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Convex Hop Domination in Graphs

Javier A. Hassan[1,*], Sergio R. Canoy, Jr.[2], Chrisley Jade Saromines[2]

[1] *Mathematics and Sciences Department, College of Arts and Sciences, MSU Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*
[2] *Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Algebra, and Analysis- PRISM, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let G be an undirected connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A set $C \subseteq V(G)$ is called convex hop dominating if for every two vertices $x, y \in C$, the vertex set of every x-y geodesic is contained in C and for every $v \in V(G) \setminus C$, there exists $w \in C$ such that $d_G(v, w) = 2$. The minimum cardinality of convex hop dominating set of G, denoted by $\gamma_{conh}(G)$, is called the convex hop domination number of G. In this paper, we show that every two positive integers a and b, where $2 \leq a \leq b$, are realizable as the connected hop domination number and convex hop domination number, respectively, of a connected graph. We also characterize the convex hop dominating sets in some graphs and determine their convex hop domination numbers.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop domination, hop domination number, convex set, convex hop dominating set, convex hop domination number

1. Introduction

Hop domination, a concept introduced and initially studied by Natarajan et al. in [18], has become one of the topics of investigation recently. So far, there is a significant number of variants of hop domination that have been defined and investigated. Some studies on hop domination, its variants, and related concepts can be found in [1], [2], [5], [8], [7], [9], [13], [14], [15], [19], [20], and [21].

Another interesting topic that had caught the attention of several researchers is convexity. Convexity is a concept that appears in many areas of mathematics (e.g. real analysis, topology, geometry, functional analysis). In Graph Theory, the concept can easily find a graph-theoretic formulation. Convexity in graphs is discussed in the book by

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Email addresses: `javier.hassan@g.msuiit.edu.ph` (J. Hassan),
`sergio.canoy@g.msuiit.edu.ph` (S. Canoy),
`chrisley.saromines@g.msuiit.edu.ph` (C. Saromines)

Buckley and Harary [3]. The concept and other types of convexity are studied in [4], [6], and [11]. The concept is also combined with many other parameters. One well-known formed combination is convex domination. This variation of domination is studied in [4], [10], [16], and [17]. In this paper, we introduce and study convex hop domination. This study is motivated by the introduction of hop domination and convex domination. Just like convex domination, we believe that this new parameter will yield significant results in the topic of domination and can lead to other interesting research directions in the future.

2. Terminology and Notation

Let $G = V(G), E(G))$ be an undirected graph. For any two vertices u and v of G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u,v)$ is called a *u-v geodesic*. The interval $I_G[u,v]$ consists of u, v, and all vertices lying on a u-v geodesic. The interval $I_G(u,v) = I_G[u,v] \setminus \{u,v\}$. Vertices u and v are adjacent (or neighbors) if $uv \in E(G)$. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if for every $v \in V(G) \setminus D$ (resp. $v \in V(G)$), there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$ (resp. $N_G(D) = V(G)$). The *domination number* (resp. *total domination number*) of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the minimum cardinality of a dominating (resp. total dominating) set in G. Any dominating (resp. total dominating) set in G with cardinality $\gamma(G)$ (resp. $\gamma_t(G)$), is called a γ-set (resp. γ_t-set) in G. If $\gamma(G) = 1$ and $\{v\}$ is a dominating set in G, then we call v a *dominating vertex* in G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u,v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v,u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The *closed hop neighborhood* of X is the set $N_G^2[X] = N_G^2(X) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* in G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality among all hop dominating sets in G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set. A hop dominating set S is *connected hop dominating* if $\langle S \rangle$ is connected. The minimum cardinality among all connected hop dominating sets of G, denoted by $\gamma_{ch}(G)$, is called the *connected hop domination number* of G. Any connected hop dominating set with cardinality equal to $\gamma_{ch}(G)$ is called a γ_{ch}-set.

A set $C \subseteq V(G)$ is *convex set* if for every two vertices $x, y \in C$, the vertex set of every x-y geodesic is contained in C, that is, $I_G[x,y] \subseteq C$. The largest cardinality of a proper convex set in G, denoted by $con(G)$, is called the *convexity number* of G. A set $C \subseteq V(G)$

is called a *convex dominating set* (resp. *convex hop dominating set*) if C is both convex and dominating (resp. convex and hop dominating). The minimum cardinality among all convex dominating (resp. convex hop dominating) sets in G, denoted by $\gamma_{con}(G)$ (resp. $\gamma_{conh}(G)$), is called the *convex domination number* (resp. *convex hop domination number*) of G. Any convex dominating (resp. convex hop dominating set) with cardinality equal to $\gamma_{con}(G)$ (resp. $\gamma_{conh}(G)$) is called a γ_{con}-set (resp. γ_{conh}-set).

A nonempty set $S \subseteq V(G)$ is *non-connecting* if for each pair of vertices $v, w \in V(G) \setminus S$ with $d_G(v,w) = 2$, it holds that $N_G(v) \cap N_G(w) \cap S = \varnothing$.

A set $S \subseteq V(G)$ is a *clique* if the subgraph $\langle S \rangle$ induced by S is a complete graph. The maximum cardinality of a clique in G, denoted by $\omega(G)$, is called the clique number of G. A clique S which is also hop dominating in G is called *clique hop dominating*. Whenever G admits a clique hop dominating set, we call the smallest cardinality of a clique hop dominating set in G, denoted by $\gamma_{clh}(G)$, the *clique hop domination number* of G.

A set $C \subseteq V(G)$ is a *pointwise non-dominating set* if for every $v \in V(G) \setminus C$, there exists $u \in C$ such that $v \notin N_G(u)$. The minimum cardinality of a pointwise non-dominating set in G, denoted by $pnd(G)$, is called a *pointwise non-domination number* of G.

A set $S \subseteq V(G)$ is a *clique pointwise non-dominating set* if S is both a clique and a pointwise non-dominating set in G. The smallest cardinality of a clique pointwise non-dominating set in G, denoted by $cpnd(G)$, is called the *clique pointwise non-domination number* of G. Any clique pointwise non-dominating set in G with cardinality $cpnd(G)$ is called a *cpnd*-set in G.

The *shadow graph* $S(G)$ of graph G is constructed by taking two copies of G, say G_1 and G_2, and then joining each vertex $u \in V(G_1)$ to the neighbors of its corresponding vertex $u' \in V(G_2)$.

For a graph G, the complementary prism, denoted by $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex in $\overline{G}$ corresponding to v. In simple terms, the graph $G\overline{G}$ is form from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every vertex $v \in V(G)$.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} \rangle + H^v$. The *lexicographic product* $G[H]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. Any non-empty set $C \subseteq V(G) \times V(H)$ can be expressed as $C = \bigcup\limits_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Specifically, $T_x = \{a \in V(H) : (x,a) \in C\}$ for each $x \in S$.

3. Results

Since every convex set in a connected graph induces a connected graph, every convex hop dominating set is connected hop dominating. We formally state a consequence of this fact here.

Remark 1. *Let G be any connected graph on n vertices. Then $\gamma_{ch}(G) \leq \gamma_{conh}(G)$.*

Remark 2. *The bound given in Remark 1 is tight. Moreover, strict inequality can also be attained.*

For tightness, consider $G = K_{1,5}$. Then $\gamma_{ch}(G) = \gamma_{conh}(G) = 2$. Next, consider the graph G in Figure 1. Let $C = \{c, d, f\}$ and $C' = \{c, d, e, f\}$. Then C and C' are γ_{ch}-set and γ_{conh}-set in G, respectively. Hence, $\gamma_{ch}(G) = 3 < 4 = \gamma_{conh}(G)$.

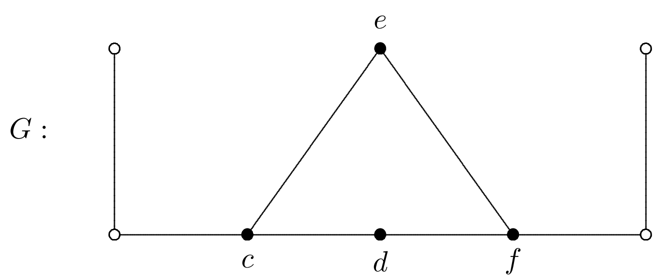

Figure 1: A graph G with $\gamma_{ch}(G) < \gamma_{conh}(G)$.

Theorem 1. *Let G be any connected graph on $n \geq 2$ vertices. Then $2 \leq \gamma_{conh}(G) \leq n$. Moreover, $\gamma_{conh}(G) = 2$ if and only if $\gamma_{ch}(G) = 2$.*

Proof. Clearly, $2 \leq \gamma_{conh}(G) \leq n$.
Suppose $\gamma_{conh}(G) = 2$. By Remark 1, $\gamma_{ch}(G) \leq \gamma_{conh}(G) = 2$. Since $\gamma_{ch}(G) \geq 2$ for any connected graph of order $n \geq 2$, it follows that $\gamma_{ch}(G) = 2$.

Conversely, suppose $\gamma_{ch}(G) = 2$, say, $S = \{x, y\}$ is a γ_{ch}-set of G. Since the graph induced by S is K_2, S is convex. Thus, S is a convex hop dominating set in G and $\gamma_{conh}(G) \leq 2$. By Remark 1, $\gamma_{conh}(G) = 2$.

□

Theorem 2. *Let a and b be positive integers such that $3 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_{ch}(G) = a$ and $\gamma_{conh}(G) = b$.*

Proof. For $a = b$, consider $G = K_a$. Then $\gamma_{ch}(G) = a = \gamma_{conh}(G)$. Suppose $a < b$. Consider the following two cases:

Case 1: $a = 3$.
Let $m = b - a$ and consider the graph G in Figure 2. Let $C = \{x_1, x_2, x_3\}$ and $C' = \{x_1, x_2, x_3, y_1, y_2, \dots, y_m\}$. Then C and C' are γ_{ch}-set and γ_{conh}-set in G, respectively. Thus, $\gamma_{ch}(G) = a$ and $\gamma_{conh}(G) = a + m = b$.

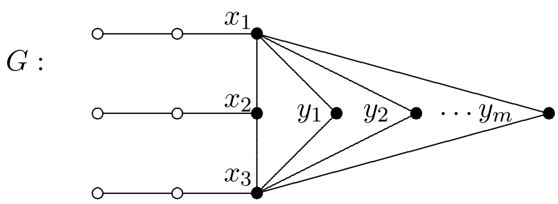

Figure 2: A graph G with $\gamma_{ch}(G) < \gamma_{conh}(G)$.

Case 2: $a \geq 4$.
Let $m = b - a$ and consider the graph G' in Figure 3. Let $D = \{x_1, x_2, \dots, x_a\}$ and $D' = \{x_1, x_2, \dots, x_a, y_1, y_2, \dots, y_m\}$. Then D and D' are γ_{ch}-set and γ_{conh}-set in G', respectively. Thus, $\gamma_{ch}(G') = a$ and $\gamma_{conh}(G') = a + m = b$.

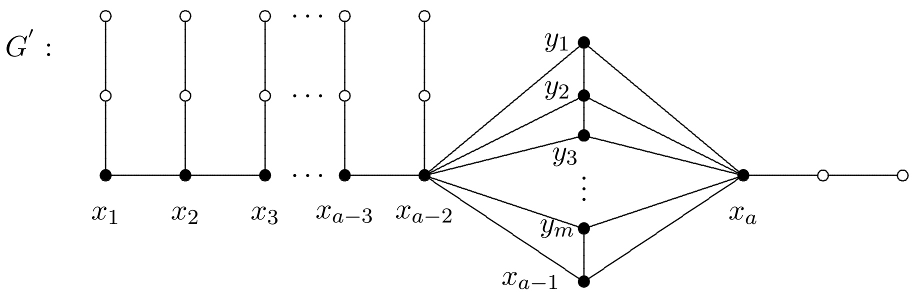

Figure 3: A graph G' with $\gamma_{ch}(G') < \gamma_{conh}(G')$.

This proves the assertion. □

Corollary 1. *Let n be a positive integer. Then there exists a connected graph G such that $\gamma_{conh}(G) - \gamma_{ch}(G) = n$. In other words, $\gamma_{conh} - \gamma_{ch}$ can be made arbitrarily large.*

Proposition 1. *Let n be any positive integer. Then each of the following holds.*

$$(i)\ \gamma_{conh}(P_n) = \begin{cases} 2 & if\ n = 2, 3, 4, 5 \\ n - 4 & if\ n \geq 6. \end{cases}$$

$$(ii)\ \gamma_{conh}(C_n) = \begin{cases} 2 & if\ n = 4,5 \\ 3 & if\ n = 3 \\ n-4 & if\ n \geq 6 \\ n - 4 if 6 \leq n \leq 9 \\ n if n \geq 10. \end{cases}$$

(iii) $\gamma_{conh}(K_n) = n$ *for all* $n \geq 1$.

Proof. (i) Clearly, $\gamma_{conh}(P_n) = 2$ for $n \in \{2,3,4,5\}$. Suppose $n \geq 6$. Let $P_n = [v_1, v_2, \ldots, v_n]$ and consider $C = \{v_3, v_4, \ldots, v_{n-3}, v_{n-2}\}$. Then C is a convex hop dominating set in P_n. Since every convex hop dominating set in P_n contains C, it follows that C is a γ_{conh}-set of P_n. Thus, $\gamma_{conh}(P_n) = n - 4$ for all $n \geq 6$.

(ii) Clearly, $\gamma_{conh}(C_n) = 2$ for $n \in \{4,5\}$ and $\gamma_{conh}(C_n) = 3$ for $n = 3$. Suppose $6 \leq n \leq 9$.. Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ and let C' be a γ_{conh}-set of C_n. We may assume that $v_1 \in C'$ and $v_n \notin C$. Then $C' = \{v_1, v_2, \ldots, v_{n-5}, v_{n-4}\}$. It follows that $\gamma_{conh}(C_n) = n - 4$ for all $6 \leq n \leq 9$.

Next, suppose that n $\geq$ $10. If$ S'$is a \gamma_{conh}$-set of C_n, then $|S'| \geq n - 4$ since S' is a connected hop dominating set. We may assume that $v_1, v_2, ..., v_{n-5}, v_{n-4} \in S'$. Then $d_{C_n}(v_1, v_{n-4}) \leq n - 5$. It follows that $v_{n-3}, v_{n-2}, v_{n-1}, v_n lie in the$ v$_1$- v_{n-4} geodesic. Since S' is convex, $S' = V(C_n)$ and $\gamma_{conh}(C_n) = n$.
(iii) Since $\gamma_{ch}(K_n) = n$ for all $n \geq 1$, it follows from Remark 1 that $\gamma_{conh}(K_n) = n$ for all $n \geq 1$. □

Theorem 3. *Let G be a connected graph of order n. Then $\gamma_{conh}(G\overline{G}) = 2$. In particular, $\{u, \overline{u}\}$ is a γ_{conh}-set of $G\overline{G}$ for any $u \in V(G)$.*

Proof. Clearly, $\gamma_{conh}(G\overline{G}) = 2$ if $n = 1$. Suppose $n \geq 2$. Let $S = \{u, \overline{u}\}$ where $u \in V(G)$ and $\overline{u} \in V(\overline{G})$. Clearly, S is a convex set. Let $w \in V(G\overline{G}) \setminus S$ and consider the following two cases:

Case 1: $w \in V(G)$.
If $uw \in E(G)$, then $d_{G\overline{G}}(\overline{u}, w) = 2$. Suppose that $uw \notin E(G)$, then $\overline{u}\ \overline{w} \in E(\overline{G})$. This implies that $d_{G\overline{G}}(\overline{u}, w) = 2$.

Case 2: $w \in V(\overline{G})$.
Let $w = \overline{z}$, where $z \in V(G)$. If $\overline{u}\ \overline{z} \in E(\overline{G})$, then $d_{G\overline{G}}(u, w) = 2$. If $\overline{u}\ \overline{z} \notin E(\overline{G})$, then $uz \in E(G)$. This means that $d_{G\overline{G}}(u, w) = 2$. Therefore, S is a convex hop dominating set in $G\overline{G}$. Since $G\overline{G}$ is non-trivial, it follows that $\gamma_{conh}(G\overline{G}) = 2$. □

If G_1 and G_2 are the copies of graph G in the definition of the shadow graph $S(G)$ and if $S_{G_1} \subseteq V(G_1)$ and $S_{G_2} \subseteq V(G_2)$, then the sets S'_{G_1} and S'_{G_2} are the sets given by

$$S'_{G_1} = \{a' \in V(G_2) : a \in S_{G_1}\} \text{ and } S'_{G_2} = \{a \in V(G_1) : a' \in S_{G_2}\}.$$

Theorem 4. *Let G be a non-trivial connected graph. Then a proper subset S of $V(S(G))$ is convex in $S(G)$ if and only if one of the following conditions holds:*

(*i*) *S is clique in G_1.*

(*ii*) *S is clique in G_2.*

(*iii*) *$S = S_{G_1} \cup S_{G_2}$ and satisfies the following conditions:*

- (*a*) *$S_{G_1} \cap S'_{G_2} = \varnothing$ and $S'_{G_1} \cap S_{G_2} = \varnothing$.*
- (*b*) *S_{G_1} and S_{G_2} are cliques in G_1 and G_2, respectively.*
- (*c*) *$S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are cliques in G_1 and G_2, respectively.*

Proof. Suppose S is convex in $S(G)$. If $S_{G_2} = \varnothing$, then $S = S_{G_1}$. Suppose S is not a clique in G_1. Then there exist $a, b \in S$ such that $d_{G_1}(a, b) = 2 = d_{S(G)}(a, b)$. It follows that $x \in S$ for all $x \in N_{G_1}(a) \cap N_{G_1}(b)$. Hence, $x' \in S$ for all $x \in N_{G_1}(a) \cap N_{G_1}(b)$. This contradicts the assumption that $S_{G_2} = \varnothing$. Therefore, S is a clique in G_1. Similarly, if $C_{G_1} = \varnothing$, then S is a clique in G_2. Hence, (*i*) and (*ii*) hold.

Next, suppose S_{G_1} and S_{G_2} are both non-empty. Then $S = S_{G_1} \cup S_{G_2}$. Suppose $S_{G_1} \cap S'_{G_2} \neq \varnothing$, say $v \in S_{G_1} \cap S'_{G_2}$. Then $v, v' \in S$. By convexity of S, $x, x' \in S$ for all $x \in N_G(v)$. This implies that $S = V(S(G))$, a contradiction. Therefore, $S_{G_1} \cap S'_{G_2} = \varnothing$. Similarly, $S'_{G_1} \cap S_{G_2} = \varnothing$, showing that (*a*) holds. Now, suppose S_{G_1} is not clique. Then there exist $a, b \in S_{G_1}$ such that $d_{G_1}(a, b) = 2 = d_{S(G)}(a, b)$. Again, by convexity of S, it follows that $x, x' \in S$ for all $x \in N_{G_1}(a) \cap N_{G_1}(b)$. This implies that $S = V(S(G))$, a contradiction. Therefore, S_{G_1} is a clique in G_1. Similarly, S_{G_2} is a clique in G_2, showing that (*b*) holds. Suppose $S_{G_1} \cup S'_{G_2}$ is not a clique in G_1. Then there exist $x, y \in S_{G_1} \cup S'_{G_2}$ such that $d_{G_1}(x, y) = 2$. Since S_{G_1} and S_{G_2} are cliques, we may assume that $x \in S_{G_1}$ and $y \in S'_{G_2}$. Then $y' \in S_{G_2}$. Let $z \in N_G(x) \cap N_G(y)$. Then $z, z' \in N_{S(G)}(x) \cap N_{S(G)}(y')$. Since S is convex, $z, z' \in S$. Since $yz, yz' \in E(S(G))$, $y \in S$ by convexity of S. This would imply that $S = V(S(G))$, a contradiction. Therefore, $S_{G_1} \cup S'_{G_2}$ is a clique in G_1. Similarly, $S'_{G_1} \cup S_{G_2}$ is a clique in G_2. Thus, (*c*) holds.

The converse is clear.

Corollary 2. *Let G be a non-trivial connected graph. Then $con(S(G)) = \omega(G)$.*

Theorem 5. *Let G be a non-trivial connected graph. Then S is a hop dominating set in $S(G)$ if and only if one of the following conditions holds:*

(*i*) *S is a hop dominating set in G_1.*

(*ii*) *S is a hop dominating set in G_2.*

(*iii*) *$S = S_{G_1} \cup S_{G_2}$ such that $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are hop dominating sets in G_1 and G_2.*

Proof. Let S be a hop dominating set in $S(G)$. Set $S_{G_1} = S \cap V(G_1)$ and $S_{G_2} = S \cap V(G_2)$. If $S_{G_2} = \varnothing$, then $S = S_{G_1}$ is a hop dominating set in G_1. If $S_{G_1} = \varnothing$, then $S = S_{G_2}$ is a hop dominating set in G_2. Hence, (i) or (ii) holds. Next, suppose $S_{G_1} \neq \varnothing$ and $S_{G_2} \neq \varnothing$. Let $x \in V(G_1) \setminus S_{G_1} \cup S'_{G_2}$. Then $x \in V(S(G)) \setminus S$. Since S is a hop dominating set in $S(G)$, there exists $y \in S$ such that $d_{S(G)}(x,y) = 2$. If $y \in S_{G_1}$, then we are done. Suppose $y \in S_{G_2}$, say $y = z'$, where $z \in V(G_1)$. Then $z \in S'_{G_2}$ and $d_{S(G)}(x,z) = d_{G_1}(x,z) = 2$. Therefore, $S_{G_1} \cup S'_{G_2}$ is a hop dominating set in G_1. Similarly, $S'_{G_1} \cup S_{G_2}$ is a hop dominating set in G_2. Hence, (iii) holds.

For the converse, suppose (i) holds. Let $a \in V(S(G)) \setminus S$. If $a \in V(G_1) \setminus S$, then there exists $b \in S$ such that $d_{G_1}(a,b) = d_{S(G)}(a,b) = 2$. Suppose $a \in V(G_2)$, say $a = v'$, where $v \in V(G_1)$. If $v \in S$, then $d_{G_1}(a,v) = d_{S(G)}(a,v) = 2$. If $v \notin S$, then there exists $w \in S$ such that $d_{G_1}(v,w) = 2$. It follows that $d_{S(G)}(a,w) = d_{S(G)}(v',w) = 2$. Therefore, S is a hop dominating set in $S(G)$. Similarly, if (ii) holds, then S is a hop dominating set in $S(G)$. Now, suppose (iii) holds. Let $y \in V(S(G)) \setminus S$. Then $y \notin S_{G_1} \cup S_{G_2}$. Suppose $y \in V(G_2) \setminus S_{G_2}$, say $y = z'$, where $z \in V(G_1)$. Then $z \notin S'_{G_2}$. If $z \in S_{G_1}$, then $d_{S(G)}(y,z) = d_{S(G)}(z',z) = 2$. Suppose $z \notin S_{G_1}$. Since $S_{G_1} \cup S'_{G_2}$ is a hop dominating set in G_1, there exists $p \in S_{G_1} \cup S'_{G_2}$ such that $d_{G_1}(p,z) = 2 = d_{S(G)}(p,z)$. If $p \in S_{G_1}$, then $p \in S$ and $d_{S(G)}(p,z') = 2$. If $p \in S'_{G_2}$, then $p' \in S_{G_2} \subseteq S$ and $d_{G_2}(p',z') = d_{S(G)}(p',z') = 2$. Therefore, S is a hop dominating set in $S(G)$. □

Corollary 3. *Let G be a non-trivial connected graph. Then $\gamma_h(S(G)) = \gamma_h(G)$.*

Theorem 6. *Let G be a non-trivial connected graph. Then S is a convex hop dominating set in $S(G)$ if and only if one of the following conditions holds:*

(i) *S is clique hop dominating set in G_1.*

(ii) *S is clique hop dominating set in G_2.*

(iii) *$S = S_{G_1} \cup S_{G_2}$ where*

- (a) *$S_{G_1} \cap S'_{G_2} = \varnothing$ and $S'_{G_1} \cap S_{G_2} = \varnothing$.*
- (b) *S_{G_1} and S_{G_2} are cliques in S_{G_1} and S_{G_2}, respectively.*
- (c) *$S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are clique hop dominating sets in S_{G_1} and S_{G_2}, respectively.*

Proof. Follows from Theorem 4 and Theorem 5.

Consider the following family of graphs:
$\mathcal{B} = \{G : G$ admits a clique hop domination$\}$. Then the following result follows from Theorem 6.

Corollary 4. *Let G be a non-trivial connected graph. Then*

$$\gamma_{conh}(S(G)) = \begin{cases} \gamma_{clh}(G) & if\ G \in \mathcal{B} \\ |V(S(G))| & if\ G \notin \mathcal{B}. \end{cases}$$

Theorem 7. *[7] Let G be a graph of order n. Then $1 \leq cpnd(G) \leq n$. Moreover,*

(i) $cpnd(G) = 1$ if and only if G has an isolated vertex.

(ii) $cpnd(G) = n$ if and only if G is a complete graph.

Corollary 5. *[7] Let n be any positive integer. Then*

(i) $cpnd(P_n) = 2$ for any $n \geq 2$.

(ii) $cpnd(C_n) = 2$ for any $n \geq 4$.

The next result is found in [13].

Theorem 8. *Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is hop dominating set in $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets in G and H, respectively.*

The following two results are obtained in [12].

Theorem 9. *Let G be a connected graph and K_n the complete graph of order n. Then a proper subset $C = S_1 \cup S_2$ of $V(G + K_n)$, where $S_1 \subseteq V(G)$ and $S_2 \subseteq V(K_n)$, is a convex set in $G + H$ if and only if S_1 induces a complete subgraph of G or $V(G) \setminus S_1$ is a non-connecting set and $S_2 = V(K_n)$.*

Theorem 10. *Let G and H be two non-complete connected graphs. Then a proper subset $C = S_1 \cup S_2$ of $V(G+H)$, where $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$, is a convex set in $G+H$ if and only if S_1 and S_2 induce complete subgraphs of G and H, respectively, where it may occur that $S_1 = \varnothing$ or $S_2 = \varnothing$.*

Theorem 11. *Let G and H be two non-complete connected graphs. A set $S \subseteq V(G+H)$ is a convex hop dominating set in $G + H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are clique pointwise non-dominating sets in G and H, respectively.*

Proof. Suppose S is a convex hop dominating set in $G+H$. Then S_G and S_H are both non-empty. Since S is a hop dominating set, S_G and S_H are pointwise non-dominating sets in G and H, respectively by Theorem 8. Since S is a convex set, S_G and S_H are cliques in G and H, respectively, by Theorem 10. Therefore, S_G and S_H are clique pointwise non-dominating sets in G and H, respectively.

Conversely, suppose that $S = S_G \cup S_H$, where S_G and S_H are clique pointwise non-dominating sets in G and H, respectively. Since S_G and S_H are pointwise non-dominating sets, $S = S_G \cup S_H$ is a hop dominating set in $G+H$ by Theorem 8. Since S_G and S_H are cliques, it follows that $S = S_G \cup S_H$ is a convex set in $G+H$ by Theorem 10. Consequently, $S = S_G \cup S_H$ is a convex hop dominating set in $G+H$. □

The next result follows from Theorem 7, Corollary 5 and Theorem 11.

Corollary 6. *Let G and H be two non-complete connected graphs. Then $\gamma_{conh}(G+H) = cpnd(G) + cpnd(H)$. In particular, we have* (*i*) $\gamma_{conh}(P_n + P_m) = 4$ *for all* $n, m \geq 3$*, and*

(*ii*) $\gamma_{conh}(C_n + C_m) = 4$ *for all* $n, m \geq 4$*.*

Theorem 12. *Let G be a connected graph and K_n the complete graph of order n. A set $S \subseteq V(G + K_n)$ is a convex hop dominating set in $G + K_n$ if and only if $S = V(K_n) \cup S_G$ where $V(G) \setminus S_G$ is a non-connecting set and S_G is a pointwise non-dominating set in G.*

Proof. Suppose $S = S_{K_n} \cup S_G$ is a convex hop dominating set of $G + K_n$. By Theorem 8, S_{K_n} and S_G are pointwise non-dominating sets of K_n and G, respectively. Hence, $S_{K_n} = V(K_n)$. Moreover, by Theorem 9, $V(G) \setminus S_G$ is a non-connecting set in G.

Conversely, suppose that $S = V(K_n) \cup S_G$ such that $V(G) \setminus S_G$ is a non-connecting set and S_G is a pointwise non-dominating set in G. Then, by Theorem 8 and Theorem 9, S is a convex hop dominating set of $G + K_n$. □

The next result follows from Theorem 12.

Corollary 7. *Let G a connected graph and K_n the complete graph of order n. Then*

$$\gamma_{conh}(G + K_n) = n + r_G,$$

where

$$r_G = \min\{|S| : V(G)\backslash S \textit{ is non-connecting and } S \textit{ is a pointwise non-dominating set in } G\}.$$

In particular, the following hold:

(*i*) $\gamma_{conh}(K_n + C_n) = \begin{cases} n+3 & if\ n = 3 \\ n+2 & if\ n \geq 4. \end{cases}$

(*ii*) $\gamma_{conh}(K_n + P_n) = n + 2$ *for all* $n \geq 2$*.*

The result that follows is a restatement of a result in [13].

Theorem 13. *Let G and H be any two graphs. A set $C \subseteq V(G)$ is a hop dominating set in $G \circ H$ if and only if $C = A \cup (\cup_{v \in V(G)} C_v)$, where $A \subseteq V(G)$ and $C_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:*

(*i*) *For each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in N_G(w)$ with $C_y \neq \varnothing$.*

(*ii*) *C_w is a pointwise non-dominating set in H^w for each $w \in V(G) \setminus N_G(A)$.*

Theorem 14. *Let G be a non-trivial connected graph and let H be any graph. Then C is a convex hop dominating set in $G \circ H$ if and only if $C = A \cup (\cup_{v \in V(G)} C_v)$, where $A \subseteq V(G)$, $C_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:*

(*i*) *For each $a \in V(G) \setminus A$, there exists $b \in A$ with $d_G(a, b) = 2$ or there exists $y \in A \cap N_G(a)$ with $C_y \neq \varnothing$.*

(*ii*) *A is a convex dominating set in G.*

(*iii*) *$C_v = \varnothing$ for each $v \in V(G) \setminus A$.*

(*iv*) *$V(H^v) \setminus C_v$ is a non-connecting in H^v for each $v \in A \cap N_G(A)$.*

(*v*) *$V(H^v) \setminus C_v$ is a non-connecting set and C_v is a pointwise non-dominating set in H^v if $A = \{v\}$ (that is, if $v \in A \setminus N_G(A)$).*

Proof. Suppose C is a convex hop dominating set in $G \circ H$. By Theorem 13(*ii*), statement (*i*) holds. Let $x, y \in A$ with $x \neq y$. Then x and y are in C. Since C is convex and $I_{G \circ H}[x, y] = I_G[x, y]$, it follows that $I_G[x, y] \subseteq A$. Hence, A is convex. Suppose A is not a dominating set in G. Then there exists $v \in V(G) \setminus N_G[A]$. By Theorem 13(*ii*), C_v is a pointwise non-dominating set in H^v. Also, by Theorem 13(*i*), there exists $x \in A$ with $d_G(v, x) = 2$ or there exists $y \in N_G(v)$ with $C_y \neq \varnothing$. Pick any $p \in C_v$ and let $q \in C$ such that $d_{G \circ H}(v, q) = 2$ ($q = x$ or $q \in C_y$). Then $v \in I_{G \circ H}(p, q)$. By convexity of C, it follows that $v \in C$, a contradiction. Thus, A is a dominating set in G. This shows that (*ii*) holds. Next, let $y \in V(G) \setminus A$. Since A is a dominating set in G, $y \in N_G(A)$. By convexity of C, $C_y = \varnothing$. Hence, (*iii*) holds. Let $v \in A$. Suppose $V(H^v) \setminus C_v$ is not a non-connecting set in H^v. Then there exist $p, q \in C_v$ such that $p \neq q$ and $N_{H^v}(p) \cap N_{H^v}(q) \cap [V(H^v) \setminus C_v] \neq \varnothing$. This implies that C is not convex, a contradiction. Therefore, $V(H^v) \setminus C_v$ is a non-connecting set in H^v, showing that (*iv*) holds. Suppose now that $v \in A \setminus N_G(A)$. Then, by Theorem 13(*ii*), C_v is a pointwise non-dominating set in H^v. Hence, (*v*) also holds.

Conversely, suppose that C has the given form and satisfies (*i*), (*ii*), (*iii*) (*iv*) and (*v*). Since (*i*) and (*v*) hold and A is a dominating set in G, the conditions (*i*) and (*ii*) of Theorem 13 hold. Thus, C is a hop dominating set in $G \circ H$. Next, let $x, y \in C$ with $x \neq y$. Let $v, w \in V(G)$ such that $x \in V(v + H^v)$ and $y \in V(w + H^w)$. Consider the following cases:

Case 1: $v = w$.
If one of x and y is v, say $x = v$, then $y \in C_v$ and $I_{G \circ H}[x, y] = \{x, y\} \subseteq C$. Suppose $x, y \in C_v$. Since $C_v \neq \varnothing$, $v \in A$ by (*iii*). By (*iv*), $V(H^v) \setminus C_v$ is a non-connecting set in H^v. Hence, $I_{G \circ H}[x, y] \subseteq C$.

Case 2: $v \neq w$.
Suppose $x = v$ and $y = w$. Since A is convex, $I_G[x, y] \subseteq A$. Since $I_{G \circ H}[x, y] = I_G[x, y]$, $I_{G \circ H}[x, y] \subseteq C$. Suppose $x = v$ and $y \in C_w$. Then $w \in A$ and, by convexity of A, $I_G[x, w] \subseteq A$. Since

$$I_{G \circ H}[x, y] = I_G[x, w] \cup I_{G \circ H}[w, y] = I_{G \circ H}[x, w] \cup \{y\},$$

it follows that $I_{G \circ H}[x, y] \subseteq C$. The same conclusion holds when $x \in C_v$ and $y = w$. Finally, let $x \in C_v$ and $y \in C_w$. Then, by (*iii*), $v, w \in A$. Again, by convexity of A, $I_{G \circ H}[v, w] = I_G[v, w]$ is contained in $A \subseteq C$. This implies that

$$I_{G \circ H}[x, y] = I_{G \circ H}[v, w] \cup I_{G \circ H}[x, v] \cup I_{G \circ H}[y, w] = I_{G \circ H}[v, w] \cup \{x, y\}$$

is contained in C.

Therefore, C is a convex set in $G \circ H$.

Accordingly, C is a convex hop dominating set in $G \circ H$. $\square$

Let G be a graph. We denote by $\mathcal{D}_G$, $\mathcal{L}_G$, and $\mathcal{I}_H$ the sets containing the dominating vertices, leaves, and isolated vertices of G, respectively. Note that if $\gamma(G) = 1$, then $|\mathcal{D}_G| \geq 1$.

Corollary 8. *Let G be a non-trivial connected graph with $\gamma(G) = 1$ and let H be any graph. Then*

$$\gamma_{conh}(G \circ H) = \begin{cases} 2, & \text{if } |\mathcal{L}_G| \geq 1 \text{ or } |\mathcal{I}_H| \geq 1 \\ 3, & \text{otherwise} \end{cases}$$

Proof. Let $v \in \mathcal{D}_G$. Suppose $|\mathcal{L}_G| \geq 1$, say $w \in \mathcal{L}_G$. Set $A_1 = \{v, w\}$. Then A_1 is a convex dominating set in G. Let $C_u = \varnothing$ for each $u \in V(G)$. Then $C_1 = A_1 \cup (\cup_{u \in V(G)} C_u) = A_1$ is a convex hop dominating set in $G \circ H$ by Theorem 14. Thus, $\gamma_{conh}(G \circ H) = 2$. Next, suppose that $|\mathcal{I}_H| \geq 1$. Pick any $p \in \mathcal{I}_{H^v}$. Then $A_2 = \{v\}$ is a convex dominating set in G. Set $C_v = \{p\}$ and let $C_u = \varnothing$ for all $u \in V(G) \setminus \{v\}$. Then $V(H^v) \setminus C_v$ is a non-connecting set and C_v is a pointwise non-dominating set in H^v. Hence, $C_2 = A_2 \cup (\cup_{z \in V(G)} C_z) = A_2 \cup C_v$ is a convex hop dominating set in $G \circ H$ by Theorem 14. It follows that $\gamma_{conh}(G \circ H) = 2$.

Suppose now that $|\mathcal{L}_G| = 0$ and $|\mathcal{I}_H| = 0$. Again, let $v \in \mathcal{D}_G$. Pick any $z \in V(G) \setminus \{v\}$ and let $A = \{v, z\}$. Then A is a convex dominating set of G. Choose any $q \in V(H^v)$ and let $C_v = \{q\}$. Put $C_x = \varnothing$ for all $x \in V(G) \setminus \{v\}$. Then $C_z = \varnothing$ and $V(H^v) \setminus C_v$ and $V(H^z) \setminus C_z$ are non-connecting sets in H^v and H^z, respectively. By Theorem 14, $C = A \cup (\cup_{y \in V(G)} C_y) = A \cup C_v$ is a convex hop dominating set in $G \circ H$. It follows that $\gamma_{conh}(G \circ H) \leq 3$. Suppose now that $C = A_0 \cup (\cup_{u \in V(G)} S_u)$ is a γ_{conh}-set of $G \circ H$. Suppose first that $|A_0| = 1$, say $A_0 = \{z\}$. Then A_0 is (convex) dominating set in G by Theorem 14(ii). Moreover, $S_u = \varnothing$ for all $u \in V(G) \setminus A_0$ by Theorem 14(iii). Since $|\mathcal{I}_H| = 0$, any pointwise non-dominating set in H^z contains at least two elements, that is, $|S_z| \geq 2$. It follows that $\gamma_{conh}(G \circ H) = |C_0| \geq 3$. Suppose that $|A_0| = 2$, say $A_0 = \{x, y\}$. If $x, y \notin \mathcal{D}_G$, then $C_x \neq \varnothing$ or $C_y \neq \varnothing$ (since x and y are not hop neighbors of a dominating vertex of G). Suppose one of x and y, say x, is a dominating vertex in G. Since $y \notin \mathcal{L}_G$, there exists a vertex $d \in N_G(y) \cap N_G(x)$. This implies that $C_x \neq \varnothing$ or $C_y \neq \varnothing$. In either case, $\gamma_{conh}(G \circ H) = |C_0| \geq 3$. Therefore, $\gamma_{conh}(G \circ H) = 3$. $\square$

For a connected graph G,

$$\gamma^h_{con}(G) = \min\{|S| : S \text{ is a convex dominating and hop dominating set in } G\}.$$

Since $V(G)$ is a convex dominating and hop dominating set, G admits a convex dominating and hop dominating set. Moreover, $\gamma_{con}(G) \leq \gamma^h_{con}(G)$.

Corollary 9. *Let G be a non-trivial connected graph with $\gamma(G) \neq 1$ and let H be any* *graph. Then*

$$\gamma_{conh}(G \circ H) = \begin{cases} \gamma_{con}(G), & \textit{if } \gamma_{con}(G) = \gamma^h_{con}(G) \\ \gamma_{con}(G) + 1, & \textit{otherwise.} \end{cases}$$

Proof. Suppose $\gamma_{con}(G) = \gamma^h_{con}(G)$. Let A be a γ^h_{con}-set of G. Then $|A| \geq 2$. Set $C_v = \varnothing$ for all $v \in V(G)$. Then, by Theorem 14, $C = A$ is a convex hop dominating set in $G \circ H$. Hence, $\gamma_{conh}(G \circ H) \leq \gamma_{con}(G)$. By Theorem 14$(ii)$, it follows that $\gamma_{conh}(G \circ H) = \gamma_{con}(G)$.

Next, suppose that $\gamma_{con}(G) < \gamma^h_{con}(G)$. Let A' be a γ_{con}-set of G. Since $\gamma(G) \neq 1$, $|A'| \geq 2$. The assumption that $\gamma_{con}(G) < \gamma^h_{con}(G)$ implies that A' is not a hop dominating set in G. Hence, there exists $v \notin N^2_G[A']$. Let $x, y \in A'$ with $x \neq y$. Since A' is a dominating set, there exists $w \in A' \cap N_G(v)$. Because A' is convex, $\langle A' \rangle$ is connected. Let $[w_1, w_2, ..., w_k]$, where $w_1 = w$ and $w_k = x$, be a w-x geodesic in $\langle A' \rangle$. Since $v \notin N^2_G[A']$, $vw_j \in E(G)$ for all $j \in \{1, 2, ..., k\}$. In particular, $vx \in E(G)$. Let $[x_1, x_2, ..., x_t]$, where $x_1 = x$ and $x_t = y$, be an x-y geodesic in $\langle A' \rangle$. Again, since $v \notin N^2_G[A']$, $vx_i \in E(G)$ for all $i \in \{1, 2, ..., t\}$. Moreover, by convexity of A', $\langle \{x_1, x_2, ..., x_t\} \rangle$ is complete (otherwise, $v \in A'$, a contradiction). Hence, $xy \in E(G)$. Thus, $\langle A' \rangle$ is complete. Pick any $w \in A'$ and $p \in V(H^w)$. Set $C_w = \{p\}$ and $C_z = \varnothing$ for all $z \in V(G) \setminus \{w\}$. Then $C' = A' \cup C_w$ is a convex hop dominating set in $G \circ H$ by Theorem 14. Hence, $\gamma_{conh}(G \circ H) \leq |C'| = \gamma_{con}(G) + 1$. Now let $C^* = A^* \cup (\cup_{v \in V(G)} R_v)$ be a γ_{conh}-set of $G \circ H$. Then A^* is a convex dominating set in G by Theorem 14. If $|A^*| > \gamma_{con}(G)$, then $|C^*| \geq |A^*| \geq \gamma_{con}(G) + 1$. Suppose $|A^*| = \gamma_{con}(G)$. Since $\gamma_{con}(G) < \gamma^h_{con}(G)$, A^* is not a hop dominating set, say $v \notin N^2_G[A^*]$. Hence, by Theorem 14(i), there exists $y \in A^* \cap N_G(v)$ with $R_y \neq \varnothing$. It follows that $\gamma_{conh}(G \circ H) = |C^*| \geq |A^*| + |R_y| \geq \gamma_{con}(G) + 1$. This establishes the desired equality. □

The next result is found in [13].

Theorem 15. *Let G and H be connected non-trivial graphs. Then $C = \bigcup_{x \in S}[\{x\} \times T_x]$ is a hop dominating set in $G[H]$ if and only if the following conditions hold.*

(i) *S is a hop dominating set in G.*

(ii) *T_x is a pointwise non-dominating set in H for each $x \in S \setminus N^2_G(S)$.*

The next result is a restatement of the one obtained by Canoy and Garces in [12].

Theorem 16. *Let G and H be connected non-complete graphs. Then $C = \bigcup_{x \in S}(\{x\} \times T_x)$ is convex in $G[H]$ if and only if S is a clique in G and T_x is a clique in H for each $x \in S$.*

Theorem 17. *Let G and H be connected non-complete graphs. Then $C = \displaystyle\bigcup_{x \in A}[\{x\} \times T_x]$, where $A \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in A$, is a convex hop dominating set in $G[H]$ if and only if $C = V(G[H])$ or C satisfies the following conditions:*

(i) *A is a clique hop dominating set in G.*

(ii) *T_x is a clique pointwise non-dominating set in H for each $x \in A$.*

Proof. If $C = V(G[H])$, then we are done. Suppose $C \neq V(G[H])$. Then A is a clique in G and T_x is a clique in H for each $x \in A$ by Theorem 16. Since C is hop dominating set, A is a hop dominating set in G by Theorem 15. Since A is a clique, $x \notin N_G^2(A)$ for all $x \in A$. Thus, T_x is a pointwise non-dominating set in H for every $x \in A$ by Theorem 15(ii). Therefore, (i) and (ii) hold.

For the converse, suppose that $C = V(G[H])$. Then C is convex hop dominating in $G[H]$. Next, suppose C satisfies i and (ii). Then by Theorem 15, C is a hop dominating set in $G[H]$. By (i), (ii) and Theorem 16, C is a convex set in $G[H]$. Hence, C is a convex hop dominating set in $G[H]$. □

In the next result, we shall consider the family $\mathcal{C}$ of graphs given by

$\mathcal{C} = \{G : G$ is a connected non-complete graph that admits a clique hop dominating set$\}$.

Corollary 10. *Let G and H be connected non-complete graphs of orders m and n, respectively. Then*

$$\gamma_{conh}(G[H]) = \begin{cases} nm & if\ G \notin \mathcal{C} \\ \gamma_{clh}(G)cpnd(H) & if\ G \in \mathcal{C}. \end{cases}$$

The next result is taken from [10].

Theorem 18. *Let G be a connected graph and K_m the complete graph of order m. A subset $C = \bigcup_{x \in S}(\{x\} \times T_x)$ of $V(G[K_m])$ is convex in $G[K_m]$ if and only if S is convex in G and $T_x = V(K_m)$ for each $x \in S \cap I_G(S)$.*

Theorem 19. *Let G be a connected graph and K_m the complete graph of order m. Then $C = \bigcup\limits_{x \in A}[\{x\} \times T_x]$, where $A \subseteq V(G)$ and $T_x \subseteq V(K_m)$ for each $x \in A$, is a convex hop dominating set in $G[K_m]$ if and only if $C = V(G[H])$ or C satisfies the following conditions:*

(i) *A is a convex hop dominating set in G.*

(ii) *$T_x = V(K_m)$ for each $x \in (A \cap I_G(A)) \cup (A \setminus N_G^2(A))$.*

Proof. Suppose C is a convex hop dominating set of $G[K_m]$. By Theorem 15 and Theorem 18, A is a convex hop dominating set in G and $T_x = V(K_m)$ for each $x \in (A \cap I_G(A)) \cup (A \setminus N_G^2(A))$. Hence, (i) and (ii) hold.

Conversely, suppose that (i) and (ii) hold. Then, by Theorem 15 and Theorem 18, C is a convex hop dominating set in $G[K_m]$. □

Corollary 11. *Let G be a connected graph and K_m the complete graph of order m. Then*

$$\gamma_{conh}(G[K_m]) = \min\{|S| + (m-1)|S^0 \cup (S \setminus N_G^2(S))| : S \textit{ is a convex hop dominating set in } G\},$$

where $S^0 = S \cap I_G(S)$.

Proof. Let $C = \bigcup_{x\in S}[\{x\} \times T_x]$ be a γ_{conh}-set of $G[K_m]$. Then S is a convex hop dominating set and $T_x = V(K_m)$ for all $x \in S^0 \cup (S \setminus N_G^2(S))$ by Theorem 19. Since C is a γ_{conh}-set, $|T_x| = 1$ for all $x \in S \setminus [S^0 \cup (S \setminus N_G^2(S))]$. It follows that

$$\begin{aligned} |C| &= \sum_{x\in[S^0\cup(S\setminus N_G^2(S))]} |T_x| + \sum_{x\in S\setminus[S^0\cup(S\setminus N_G^2(S))]} |T_x| \\ &= m|S^0 \cup (S \setminus N_G^2(S))| + |S| - |S^0 \cup (S \setminus N_G^2(S))| \\ &= |S| + (m-1)|S^0 \cup (S \setminus N_G^2(S))|. \end{aligned}$$

This proves the desired equality. □

It is worth mentioning that the value of the parameter given in Corollary 11 is not necessarily attained when S is a γ_{conh}-set in G. To see this, consider $P_5[K_3]$. It is easily verified that $\gamma_{conh}(P_5) = 2$. If S is γ_{conh}-set in P_5, then $\langle S\rangle = K_2$ and $S^0 \cup (S\setminus N_G^2(S)) = S$. Hence, $|S| + (3-1)|S^0 \cup (S \setminus N_G^2(S))| = 6$. However, by taking any three consecutive vertices of P_5, one can see that $\gamma_{conh}(P_5[K_3]) = 5$.

4. Conclusion

The concept of convex hop domination has been introduced and initially investigated in this study. Graphs which attained some specific convex hop domination number have been characterized. The convex hop domination number of the complementary prism has been obtained and necessary and sufficient conditions for a subset to be convex hop dominating in the shadow graph, join, corona, and lexicographic product of two graphs have been obtained. These characterizations have been used to obtain bounds or exact value of the convex hop domination number of each of these graphs. The concept can be studied for other interesting graphs. Moreover, it is conjectured that the convex hop domination problem is NP-complete.

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