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On Gaussian Fibonacci Functions with periodicity

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Abstract. In this work, Gaussian Fibonacci functions with the use of the (ultimately) periodicity and exponential Gaussian Fibonacci functions are also discussed. Especially, by giving a non-negative real valued function, several exponential Gaussian Fibonacci functions are attained.

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1. Introduction

Fibonacci numbers have many applications in different disciplines such as in mathematics, philosophy, physics, art, architecture etc, where can be found in [2, 3, 7]. A series of the Fibonacci numbers is $1, 1, 2, 3, 5, 8, \ldots$, where the first two initiated numbers are 1 and every other number comes from the sum of the two preceding numbers. In 1963, Fibonacci numbers were examined on the complex plane and some interesting properties about them are established [1]. By the same strategy of finding the Fibonacci numbers, Gaussian Fibonacci numbers GF_n are defined recursively by $GF_n = GF_{n-1} + GF_{n-2}$, where $GF_0 = i, GF_1 = 1$, and $n \geq 2$ [6].
In [4], it is showed that if f_G is a Gaussian Fibonacci function, we have that $\lim_{x\to\infty} \frac{f_G(x+1)}{f_G(x)} = \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$. Similarly, it is showed that if f_G is a Gaussian Fibonacci function and f is a Fibonacci function, then $\lim_{x\to\infty} \frac{f_G(x+1)}{f(x)} = \phi + i$, where $\phi = \frac{1+\sqrt{5}}{2}$.
The Fibonacci functions with periodicity is studied in [5]. In this paper, Gaussian Fibonacci functions with periodicity are discussed and studied as well as discussing the exponential Gaussian Fibonacci functions, Especially, by giving a non-negative real valued function, several exponential Gaussian Fibonacci functions are obtained.

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2. Preliminaries

Definition 1. *[4] A Gaussian function f_G on the real numbers $\mathbb{R}$ is said to be a Gaussian Fibonacci function if it satisfies the formula*

$$f_G(x+2) = f(x+2) + f(x+1)i,$$

where f is a Fibonacci function and for any $x \in \mathbb{R}$.

Remark 1. *[4] For all $n \geq 0$. The full Gaussian Fibonacci sequence, where $Gu_0 = i$ and $Gu_1 = 1$, are formed by the following formula*

$$Gu_{-n} = (-1)^n * i * G_{n+1}.$$

Then the full Gaussian Fibonacci sequence, where $Gu_n = GF_n$ the n^{th} Gaussian Fibonacci numbers, are: $\ldots, -3+5i, 2-3i, -1+2i, 1-i, i, 1, 1+i, 2+i, 3+2i, 5+3i, \ldots$.

Example 1. *[4] Let $\{Gu_n\}_{n=-\infty}^{\infty}$ and $\{Gv_n\}_{n=-\infty}^{\infty}$ be full Gaussian Fibonacci sequences. We define a function f_G by $f_G(x) := Gu_{\lfloor x \rfloor} + Gv_{\lfloor x \rfloor}t$ and $f(x) := u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor}t$, where $t = x - \lfloor x \rfloor \in (0,1)$ and $x \in \mathbb{R}$. Then*

$$f_G(x+2) = Gu_{\lfloor x+2 \rfloor} + Gv_{\lfloor x+2 \rfloor}t = Gu_{\lfloor x \rfloor+2} + Gv_{\lfloor x \rfloor+2}t$$

by the fact that $Gu_{\lfloor x \rfloor+2} = u_{\lfloor x \rfloor+2} + iu_{\lfloor x \rfloor+1}$ and $Gv_{\lfloor x \rfloor+2} = v_{\lfloor x \rfloor+2} + iv_{\lfloor x \rfloor+1}$, we obtain that

$$\begin{aligned} f_G(x+2) &= (u_{\lfloor x \rfloor+2} + iu_{\lfloor x \rfloor+1}) + (v_{\lfloor x \rfloor+2} + iv_{\lfloor x \rfloor+1})t \\ &= (u_{\lfloor x+2 \rfloor} + v_{\lfloor x+2 \rfloor}t) + i(u_{\lfloor x+1 \rfloor} + v_{\lfloor x+1 \rfloor}t) \\ &= f(x+2) + f(x+1)i. \end{aligned}$$

Therefore, f_G is a Gaussian Fibonacci function.

Example 2. *Let $\phi(t)$, $\psi(t)$ be any real valued functions which are defined on $[0,1)$ and let $\{Gu_{-n}\}$ and $\{Gv_{-n}\}$ be Gaussian Fibonacci sequences. Define a map $f_G(x) := Gu_{\lfloor x \rfloor}\phi(t) + Gv_{\lfloor x \rfloor}\psi(t)$, and $f(x) = u_{\lfloor x \rfloor}\phi(t) + v_{\lfloor x \rfloor}\psi(t)$ where $t = x - \lfloor x \rfloor \in [0,1)$. Then $f_G(x+2) := Gu_{\lfloor x+2 \rfloor}\phi(t) + Gv_{\lfloor x+2 \rfloor}\psi(t)$. Since $\lfloor x+2 \rfloor = \lfloor x \rfloor + 2$ and hence $x + 2 - \lfloor x+2 \rfloor = x - \lfloor x \rfloor$, we obtain*

$$\begin{aligned} f_G(x+2) &= Gu_{\lfloor x \rfloor+2}\phi(t) + Gv_{\lfloor x \rfloor+2}\psi(t) \\ &= (u_{\lfloor x \rfloor+2} + iu_{\lfloor x \rfloor+1})\phi(t) + (v_{\lfloor x \rfloor+2} + iv_{\lfloor x \rfloor+1})\psi(t) \\ &= (u_{\lfloor x \rfloor+2}\phi(t) + v_{\lfloor x \rfloor+2}\psi(t)) + i(u_{\lfloor x \rfloor+1}\phi(t) + v_{\lfloor x \rfloor+1}\psi(t)) \\ &= f(x+2) + if(x+1). \end{aligned}$$

Therefore, $f_G(x)$ is a Gaussian Fibonacci function.

By using the concept of an f_G-even and f_G-odd functions, we attain some Gaussian Fibonacci functions which are discussed in [1]

Definition 2. *[4] Let $c(x)$ be real-valued function of a real variable such that $c(x)h(x) \equiv 0$ and $h(x)$ is continuous, then $h(x) = 0$. The function $c(x)$ is said to be f_G-even function (resp., f_G-odd function) if $c(x+1) = c(x)$ (resp., $c(x+1) = -c(x)$) for any $x \in \mathbb{R}$.*

Theorem 1. *[4] Let $f_G(x) = c(x)g_G(x)$ be a function and $f(x) = c(x)g(x)$ be a Fibonacci function, where $c(x)$ is an f_G-even function and $g_G(x)$ and $g(x)$ are continuous functions. Then $f_G(x)$ is a Gaussian Fibonacci function if and only if $g_G(x)$ is a Gaussian Fibonacci function.*

Note that if a Gaussian Fibonacci function is differentiable on $\mathbb{R}$, then its derivative is also a Gaussian Fibonacci function.

Proposition 1. *Let f_G be a Gaussian Fibonacci function. If we define $g_G(x) := f_G(x+t)$ and $g(x) := f(x+t)$ where $t \in \mathbb{R}$, for any $x \in \mathbb{R}$. If g is a Gaussian Fibonacci function, then g_G is also a Gaussian Fibonacci function.*

Theorem 2. *[4] If $f_G(x)$ is a Gaussian Fibonacci function, then the limit of quotient $\frac{f_G(x+1)}{f_G(x)}$ exists.*

Corollary 1. *[4] If $f_G(x)$ is a Gaussian Fibonacci function, then*

$$\lim_{x\to\infty} \frac{f_G(x+1)}{f_G(x)} = \frac{1+\sqrt{5}}{2} = \phi.$$

3. Gaussian Fibonacci functions with periodicity

In this section, several results of Gaussian Fibonacci functions with periodicity is obtained.

Theorem 3. *Let $f_G(x), g_G(x)$ be Gaussian Fibonacci functions with $g_G(x) = a_G(X)f_G(x)$. If $a_G(x+1) \neq a_G(X)$ for all $x \in \mathbb{R}$, then*

$$\lim_{x\to\infty} \frac{a_G(x+1)}{a_G(x)} = 1.$$

Proof. Since $a_G(x+1) \neq a_G(X)$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} a_G(x+1)[f(x+1) + if(x)] &= a_G(x+1)f_G(x+1) \\ &= g_G(x+1) \\ &= g(x+1) + ig(x) \\ &= a_G(x+1)f(x+1) + ia_G(x)f(x) \end{aligned}$$

Comparing the two sides, we obtain

$$\lim_{x\to\infty} \frac{a_G(x+1)}{a_G(x)} = 1.$$

Corollary 2. *Let $f_G(x), g_G(x)$ be Gaussian Fibonacci functions with $g_G(x) = a_G(X)f_G(x)$. If $a_G(x+p) \neq a_G(x)$ for all $x \in \mathbb{R}$, thwn*

$$\lim_{x\to\infty} \frac{a_G(x+p)}{a_G(x)} = 1.$$

Proof. The proof is similar to the proof of the Proposition 3.

Corollary 3. *Let $f_G(x)$ and $g_G(x)$ be Gaussian Fibonacci functions with $g_G(x) = a(x)f_G(x)$ for some $a(x)$. If $y > 0$, then*

$$\lim_{x\to\infty} \frac{a(x+y)}{a(x)} = \lim_{x\to\infty} \frac{a(x+y)}{a(x+y-\lfloor y \rfloor)}$$

Proof.

$$\begin{aligned}
\lim_{x\to\infty} \frac{a(x+y)}{a(x)} &= \lim_{x\to\infty} \frac{a(x+y)a(x+y-\lfloor y \rfloor)}{a(x+y-\lfloor y \rfloor)a(x)} \\
&= \lim_{x\to\infty} \frac{a(x+y)}{a(x+y-\lfloor y \rfloor)} \lim_{x\to\infty} \frac{a(x+y-\lfloor y \rfloor)}{a(x)} \\
&= \lim_{x\to\infty} \frac{a(x+y-\lfloor y \rfloor + \lfloor y \rfloor)}{a(x+y-\lfloor y \rfloor)} \lim_{x\to\infty} \frac{a(x+y-\lfloor y \rfloor)}{a(x)} \\
&= \lim_{x\to\infty} \frac{a(x+y)}{a(x+y-\lfloor y \rfloor)}
\end{aligned}$$

Definition 3. *A map $t_G(x)$ is said to be Gaussian ultimately periodic of period $p > 0$ if*

$$\lim_{x\to\infty} \frac{t_G(x+p)}{t_G(x)} = 1.$$

Note that $a(x)$ discussed in Proposition 3 is Gaussian ultimately periodic of period 1.

Example 3. *Let $t_G(x) := mx + b$. If $m \neq 0$,then*

$$\lim_{x\to\infty} \frac{t_G(x+p)}{t_G(x)} = \lim_{x\to\infty} \frac{m(x+p)+b}{mx+b} = 1,$$

showing that $t_G(x)$ is a Gaussian ultimately periodic of period p for all $p > 0$.

Using Example 3, we obtain the following example.

Example 4. *If $t_G(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $t_G(x)$ is a Gaussian ultimately periodic of period p for all $p > 0$.*

Example 5. *If* $t_G(x) = \cos(x)$, *then*

$$\begin{aligned}\lim_{x\to\infty}\frac{t_G(x+p)}{t_G(x)} &= \lim_{x\to\infty}\frac{\cos(x+p)}{\cos(x)}\\ &= \lim_{x\to\infty}\frac{\cos(x)\cos(p)+\sin(x)\sin(p)}{\cos(x)}\\ &= \cos(p)+\sin(p)\lim_{x\to\infty}\tan(x).\end{aligned}$$

Since $\lim_{x\to\infty}\tan(x)$ *does not exist,* $t_G(x)$ *is not a Gaussian ultimately periodic of period* $p > 0$ *unless* $\sin(p) = 0$ *and* $\cos(p) = 1$.

Proposition 2. *If* $a_G(x)$ *and* $b_G(x)$ *are Gaussian ultimately periodic of period* $p > 0$, *then* $\alpha a_G(x) + \beta b_G(x)$ *is also a Gaussian ultimately periodic of period* $p > 0$, *for all* $\alpha, \beta > 0$

Proof. Since $a_G(x)$ and $b_G(x)$ are Gaussian ultimately periodic of period $p > 0$, there exist $\epsilon_1(x), \epsilon_2(x) > 0$ such that $\frac{a_G(x+p)}{a_G(p)} = 1 + \epsilon_1(x)$ and $\frac{b_G(x+p)}{b_G(p)} = 1 + \epsilon_2(x)$ where $\epsilon_1(x), \epsilon_2(x) \to 0$. We know that $\frac{1+\epsilon_1(x)}{1+\epsilon_2(x)} = 1 + \epsilon(x)$. In fact, $\epsilon(x) = \frac{\epsilon_1(x)-\epsilon_2(x)}{1+\epsilon_1(x)} \to 0$. This shows that

$$\begin{aligned}\frac{\alpha a_G(x+p)+\beta b_G(x+p)}{\alpha a_G(x)+\beta b_G(x)} &= \frac{1+\frac{\beta b_G(x+p)}{\alpha a_G(x+p)}}{1+\frac{\beta b_G(x)}{\alpha a_G(x)}}\frac{\alpha a_G(x+p)}{\alpha a_G(x)}\\ &= \frac{1+\frac{\beta(1+\epsilon_2(x))b_G(x)}{\alpha(1+\epsilon_1(x))a_G(x)}}{1+\frac{\beta b_G(x)}{\alpha a_G(x)}}\frac{a_G(x+p)}{a_G(x)}\\ &\to \frac{a_G(x+p)}{a_G(x)}\\ &\to 1.\end{aligned}$$

Hence, the proposition is proved.

Proposition 3. *If* $a_G(x)$ *and* $b_G(x)$ *are Gaussian ultimately periodic of period* $p > 0$, *then* $a_G(x)b_G(x)$ *is also a Gaussian ultimately periodic of period* $p > 0$.

Proof. This can be prove by the following equation:

$$\lim_{x\to\infty}\frac{a_G(x+p)b_G(x+p)}{a_G(x)b_G(x)} = \lim_{x\to\infty}\frac{a_G(x+p)}{a_G(x)}\lim_{x\to\infty}\frac{b_G(x+p)}{b_G(x)} = 1.$$

Note that $\mathcal{GU}_p$ is the collection of all functions which are Gaussian ultimately periodic of period $p > 0$.

Proposition 4. *If* $a_G(x) \in \mathcal{GU}_p$ *and* $a_G(x) \neq 0$ *for all* $x \in [\lambda, \infty)$, *then* $\frac{1}{a_G(x)} \in \mathcal{GU}_p$.

Proof.

$$\lim_{x\to\infty}\frac{\frac{1}{a_G(x+p)}}{\frac{1}{a_G(x)}} = \lim_{x\to\infty}\frac{a_G(x)}{a_G(x+p)} = 1.$$

A map f_G which is defined on the set of all real numbers $\mathbb{R}$ is said to be Gaussian periodic of period $p > 0$ if $f_G(x+p) = f_G(x)$ for all $x \in \mathbb{R}$. It is obvious that every Gaussian map of period of periodic 1 is Gaussian ultimately periodic of period p.

Proposition 5. *Let $f_G(x)$ be a Gaussian Fibonacci function and $f(x)$ be a Fibonacci function and let $a_G(x)$ be a Gaussian periodic of period 1. If $g_G(x) := a_G(x)f_G(x)$ and $g(x) := a_G(x)f(x)$, then $g_G(x)$ is a Gaussian Fibonacci function.*

Proof. Given $x \in \mathbb{R}$. Since $a_G(x)$ is a Gaussian periodic of period 1, we have

$$\begin{aligned} g_G(x+2) &= a_G(x+2)f_G(x+2) \\ &= a_G(x)[f(x+2) + if(x+1)] \\ &= a_G(x)f(x+2) + ia_G(x)f(x+1) \\ &= a_G(x+2)f(x+2) + ia_G(x+1)f(x+1) \\ &= g(x+2) + ig(x+1). \end{aligned}$$

Hence, $g_G(x)$ is a Gaussian Fibonacci function.

We ask the following question: Are there a Gaussian Fibonacci function $f_G(x)$ and a function $a_G(x)$ which is a Gaussian ultimately periodic of period 1 but not periodic of period 1 such that $g_G(x) = a_G(X)f_G(x)$ is also a Gaussian Fibonacci function?

4. Exponential Gaussian Fibonacci functions

Consider a Gaussian map $T_G(x) = \frac{\ln(x+i)}{\ln(x)}$ with domain $D = (0,\infty) \setminus \{1\}$. If we let $C := \mathbb{C} \setminus [0,1]$, then $T_G : D \to C$ is a bijective function.

Proposition 6. *If $g_G(x) = A(x)^{f(x)}$ is a Gaussian Fibonacci function where $A(x) > 0$, then there exists $\gamma(x) \in C$ such that*

$$\frac{g_G(x+2)}{g(x+1)} = \left[\frac{g(x+2)}{g(x+1)}\right]^{\gamma(x)}.$$

Proof. If $g_G(x) = A(x)^{f(x)}$, $A(x) > 0$, then $g_G(x) > 0$. Assume

$$\frac{g_G(x+2)}{g(x+1)} = \left[\frac{g(x+2)}{g(x+1)}\right]^{\gamma(x)}$$

for some $\gamma(x)$. If we let $B(x) := \frac{g(x+2)}{g(x+1)}$, then

$$B(x)^{\gamma(x)} = \frac{g_G(x+2)}{g(x+1)} = \frac{g(x+2) + ig(x+1)}{g(x+1)} = B(x) + i.$$

It follows that

$$\gamma(x) = \frac{\ln(B(x)+i)}{\ln B(x)} = \frac{\ln\left(\frac{g(x+2)}{g(x+1)} + i\right)}{\ln\left(\frac{g(x+2)}{g(x+1)}\right)} = T_G(B(x)) \in C.$$

This has proved the proposition.

Proposition 7. *There is no Gaussian Fibonacci function $f_G(x)$ such that $g_G(x) = A^{f_G(x)}$ and $g(x) = A^{f(x)}$, $A > 0$ where $f_G(x)$ and $f(x)$ are differentiable and $g_G(x)$ and $g(x)$ are Gaussian Fibonacci function and Fibonacci function, respectively.*

Proof. Suppose that $f_G(x)$ is a Gaussian Fibonacci function. Since $f_G(x)$ is differentiable, we have

$$f'_G(x+2) = f'(x+2) + if'(x+2) \tag{1}$$

Since $g_G(x)$ is a Gaussian Fibonacci function, then $g_G(x+2) = g(x+2) + ig(x+1)$. Since $g_G(x+2) = A^{f_G(x)}$ and $g(x) = A^{f(x)}$ and $f_G(x)$ and $f_G(x)$ are differential, $g'_G(x+2) = g'(x+2) + ig'(x+2)$, *i.e.*, $g'_G(x)$ is also a Gaussian Fibonacci function. It follows from $g'_G(x) = g_G(x)\ln A f'_G(x)$ and $g'(x) = g(x)\ln A f'(x)$ that

$$\begin{aligned} g_G(x+2)\ln A f'_G(x+2) &= g'_G(x+2) \\ &= g'(x+2) + ig'(x+1) \\ &= g(x+2)\ln A f'(x+2) + ig(x+1)\ln A f'(x+1) \end{aligned}$$

Comparing the two sides, we obtain

$$f'_G(x+2) = \frac{g(x+2)}{g_G(x+2)} f'(x+2) + i\frac{g(x+1)}{g_G(x+2)} f'(x+1) \tag{2}$$

From equations (1) and (2), we obtain

$$\left[\frac{g(x+2)}{g_G(x+2)} - 1\right] f'(x+2) + i\left[\frac{g(x+1)}{g_G(x+2)} - 1\right] f'(x+1) = 0$$

This implies that

$$\frac{f'(x+2)}{if'(x+1)} = \frac{g(x+1) - g_G(x+2)}{g(x+2) - g_G(x+2)} = \frac{g(x+1) - g_G(x+2)}{ig(x+1)} = \frac{1}{i} - \frac{g(x+2)}{ig(x+1)} - 1$$

This follows that

$$\phi = \lim_{x\to\infty} \frac{f'(x+2)}{f'(x+1)} = 1 - \lim_{x\to\infty} \frac{g(x+2)}{g(x+1)} - i = 1 - \phi - i$$

Which is contradiction because $\phi = \frac{1+\sqrt{5}}{2}$ but we obtain that $\phi = \frac{1-i}{2}$. Hence, $f_G(x)$ is a Gaussian Fibonacci function.

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