline{v}\}\}$, $B = \{v \in V_0 \cap V(G) : \overline{v} \in V_2 \text{ and } |V_2 \cap N_{G\overline{G}}(v)| = 2\}$ and $C = \{v \in V_1 \cap V(G) : (V_2 \cup V_3) \cap N_{G\overline{G}}(v) = \{\overline{v}\}$. Define $g = (V_0^*, V_1^*, V_2^*, V_3^*)$ on $V(G)$ by

$$g(x) = \begin{cases} f(x), & \text{if } x \in V(G) \setminus (A \cup B \cup C); \\ 2, & \text{if } x \in A \cup C; \\ 1, & \text{if } x \in B. \end{cases}$$

It follows that $g \in DRD(G)$ with $V_0^* = (V_0 \cap V(G)) \setminus (A \cup B)$, $V_1^* = B \cup [(V_1 \cap V(G)) \setminus C]$, $V_2^* = [V_2 \cap V(G)] \cup A \cup C$ and $V_3^* = V_3 \cap V(G)$. Moreover,

$$\gamma_{dR}(G\overline{G}) = \omega_G(g) + \sum_{x \in V(\overline{G})} f(x) - 2|A| - |B| - |C| \geq \omega_G(g) + 1 \geq \gamma_{dR}(G) + 1.$$

To show sharpness of the lower bound, note that by Proposition 14, $\gamma_{dR}(K_1\overline{K_1}) = 3 = 1 + \gamma_{dR}(K_1)$. For the upper bound, pick $G = K_n$, $n \geq 3$. Observe that $\gamma_{dR}(G\overline{G}) = 3 + 2(n-1) = \rho$. □

Corollary 2. *Let G be a nontrivial graph with isolated vertex v. Then*

$$\gamma_{dR}(G\overline{G}) = 3 + \gamma_{dR}(G - v).$$

Proof. Let f be a γ_{dR}-function of $G - v$. Extend f to a function on $V(G\overline{G})$ by defining $f(\overline{v}) = 3$ and $f(x) = 0$ for all $x \in V(\overline{G}) \cup \{v\}$. Since $f \in DRD(G\overline{G})$, $\gamma_{dR}(G\overline{G}) \leq 3 + \gamma_{dR}(G - v)$.

On the other hand, by Proposition 9, $\gamma_{dR}(G) = 2 + \gamma_{dR}(G - v)$. Thus, $3 + \gamma_{dR}(G - v) = 1 + \gamma_{dR}(G) \leq \gamma_{dR}(G\overline{G})$ by Theorem 1. □

Let G and H be graphs with disjoint vertex sets. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex of the i^{th} copy of H. For convenience, we write H^v to denote the copy of H joined to v and write $H^v + v = H^v + \langle\{v\}\rangle$. If $H = \{u\}$, then $V(H^v) = \{u^v\}$.

Given a function $f = (V_0, V_1, V_2, V_3)$ on $V(G \circ H)$, we write for each $v \in V(G)$, $V_i^v = V_i \cap V(H^v)$ for all $i = 0, 1, 2, 3$. Observe also that

$$\omega_{G \circ H}(f) = \sum_{v \in V(G)} \omega_{H^v + v}(f|_{H^v + v}).$$

Proposition 15. *Let G be a nontrivial connected graph and H any graph, and let $f = (V_0, V_1, V_2, V_3)$ be a function on $V(G \circ H)$. Then $f \in DRD(G \circ H)$ if and only if each of the following holds for f:*

- (i) *For each $v \in (V_0 \cup V_1) \cap V(G)$, $f|_{H^v} \in DRD(H^v)$. Moreover, for each $v \in V_0 \cap V(G)$, if $|V_2^v| = 1$ and $|V_3^v| = 0$, then $|(V_2 \cup V_3) \cap N_G(v)| \geq 1$.*
- (ii) *For each $v \in V_2 \cap V(G)$, $V_2^v \cup V_3^v$ dominates V_0^v.*

Proof. Assume that $f \in DRD(G \circ H)$ and let $v \in (V_0 \cup V_1) \cap V(G)$. To show that $f|_{H^v} \in DRD(H^v)$, first let $u \in V_0^v$. Note that $N_{G \circ H}(u) = \{v\} \cup N_{H^v}(u)$. If $|V_2 \cap N_{G \circ H}(u)| \geq 2$, then $|V_2^v \cap N_{H^v}(u)| \geq 2$. On the other hand, if $|V_3 \cap N_{G \circ H}(u)| \geq 1$, then $|V_3^v \cap N_{H^v}(u)| \geq 1$. Next, let $u \in V_1^v$. Then there exists $w \in V_2 \cup V_3$ such that $w \in N_{G \circ H}(u)$. Necessarily, $w \in V_2^v \cup V_3^v$ and $w \in N_{H^v}(u)$. Therefore, $f|_{H^v} \in DRD(H^v)$.

Now, let $v \in V_0 \cap V(G)$ and, suppose that $|V_2^v| = 1$ and $V_3^v = \varnothing$. If $u \in V_3 \cap N_{G \circ H}(v)$, then $u \in V_3 \cap N_G(v)$. Suppose that $|V_2 \cap N_{G\circ H}(v)| \geq 2$. Since $|V_2^v| = 1$, $|V_2 \cap N_G(v)| \geq 1$. This completely proves (i).

To prove (ii), let $v \in V_2 \cap V(G)$ and $u \in V_0^v$. Suppose there exists $\{w, z\} \subseteq V_2 \cap N_{G\circ H}(u)$. If $w = v$, then $z \in V_2^v$ and $zu \in E(H^v)$. Suppose there exists $w \in V_3 \cap N_{G\circ H}(u)$. Then as $w \neq v$, $w \in V_3^v$ and $wu \in E(H^v)$. This means that $V_2^v \cup V_3^v$ dominates V_0^v.

Conversely, assume that (i) and (ii) all hold for f. Let $u \in V_0$, and let $v \in V(G)$ for which $u \in V(H^v + v)$. First, suppose that $u = v$. If $V_3^v \neq \varnothing$ and $w \in V_3^v$, then $w \in V_3 \cap N_{G\circ H}(u)$. Suppose that $V_3^v = \varnothing$. Since $f|_v \in DRD(H^v)$, $V_2^v \neq \varnothing$. If $|V_2^v| \geq 2$, then $|V_2 \cap N_{G\circ H}(u)| \geq 2$. Suppose that $|V_2^v| = 1$. By Condition (i), $|\,(V_2 \cup V_3) \cap N_G(v)| \geq 1$. This means that $|V_2 \cap N_{G\circ H}(u)| \geq 2$ or $|V_3 \cap N_{G\circ H}(u)| \geq 1$.

Next, suppose that $u \in V_0^v$. If $v \in V_3$, then $|V_3 \cap N_{G\circ H}(u)| \geq 1$. If $v \in V_0 \cup V_1$, then by Condition (i), $|V_2^v \cap N_{H^v}(u)| \geq 2$ or $|V_3^v \cap N_{H^v}(u)| \geq 1$. This means that $|V_2 \cap N_{G\circ H}(u)| \geq 2$ or $|V_3 \cap N_{G\circ H}(u)| \geq 1$. Now, suppose that $v \in V_2$. By Condition (ii), there exists $w \in V_2^v \cup V_3^v$ for which $w \in N_{H^v}(u)$. If $w \in V_3^v$, then $|V_3 \cap N_{G\circ H}(v)| \geq 1$. If $w \in V_2^v$, then $\{w, v\} \subseteq V_2 \cap N_{G\circ H}(u)$.

Finally, let $u \in V_1$. If $u \in V(G)$, then since $f|_{H^v} \in DRD(H^u)$ (by (i)), $V_2^u \cup V_3^u \neq \varnothing$, say $w \in V_2^u \cup V_2^u$. Then $w \in (V_1 \cup V_2) \cap N_{G\circ H}(u)$. Suppose that $u \in V(H^v)$ for some $v \in V(G)$. If $v \in V_2 \cup V_3$, then $v \in (V_1 \cup V_2) \cap N_{G\circ H}(u)$. If $v \in V_0 \cup V_1$, then as $f|_v \in DRD(H^v)$ (by (i)), there exists $w \in V_2^v \cup V_3^v$ such that $w \in N_{H^v}(u)$. This means that $w \in V_2 \cup V_3$ and $w \in N_{G\circ H}(u)$. Therefore, $f \in DRD(G \circ H)$. □

Corollary 3. *Let G be a nontrivial connected graph of order n. Then*

(i) $\gamma_{dR}(G \circ K_1) = 3n - \max\{|V_0| : f = (V_0, V_1, V_2.V_3) \in DRD(G)\}$.

(ii) $\gamma_{dR}(G \circ H) = 3n$ for all nontrivial graphs H.

Proof. For (i): Let $\alpha = 3n - \max\{|V_0| : f = (V_0, V_1, V_2.V_3) \in DRD(G)\}$ and put $V(K_1) = \{u\}$. Let $f = (V_0, V_1, V_2, V_3) \in DRD(G)$ for which $|V_0|$ is maximum. Define $V_0^* = V_0 \cup \{u^v : v \in V_3\}$, $V_1^* = V_1 \cup \{u^v : v \in V_2\}$, $V_2^* = V_2 \cup \{u^v : v \in V_0 \cup V_1\}$ and $V_3^* = V_3$. By Proposition 15, $g = (V_0^*, V_1^*, V_2^*, V_3^*) \in DRD(G \circ K_1)$. Thus, $\gamma_{dR}(G \circ K_1) \leq 3(n - |V_0|) + 2|V_0| = 3n - |V_0| = \alpha$.

To get the other inequality, let $f = (V_0, \varnothing, V_2, V_3)$ be a γ_{dR}-function of $G \circ K_1$. First, we claim that $V_2 \cap V(G) = \varnothing$. Suppose not, and let $w \in V_2 \cap V(G)$. Since f is a γ_{dR}-function, $u^w \in V_1$, a contradiction to the choice of f. Next, we claim that $f|_G \in DRD(G)$. Let $v \in V_0 \cap V(G)$. If $u^v \in V_3$, then $g = (V_0 \setminus \{v\}, \{v, u^v\}, V_2, V_3 \setminus \{u^v\}) \in DRD(G \circ K_1)$ with $\omega_{G\circ K_1}(g) = \omega_{G\circ K_1}(f) - 1$, a contradiction. Thus, $u^v \in V_2$. Since $V_2 \cap V(G) = \varnothing$, there exists $w \in V_3 \cap V(G)$ for which $vw \in E(G)$. Since $V_1 \cap V(G) = \varnothing$, $f|_G = (V_0^*, V_1^*, V_2^*, V_3^*) \in DRD(G)$ with $V_0^* = V_0 \cap V(G)$, $V_1^* = V_2^* = \varnothing$ and $V_3^* = V_3$. Observe also that for each

$v \in V(G)$, either $u^v \in V_0$ or $u^v \in V_2$. More precisely, $u^v \in V_0$ if and only if $v \in V_3$ and $u^v \in V_2$ if and only if $v \in V_0$. Thus

$$\begin{aligned}\gamma_{dR}(G \circ K_1) = \omega_{G\circ K_1}(f) &= 3|V_3| + 2|V_0 \cap V(G)| \\ &= 3|V_3| + 3|V_0 \cap V(G)| - |V_0 \cap V(G)| \\ &= 3n - |V_0^*| \\ &\geq \alpha.\end{aligned}$$

For (ii): By Proposition 15, $f = (\cup_{v\in V(G)} V(H^v), \varnothing, \varnothing, V(G)) \in DRD(G \circ H)$. Thus,

$$\gamma_{dR}(G \circ H) \leq 3|V(G)| = 3n.$$

On the other hand, if $f = (V_0, V_1, V_2, V_3) \in DRD(G \circ H)$, then $\omega_{H^v+v}(f|_{H^v+v}) \geq 3$ for each $v \in V(G)$. Thus,

$$\gamma_{dR}(G \circ H) = \omega_{G\circ H}(f) = \sum_{v\in V(G)} \omega_{H^v+v}(f|_{H^v+v}) \geq 3n.$$

□

The succeeding corollary, which are found in [19], are immediate consequences of Corollary 3(i).

Corollary 4. [19]

(i) $\gamma_{dR}(P_n \circ K_1) = \begin{cases} \frac{7n}{3}, & \text{if } n = 3k, \\ \frac{7n+2}{3}, & \text{if } n = 3k+1, \\ \frac{7n+1}{3}, & \text{if } n = 3k+2. \end{cases}$

(ii) $\gamma_{dR}(C_n \circ K_1) = \begin{cases} \frac{7n}{3}, & \text{if } n = 3k, \\ \frac{7n+2}{3}, & \text{if } n = 3k+1, \\ \frac{7n+1}{3}, & \text{if } n = 3k+2. \end{cases}$

(iii) $\gamma_{dR}(K_n \circ K_1) = 2n + 1$.

(iv) $\gamma_{dR}(K_{p,q} \circ K_1) = \begin{cases} 2(p+q)+1, & \text{if } p = 1 \text{ or } q = 1, \\ 2(p+q+1), & \text{otherwise.} \end{cases}$

The *lexicographic product* of graphs G and H is the graph $G[H]$ with $V(G[H])= V(G) \times V(H)$ and $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if and only if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

For $S \subseteq V(G[H])$, we write

$$S_G = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}.$$

S_G is referred to as the *G-projection* of S in $G[H]$. For a graph G, we define

$$\mathcal{C}_G = \{f = (V_0, \varnothing, V_2, V_3) \in DRD(G) : V_2 \setminus N_G(V_2 \cup V_3) = \varnothing\}.$$

Since $f = (\varnothing, \varnothing, V(G), \varnothing) \in \mathcal{C}_G$, $\mathcal{C}_G \neq \varnothing$.

Proposition 16. *Let G be a connected noncomplete graph and H any nontrivial graph with $\gamma(H) = 1$. Then*

$$\gamma_{dR}(G[H]) \leq \min\{\omega_G(f) : f = (V_0, \varnothing, V_2, V_3) \in \mathcal{C}_G\}.$$

Moreover, this upper bound is sharp.

Proof. Put $\alpha = \min\{\omega_G(f) : f = (V_0, \varnothing, V_2, V_3) \in \mathcal{C}_G\}$, and let $v \in V(H)$ for which $N_H[v] = V(H)$. Let $f = (V_0, \varnothing, V_2, V_3) \in \mathcal{C}_G$. Put $V_1^* = \varnothing$, $V_2^* = V_2 \times \{v\}$, $V_3^* = V_3 \times \{v\}$ and $V_0^* = V(G[H]) \setminus (V_2^* \cup V_3^*)$. Let $(x, y) \in V_0^*$. If $x \in V_3$, then $(x, v) \in V_3^* \cap N_{G[H]}((x, y))$.

Suppose that $x \in V_2$. Then $y \neq v$. Since $f \in \mathcal{C}_G$, there exists $w \in V_2 \cap N_G(x)$ or there exists $z \in V_3 \cap N_G(x)$. The former implies that $(x, v), (w, v) \in V_2^* \cap N_{G[K_p]}((x, y))$. The latter, on the other hand, implies that $(z, v) \in V_3^* \cap N_{G[K_p]}((x, y))$.

Finally, suppose that $x \in V_0$. Since $f \in DRD(G)$, there exists $u \in V_3 \cap N_G(x)$ or there exist distinct $w, z \in V_2 \cap N_G(x)$. This means that $(u, v) \in V_3^* \cap N_{G[K_p]}((x, y))$ or we have distinct $(w, v), (z, v) \in N_{G[H]}((x, y))$.

Accordingly, $g = (V_0^*, V_1^*, V_2^*, V_3^*) \in DRD(G[H])$. Moreover,

$$\omega_{G[H]}(g) = 2|V_2^*| + 3|V_3^*| = 2|V_2| + 3|V_3| = \omega_G(f).$$

Therefore, $\gamma_{dR}(G[H]) \leq \omega_G(f)$. Since f is arbitrary, $\gamma_{dR}(G[H]) \leq \alpha$.

To show sharpness, consider the lexicographic product of $G = P_4 = [v_1, v_2, v_3, v_4]$ and $H = P_3$ as shown in Figure 3. We have for this case, $\gamma_{dR}(G[H]) = 6 = \omega(f)$, where $f = (\{v_2, v_3\}, \varnothing, \varnothing, \{v_1, v_4\})$. □

The example presented in the proof of Proposition 16 also shows that $\min\{\omega_G(f) : f = (V_0, \varnothing, V_2, V_3) \in \mathcal{C}_G\}$ need not be determined by a γ_{dR}-function f of G.

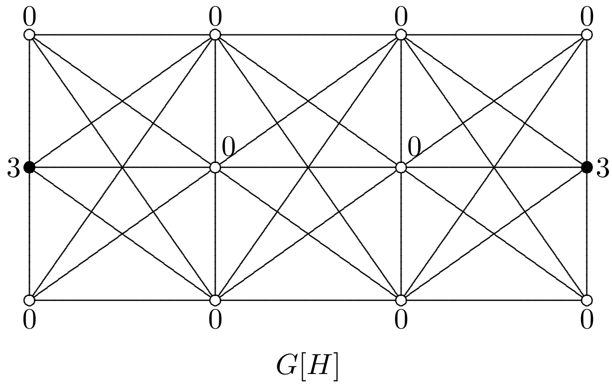

Figure 3: Graph $G[H]$ where $G = P_4$ and $H = P_3$

Proposition 17. *Let G be a connected noncomplete graph and $p \geq 2$. Then*

$$\gamma_{dR}(G[K_p]) = \min\{\omega_G(f) : f = (V_0, \varnothing, V_2, V_3) \in \mathcal{C}_G\}.$$

Proof. Put $\alpha = \min\{\omega_G(f) : f = (V_0, \varnothing, V_2, V_3) \in \mathcal{C}_G\}$, and let $v \in V(K_p)$. By Proposition 16, $\gamma_{dR}(G[K_p]) \leq \alpha$.

To get the other inequality, let $f = (V_0, V_1, V_2, V_3)$ be a γ_{dR}-function of $G[K_p]$. We assume that $V_1 = \varnothing$. First, we claim the following:

(*i*) $(V_2)_G \cap (V_3)_G = \varnothing$;

(*ii*) For each $x \in (V_2)_G$, $|\{y : (x, y) \in V_2\}| = 1$; and

(*iii*) For each $x \in (V_3)_G$, $|\{y : (x, y) \in V_3\}| = 1$.

For suppose that $u \in (V_2)_G \cap (V_3)_G$, and let $w \in V(K_p)$ for which $(u, w) \in V_2$. Then $f^* \in DRD(G[K_p])$, where f^* is defined on $V(G[K_p])$ by $f^*((x, y)) = f((x, y))$ for all $(x, y) \in V(G[K_p]) \setminus \{(u, w)\}$ and $f^*((u, w)) = 0$. This is a contradiction since $\omega_{G[K_p]}(f^*) < \omega_{G[K_p]}(f)$ and f is a γ_{dR}-function. This proves claim (*i*). To prove (*ii*), suppose that for some $u \in (V_2)_G$, we have $(u, w), (u, t) \in V_2$. Then $f^* \in DRD(G[K_p])$, where f^* is defined on $V(G[K_p])$ by $f^*((u, w)) = 3$, $f^*((u, t)) = 0$ and $f^*((x, y)) = f((x, y))$ for all $(x, y) \in V(G[K_p]) \setminus \{(u, w), (u, t)\}$. Since $\omega_{G[K_p]}(f^*) < \omega_{G[K_p]}(f)$, this is a contradiction. Claim (*iii*) is clear.

Let $A = (V_2)_G$, $B = (V_3)_G$ and $C = V(G) \setminus (A \cup B)$, and define $V_0^* = V(G[K_p]) \setminus ((A \cup B) \times \{v\})$, $V_1^* = \varnothing$, $V_2^* = A \times \{v\}$ and $V_3^* = B \times \{v\}$. Define the function $g = (V_0^*, \varnothing, V_2^*, V_3^*)$ on $V(G[K_p])$. More specifically,

$$g((x, y)) = \begin{cases} 3, & \text{if } x \in B \text{ and } y = v, \\ 2, & \text{if } x \in A \text{ and } y = v, \\ 0, & \text{else} \end{cases}$$

Let $(x,y) \in V_0^*$. We consider the following cases:

Case 1: Assume $x \in A$ and $y \neq v$. Since $p \geq 2$, Claim (ii) implies that there exists $w \in V(K_p)$ for which $(x,w) \in V_0$. Thus, there exists $(a,b) \in V_3 \cap N_{G[K_p]}((x,w))$ or there exist distinct $(c,d),(e,f) \in V_2 \cap N_{G[K_p]}((x,w))$. If the former holds, then $(a,v) \in V_3^* \cap N_{G[K_p]}((x,y))$. Suppose the latter holds. By Claim (ii), $c \neq e$ so that we have distinct points $(c,v),(e,v) \in V_2^* \cap N_{G[K_p]}((x,y))$. We note here that it is possible to have $x = c$ or $x = e$.

Case 2: Assume $x \in B$ and $y \neq v$. Then $(x,v) \in V_3^* \cap N_{G[K_p]}((x,y))$.

Case 3: Assume $x \in (V_0)_G \setminus (A \cup B)$. Then $(x,w) \in V_0$ for all $w \in V(K_p)$. Since $f \in DRD(G[K_p])$, there exists $(a,b) \in V_3 \cap N_{G[K_p]}((x,y))$ or there exist distinct $(c,d),(e,f) \in V_2 \cap N_{G[K_p]}((x,y))$. If the former holds, then $(a,v) \in V_3^* \cap N_{G[K_p]}((x,y))$. Suppose the latter holds. By Claim (ii), x, c and e are distinct vertices of G and $(c,v),(e,v) \in V_2^* \cap N_{G[K_p]}((x,y))$.

All of the above imply that $g \in DRD(G[K_p])$. Since f is a γ_{dR}-function, $\omega_{G[K_p]}(g) = \omega_{G[K_p]}(f)$. Thus, $\omega_{G[K_p]}(f) \geq \omega_{G[K_p]}(g) = 2|A| + 3|B|$.

Now consider the function $h = (C, \varnothing, A, B)$ on $V(G)$. Let $x \in C$. Then, in particular, $(x,v) \in V_0^*$. Thus, there exists $u \in B$ such that $(u,v) \in N_{G[K_p]}((x,v))$ or there exist distinct $w,z \in A$ for which $(w,v),(z,v) \in N_{G[K_p]}((x,v))$. This means that there exists $u \in B \cap N_G(x)$ or there exist distinct $w,z \in A \cap N_G(x)$. Therefore, $h \in DRD(G)$ with $\omega_G(h) = 2|A| + 3|B|$. Let $x \in A \setminus N_G(A \cup B)$, and pick $y \in V(K_p) \setminus \{v\}$. Then $(x,y) \in V_0^*$. In view of Claim(iii), there exists $(w,z) \in V_2^* \cap N_{G[K_p]}((x,y))$. This means that either $w = x$ or $w \in N_G(x)$, a contradiction. Thus, $A \setminus N_G(A \cup B) = \varnothing$ and $h \in \mathcal{C}_G$. Finally, therefore, $\gamma_{dR}(G[K_p]) \geq \omega_G(h) \geq \alpha$. □

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