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1-Movable Resolving Hop Domination in Graphs

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Abstract. Let G be a connected graph. A set $W \subseteq V(G)$ is a *resolving hop dominating set* of G if W is a resolving set in G and for every vertex $v \in V(G) \setminus W$ there exists $u \in W$ such that $d_G(u,v) = 2$. A set $S \subseteq V(G)$ is a 1-*movable resolving hop dominating* set of G if S is a resolving hop dominating set of G and for every $v \in S$, either $S \setminus \{v\}$ is a resolving hop dominating set of G or there exists a vertex $u \in ((V(G) \setminus S) \cap N_G(v))$ such that $(S \setminus \{v\}) \cup \{u\}$ is a resolving hop dominating set of G. The 1-*movable resolving hop domination number* of G, denoted by $\gamma^1_{mRh}(G)$ is the smallest cardinality of a 1-movable resolving hop dominating set of G. This paper presents the characterization of the 1-movable resolving hop dominating sets in the join, corona and lexicographic product of graphs. Furthermore, this paper determines the exact value or bounds of their corresponding 1-movable resolving hop domination number.

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Key Words and Phrases: 1-movable resolving hop dominating set, 1-movable resolving hop domination number, hop dominated superclique, join, corona, lexicographic product

1. Introduction

Dominating sets in graphs have been studied extensively and there have been many published studies that have introduced different variants of domination in graphs [7, 13]. In 2015, Natarajan and Ayyaswamy [12] studied the concept of hop domination in graphs and the hop domination number.

Movable resolving domination in graphs was studied in [11] and the resolving hop domination sets in graphs was introduced in [10]. Other variations of resolving sets can be found in [2, 3, 6] and resolving dominating sets in [1, 4, 5, 9, 14]. This paper introduces and characterizes the concept of 1-movable resolving hop domination in graphs.

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We consider connected graphs that are finite, simple, and undirected. For elementary Graph Theory concepts, it is recommended that readers refer to [8].

Let $G = \big(V(G), E(G)\big)$ be a graph. $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is a *neighborhood* of v. An element $u \in N_G(v)$ is called a *neighbor* of v. $N_G[v] = N_G(v) \cup \{v\}$ is a *closed neighborhood* of v. The degree of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$.

The *distance* $d_G(u, v)$ of two vertices u, v in G is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

A set $S \subseteq V(G)$ is a *dominating set* if every $u \in V(G) \setminus S$ is adjacent to at least one vertex $v \in S$. The *domination number* of a graph G, denoted by $\gamma(G)$, is given by $\gamma(G) = min\{|S| : S \text{ is a dominating set of } G\}$.

A set $S \subseteq V(G)$ is a *total dominating set* if every vertex in graph G is adjacent to some vertex of S. The minimum cardinality of a total dominating set in G is the *total domination number* of G, denoted by $\gamma_t(G)$, and we refer to such a set as γ_t-set of G.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u, 2] = N_G(u, 2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X, 2) = \bigcup_{u \in X} N_G(u, 2)$. The *closed hop neighborhood* of X in G is the set $N_G[X, 2] = N_G(X, 2) \cup X$.

A set $S \subseteq V(G)$ is a *total hop dominating set* of G if for every $v \in V(G)$, there exists $u \in S$ such that $d_G(u, v) = 2$. That is, S is a hop dominating set of G and for all $z \in S$, $N_G(z, 2) \cap S \neq \varnothing$. The smallest cardinality of a total hop dominating set of G, denoted by $\gamma_{th}(G)$, is called the *total hop domination number* of G. Any total hop dominating set with cardinality equal to $\gamma_{th}(G)$ is called a γ_{th}-set.

A set $S \subseteq V(G)$ is a *locating set* of G if for every two distinct vertices u and v of $V(G)\backslash S$, $N_G(u) \cap S \neq N_G(v) \cap S$. The *locating number* of G, denoted by $ln(G)$, is the smallest cardinality of a locating set of G. A locating set of G of cardinality $ln(G)$ is referred to as *ln-set* of G. A set $S \subseteq V(G)$ is a *strictly locating set* of G if it is a locating set of G and $N_G(u) \cap S \neq S$ for all $u \in V(G)\backslash S$. The *strictly locating number* of G, denoted by $sln(G)$, is the smallest cardinality of a strictly locating set of G. A strictly locating set of G of cardinality $sln(G)$ is referred to as a *sln-set* of G.

A locating (resp. strictly locating) subset S of $V(G)$ is a 1-*movable locating* (resp. 1-*movable strictly locating*) set of G if for every $v \in S$, either $S \setminus \{v\}$ is a locating (resp. strictly locating) set of G or there exists a vertex $u \in ((V(G) \setminus S) \cap N_G(v))$ such that $(S \setminus \{v\}) \cup \{u\}$ is a locating (resp. strictly locating) set of G. The minimum cardinality of a 1-movable locating (resp. 1-movable strictly locating) set of G, denoted by $mln(G)$(resp.

$msln(G)$) is the 1-*movable location number* (resp. 1-*movable strictly location number*) of G. Any 1-movable locating (resp. 1-movable strictly locating) set of cardinality $mln(G)$ (resp. $msln(G)$) is referred to as mln-set (resp. $msln$-set) of G.

A vertex x of a graph G is said to *resolve two vertices* u and v of G if $d_G(x,u) \neq d_G(x,v)$. For an ordered set $W = \{x_1, ..., x_k\} \subseteq V(G)$ and a vertex v in G, the $k-vector$

$$r_G(v/W) = (d_G(v,x_1), d_G(v,x_2), \cdots, d_G(v,x_k))$$

is called the representation of v with respect to W. The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W. The *metric dimension* of G, denoted by, $dim(G)$, is the minimum cardinality over all resolving sets of G. A resolving set of cardinality $dim(G)$ is called *basis*.

A set $S \subseteq V(G)$ is a *resolving hop dominating set* of G if S is both a resolving set and a hop dominating set. The minimum cardinality of a resolving hop dominating set of G, denoted by $\gamma_{Rh}(G)$, is called the *resolving hop domination number* of G. Any resolving hop dominating set with cardinality equal to $\gamma_{Rh}(G)$ is called a γ_{Rh}-set.

A set $S \subseteq V(G)$ is a 1-*movable resolving hop dominating* set of G if S is a resolving hop dominating set of G and for every $v \in S$, either $S \setminus \{v\}$ is a resolving hop dominating set of G or there exists a vertex $u \in ((V(G) \setminus S) \cap N_G(v))$ such that $(S \setminus \{v\}) \cup \{u\}$ is a resolving hop dominating set of G. The 1-*movable resolving hop domination number* of G, denoted by $\gamma^1_{mRh}(G)$ is the smallest cardinality of a 1-movable resolving hop dominating set of G. Any 1-movable resolving hop dominating set of cardinality $\gamma^1_{mRh}(G)$ is referred to as a γ^1_{mRh}-set of G.

2. Preliminary Results

Remark 1. *Every* 1-*movable resolving hop dominating set of* G *is a resolving hop dominating set. Thus,*

$$2 \leq \gamma_{Rh}(G) \leq \gamma^1_{mRh}(G).$$

Remark 2. *Every* 1-*movable resolving hop dominating set of* G *is a hop dominating set. Thus,*

$$2 \leq \gamma_h(G) \leq \gamma^1_{mRh}(G).$$

Remark 3. *Every* 1-*movable resolving hop dominating set of* G *is a resolving set. Thus,*

$$1 \leq dim(G) \leq \gamma^1_{mRh}(G).$$

Consider $G = P_5$ where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ with $deg(v_1) = deg(v_5) = 1$ and $N_G(v_3) = \{v_2, v_4\}$. Let $S_1 = \{v_1\}$, $S_2 = \{v_2, v_3\}$ and $S_3 = V(G)$. Then, S_1 is a resolving set of G, S_2 is a hop dominating set and a resolving set of G and S_3 is a 1-movable resolving hop dominating set of G. It can be verified that $dim(G) = 1$, $\gamma_h(G) = 2$, $\gamma_{Rh}(G) = 2$ and $\gamma^1_{mRh}(G) = 5$. Hence for $G = P_5$, Remarks 1, 2 and 3 holds.

Proposition 1. *Let G be a nontrivial connected graph. Then G admits a 1-movable resolving hop dominating set if and only if $\gamma(G) \neq 1$.*

Proof: Suppose G has a 1-movable resolving hop dominating set S. Suppose further that $\gamma(G) = 1$. Let $A = \{x \in V(G) : \{x\}$ is a dominating set of $G\}$. Then $A \neq \varnothing$ since $\gamma(G) = 1$. Since S is a hop dominating set, $A \subseteq S$. Let $x \in A$. Then $S \setminus \{x\}$ and $(S \setminus \{x\}) \cup \{y\}$ for each $y \in V(G) \setminus S$ are not hop dominating sets of G. Thus, S is not a 1-movable resolving hop dominating set, a contradiction.

Conversely, suppose that $\gamma(G) \neq 1$. Let $S = V(G)$. Then S is a resolving hop dominating set of G. For each $x \in S$, $S \setminus \{x\}$ is a resolving set of G. Also, since $\{x\}$ is not a dominating set, there exists $y \in (S \setminus \{x\}) \cap N_G(x, 2)$. Hence, $S \setminus \{x\}$ is a hop dominating set of G. Therefore, $S \setminus \{x\}$ is a resolving hop dominating set of G for each $x \in S$. Accordingly, S is a 1-movable resolving hop dominating set of G. □

As a consequence of Proposition 1 the next result follows.

Corollary 1. *A graph G does not admit a 1-movable resolving hop dominating set if and only if $G = K_1 + H$ for any graph H.*

Proposition 2. *Let G be a connected graph and S a 1-movable resolving hop dominating set of G. Then for all $z \in S$, $N_G(z, 2) \cap S \neq \varnothing$ and for each $x \in V(G) \setminus S$, $|N_G(x, 2) \cap S| \geq 1$ and there exists $w \in (V(G) \setminus S) \cap N_G(x, 2) \cap N_G(v)$ whenever $N_G(x, 2) \cap S = \{v\}$.*

Proof: Let S be a 1-movable resolving hop dominating set of G and $z \in S$. Suppose $N_G(z, 2) \cap S = \varnothing$. Then $S \setminus \{z\}$ and $(S \setminus \{z\}) \cup \{u\}$ where $u \in (V(G) \setminus S) \cap N_G(z)$ are not hop dominating sets of G since z has no hop neighbor in both sets, a contradiction. Thus, $N_G(z, 2) \cap S \neq \varnothing$. Now, let $x \in V(G) \setminus S$. Since S is hop dominating, $N_G(x, 2) \cap S \neq \varnothing$. Suppose $|N_G(x, 2) \cap S| = 1$. Let $v \in N_G(x, 2) \cap S$. Then $S \setminus \{v\}$ is not hop dominating, since x has no hop neighbor in $S \setminus \{v\}$. It follows that $(S \setminus \{v\}) \cup \{w\}$ for some $w \in (V(G) \setminus S) \cap N_G(v)$ is a resolving hop dominating set of G. Hence, x must be a hop neighbor of w and so $w \in (V(G) \setminus S) \cap N_G(x, 2) \cap N_G(v)$. □

As a consequence of Proposition 2, the next corollary follows.

Corollary 2. *Every 1-movable resolving hop dominating set is a total hop dominating set. Moreover, $\gamma_{th}(G) \leq \gamma^1_{mRh}(G)$.*

3. On 1-Movable Resolving Hop Domination in the Join of Graphs

Let A and B be sets which are not necessarily disjoint. The *disjoint union* of A and B, denoted by $A \dot{\cup} B$, is the set obtained by taking the union of A and B treating each element in A as distinct from each element in B. The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 with disjoint vertex-sets $V(G_1)$ and $V(G_2)$, respectively, is the graph G with $V(G) = V(G_1) \dot{\cup} V(G_2)$ and $E(G) = E(G_1) \dot{\cup} E(G_2)$. The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G + H) = E(G) \dot{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 1. [10] Let G and H be nontrivial connected graphs. A set $W \subseteq V(G+H)$ is a resolving hop dominating set of $G+H$ if and only if $W = W_G \cup W_H$ where W_G and W_H are strictly locating sets of G and H, respectively.

As an illustration, consider the graph $P_3 + P_3$ in Figure 1. It is easy to verify that $sln(P_3) = 2$, and by Theorem 1, the set of shaded vertices is a resolving hop dominating set of $P_3 + P_3$. It follows that $\gamma_{Rh}(P_3 + P_3) = 4$.

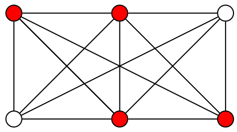
$P_3 + P_3$: 

Figure 1: Graph $P_3 + P_3$ with $\gamma_{Rh}(P_3 + P_3) = 4$

Theorem 2. *Let G and H be connected graphs with $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. A set $W \subseteq V(G+H)$ is a 1-movable resolving hop dominating set of $G+H$ if and only if $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$ are 1-movable strictly locating sets of G and H, respectively, and one of the following statements holds:*

(i) *For each $u \in W_G$, $W_G \setminus \{u\}$ and $W_H \cup \{v\}$ are strictly locating sets of G and H, respectively, for some $v \in V(H) \setminus W_H$;*

(ii) *For each $q \in W_H$, $W_H \setminus \{q\}$ and $W_G \cup \{b\}$ are strictly locating sets of H and G, respectively, for some $b \in V(G) \setminus W_G$.*

Proof: Suppose that $W \subseteq V(G+H)$ is a 1-movable resolving hop dominating set of $G+H$. Then W is resolving hop dominating. By Theorem 1, $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$ are strictly locating sets of G and H, respectively. Moreover, since G and H are connected graphs with $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$, $W_G \neq \varnothing$ and $W_H \neq \varnothing$. Let $x \in W_G$. By assumption, $W \setminus \{x\} = (W_G \setminus \{x\}) \cup W_H$ or $(W \setminus \{x\}) \cup \{w\} = [(W_G \setminus \{x\} \cup \{w\})] \cup W_H$ for some $w \in N_G(x) \cap (V(G) \setminus W_G)$ or $(W \setminus \{x\}) \cup \{z\} = (W_G \setminus \{x\}) \cup (W_H \cup \{z\})$ for some $z \in V(H) \setminus W_H$ is a resolving hop dominating set of $G+H$. Thus, by Theorem 1, $W_G \setminus \{x\}$ or $(W_G \setminus \{x\}) \cup \{w\}$ is a strictly locating set of G. This implies that W_G is a 1-movable-strictly locating set of G. Similarly, W_H is a 1-movable strictly locating set of H.

Now, let $u \in W_G$. Since W is a 1-movable resolving hop dominating set, $W \setminus \{u\} = (W_G \setminus \{u\}) \cup W_H$ or $(W \setminus \{u\}) \cup \{r\} = [(W_G \setminus \{u\}) \cup \{r\}] \cup W_H$ for some $r \in N_G(u) \cap (V(G) \setminus W_G)$ or $(W \setminus \{u\}) \cup \{v\} = (W_G \setminus \{u\}) \cup (W_H \cup \{v\})$ for some $v \in V(H) \setminus W_H$ is a resolving hop dominating set of $G+H$. It follows from Theorem 1 that $W_G \setminus \{u\}$ and $W_H \cup \{v\}$ are strictly locating sets of G and H, respectively. Thus, (i) holds. Similarly, (ii) holds.

For the converse, suppose that W_G and W_H are 1-movable strictly locating sets of G and H, respectively. Suppose (i) holds. Then $W = W_G \cup W_H$ is a resolving hop dominating set of $G+H$ by Theorem 1. Let $u \in W$. If $u \in W_G$, then by assumption and Theorem 1, $W \setminus \{u\} = (W_G \setminus \{u\}) \cup W_H$ or $(W \setminus \{u\}) \cup \{w\} = [(W_G \setminus \{u\}) \cup \{w\}] \cup W_H$ for some $w \in N_G(u) \cap (V(G) \setminus W_G)$ or $W \setminus \{u\} \cup \{z\} = (W_G \setminus \{u\}) \cup (W_H \cup \{z\})$ for some $z \in V(H \setminus W_H)$ is a resolving hop dominating set of $G+H$. Now, suppose that $u \in W_H$. Since W_G and W_H are 1-movable strictly locating sets of G and H, respectively, it follows from Theorem 1 that $W \setminus \{u\} = (W_H \setminus \{u\}) \cup W_G$ or $(W \setminus \{u\}) \cup \{y\} = [(W_H \setminus \{u\}) \cup \{y\}] \cup W_G$ for some $y \in N_H(u) \cap (V(H) \setminus W_H)$ is a resolving hop dominating set of $G+H$. Therefore, W is a 1-movable resolving hop dominating set of $G+H$. Similarly, W is a 1-movable resolving hop dominating set of $G+H$ if (ii) holds. □

Corollary 3. *Let G and H be nontrivial connected graphs with $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. If G and H have 1-movable strictly locating sets, then*

$$\gamma^1_{mRh}(G+H) \leq msln(G) + msln(H).$$

Proof: Suppose G and H have 1-movable strictly locating sets. Let W_G and W_H be $msln$-sets of G and H, respectively. Then $W = W_G \cup W_H$ is a 1-movable resolving hop dominating set of $G+H$ by Theorem 2. Thus,

$$\gamma^1_{mRh}(G+H) \leq |W| = |W_G| + |W_H| = msln(G) + msln(H). \quad \square$$

4. On 1-Movable Resolving Hop Domination in the Corona of Graphs

The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v, v \in V(G)$.

Theorem 3. [10] Let G and H be nontrivial connected graphs. Then $W \subseteq V(G \circ H)$ is a resolving hop dominating set of $G \circ H$ if and only if $W \cap V(H^v) \neq \varnothing$ for every $v \in V(G)$ and $W = A \cup B \cup D$ where $A \subseteq V(G)$,

$$B = \cup \{B_v : v \in V(G) \cap N_G(A) \text{ and } B_v \text{ is a locating set of } H^v\} \text{ and}$$
$$D = \cup \{D_u : u \in V(G) \setminus N_G(A) \text{ and } D_u \text{ is a strictly locating set of } H^u\}.$$

As an illustration, consider the graph $P_3 \circ P_4$ in Figure 2 and let $G = P_3$ and $H = P_4$. It can be easily verified that $ln(P_4) = sln(P_4) = 2$ and by Theorem 3, the set of shaded vertices is a resolving hop dominating set of $P_3 \circ P_4$. It can be verified that $\gamma_{Rh}(P_3 \circ P_4) = 6$.

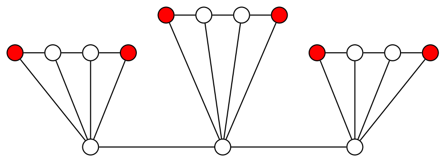


Figure 2: Graph $P_3 \circ P_4$ with $\gamma_{Rh}(P_3 \circ P_4) = 6$

Theorem 4. *Let G and H be nontrivial connected graphs. Then $W \subseteq V(G \circ H)$ is a 1-movable resolving hop dominating set of $G \circ H$ if and only if $W \cap V(H^v) \neq \varnothing$ for every $v \in V(G)$ and*

$$W = A \cup \left(\bigcup_{v \in N_G(A)} B_v \right) \cup \left(\bigcup_{u \in V(G) \setminus N_G(A)} D_u \right)$$

where $A \subseteq V(G)$, $B_v \subseteq V(H^v)$ for all $v \in V(G) \cap N_G(A)$ and $D_u \subseteq V(H^u)$ for all $u \in V(G) \setminus N_G(A)$ are 1-movable locating and 1-movable strictly locating sets of H^v and H^u, respectively.

Proof: Suppose that $W \subseteq V(G \circ H)$ is a 1-movable resolving hop dominating set of $G \circ H$. Then W is a resolving hop dominating set. By Theorem 3, $W \cap V(H^v) \neq \varnothing$ and $W \cap V(H^v)$ is a locating set of H^v for all $v \in V(G)$. Let $A = W \cap V(G)$, $B_v = W \cap V(H^v)$ for all $v \in V(G) \cap N_G(A)$ and $D_u = W \cap V(H^u)$ for all $u \in V(G) \setminus N_G(A)$. By Theorem 3, B_v is a locating set of H^v and D_u is a strictly locating set of H^u. Let $x \in B_v$. Since W is a 1-movable resolving hop dominating set and $x \in W$, either $W \setminus \{x\}$ is a resolving hop dominating set of $G \circ H$ or there exists $y \in (V(G \circ H) \setminus W) \cap N_{G \circ H}(x)$ such that $(W \setminus \{x\}) \cup \{y\}$ is a resolving hop dominating set of $G \circ H$. Note that

$$W \setminus \{x\} = (B_v \setminus \{x\}) \cup \left(\bigcup_{u \in V(G) \setminus \{v\}} D_u^* \right) \cup A$$

and $(W \setminus \{x\}) \cup \{y\}$ is equal to $((B_v \setminus \{x\}) \cup \{y\}) \cup \left(\bigcup_{u \in V(G) \setminus \{v\}} D_u^* \right) \cup A$ if $y \in V(H^v) \setminus B_v$ or equal to $(B_v \setminus \{x\}) \cup \left(\bigcup_{u \in V(G) \setminus \{v\}} D_u^* \right) \cup (A \cup \{y\})$ if $y = v \in V(G) \setminus A$. Hence, either $B_v \setminus \{x\}$ is a locating set of H^v or $(B_v \setminus \{x\}) \cup \{y\}$ for some $y \in (V(H^v) \setminus B_v) \cap N_{H^v}(x)$ is a locating set of H^v. Thus, B_v is a movable locating set of H^v. The proof that D_u is a 1-movable strictly locating set of H^u is similar.

For the converse, suppose that W is a set described above. Then by Theorem 3, W is a resolving hop dominating set. Let $x \in W$ and let $v \in V(G)$ such that $x \in V(\langle v \rangle + H^v)$. Suppose that $x \neq v$. Consider the following cases.

Case 1. $v \in V(G) \cap N_G(A)$

Then $x \in B_v$ and $W \setminus \{x\} = (B_v \setminus \{x\}) \cup \left(\bigcup_{u \in V(G) \setminus \{v\}} D_u \right) \cup A$ or $(W \setminus \{x\}) \cup \{y\}$ for some $y \in (V(G \circ H) \setminus W) \cap N_{G \circ H}(x)$ is a resolving hop dominating set by Theorem 3.

Case 2. $v \in V(G) \setminus N_G(A)$

Then $x \in D_v$ and $W \setminus \{x\} = (D_v \setminus \{x\}) \cup \left(\bigcup_{u \in V(G) \setminus \{v\}} B_u \right) \cup A$ or $(W \setminus \{x\}) \cup \{y\}$ is a resolving hop dominating set by Theorem 3.

Therefore W is a 1-movable resolving hop dominating set of $G \circ H$. □

Corollary 4. *Let G and H be nontrivial connected graphs where $|V(G)| = p$. Then*

$$\gamma^1_{mrRh}(G \circ H) \leq \min \{p(msln(H)), \gamma_t(G) + p(mln(H))\}.$$

Proof: Let $W \subseteq V(G \circ H)$ be a 1-movable resolving hop dominating set of $G \circ H$. Then $W \cap V(H^v) \neq \varnothing$ and $W \cap V(H^v)$ is a 1-movable locating set for each $v \in V(G)$ and

$$W = A \cup \left(\bigcup_{v \in N_G(A)} B_v \right) \cup \left(\bigcup_{u \in V(G) \setminus N_G(A)} D_u \right)$$

where $A \subseteq V(G)$ and B_v and D_u satisfy the given properties in Theorem 4. Consider the following cases for set A.

Case 1. $A = \varnothing$

Then $N_G(A) = \varnothing$. Let $D_u = W \cap V(H^u)$ be an $msln$-set of H^u for each $u \in V(G)$. Thus, $W = \left(\bigcup_{u \in V(G)} D_u \right)$ is a 1-movable resolving hop dominating set of $G \circ H$ by Theorem 4. Implying that,

$$\gamma^1_{mRh}(G \circ H) \leq |W| = |V(G)||D_u| \leq p(msln(H)).$$

Case 2. A is a γ_t-set of G

Then $N_G(A) = V(G)$. Let $B_v = W \cap V(H^v)$ be an mln-set of H^v for each $v \in V(G)$. Hence, $W = A \cup \left(\bigcup_{v \in V(G)} B_v \right)$ is a 1-movable resolving hop dominating set of $G \circ H$ by Theorem 4. It follows that

$$\gamma^1_{mRh}(G \circ H) \leq |W| = |A| + |V(G)||B_v| = \gamma_t(G) + p(mln(H)).$$

Therefore,

$$\gamma^1_{mrRh}(G \circ H) \leq \min \{p(msln(H)), \gamma_t(G) + p(mln(H))\}.$$ □

5. On 1-Movable Resolving Hop Domination in the Lexicographic Product of Graphs

The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

Theorem 5. [10] Let G and H be nontrivial connected graphs with $\triangle(H) \leq |V(H)| - 2$. Then $W = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving hop dominating set of $G[H]$ if and only if

(i) $S = V(G)$;

(ii) T_x is a locating set for every $x \in V(G)$;

(iii) T_x or T_y is a strictly locating set of H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$;

(iv) T_x or T_y is a (locating) dominating set of H whenever x and y are nonadjacent vertices of G with $N_G(x) = N_G(y)$; and

(v) T_x is a strictly locating set of H for each $x \in S \setminus N_G(S, 2)$.

The set of shaded vertices in the lexicographic product $P_3[P_4]$ in Figure 3 where $G = P_3$ and $H = P_4$ satisfies the conditions in Theorem 5 and thus it is a resolving hop dominating set of $G[H]$. In fact, the set of vertices that are not shaded is also a resolving hop dominating set of $G[H]$.

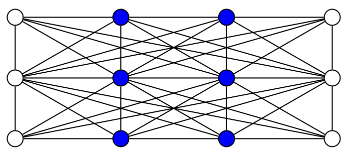


Figure 3: Resolving hop dominating sets of $P_3[P_4]$

Theorem 6. *Let G and H be nontrivial connected graphs with $\triangle(H) \leq |V(H)| - 2$. Then $W = \bigcup_{x \in S} (\{x\} \times T_x)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 1-movable resolving hop dominating set of $G[H]$ if and only if the following conditions hold:*

(i) $S = V(G)$.

(ii) T_x is a 1-movable locating set for each $x \in S$.

(*iii*) $T_x \setminus \{a\}$ *or* T_y *is a strictly locating set of* H *whenever* x *and* y *are adjacent vertices of* G *with* $N_G[x] = N_G[y]$ *and for each* $a \in T_x$.

(*iv*) $T_x \setminus \{a\}$ *or* $T_x \setminus \{a\} \cup \{b\}$ *or* T_y *is a (locating) dominating set of* H *whenever* x *and* y *are nonadjacent vertices of* G *with* $N_G(x) = N_G(y)$ *and for each* $a \in T_x$ *and for some* $b \in N_H(a)$.

(*v*) $T_x \setminus \{a\}$ *or* $T_x \setminus \{a\} \cup \{b\}$ *is a strictly locating set of* H *for each* $x \in S \setminus N_G(S, 2)$ *and for each* $a \in T_x$ *and for some* $b \in N_H(a)$.

Proof: Suppose W is a 1-movable resolving hop dominating set of $G[H]$. Then by Theorem 5, $S = V(G)$ and T_x is a locating set of H for each $x \in V(G)$. Let $a \in T_x$. Then $(x, a) \in W$. Since W is a 1-movable resolving hop dominating set, either

$$W \setminus \{(x,a)\} = \left[\bigcup_{v \in S \setminus \{x\}} (\{v\} \times T_v) \right] \cup [\{x\} \times (T_x \setminus \{a\})]$$

or

$$(W \setminus \{(x,a)\}) \cup \{(x,b)\} = \left[\bigcup_{z \in S \setminus \{x\}} (\{z\} \times T_z) \right] \cup [\{x\} \times (T_x \setminus \{a\} \cup \{b\})]$$

for some $b \in N_H(a) \cap (V(H) \setminus T_x)$ or

$$(W \setminus \{(x,a)\}) \cup \{(y,u)\} = \left[\bigcup_{p \in S \setminus \{(x,y)\}} (\{p\} \times T_p) \right] \cup [\{x\} \times (T_x \setminus \{a\})] \cup [\{y\} \times (T_y \cup \{u\})]$$

for some $y \in V(G) \cap N_G(x)$ and $u \in V(H) \setminus T_y$ is a resolving hop dominating set of $G[H]$.

By Theorem 5, $T_x \setminus \{a\}$ or $(T_x \setminus \{a\}) \cup \{b\}$ is a locating set of H for each $a \in T_x$ and for some $b \in N_H(a) \cap (V(H) \setminus T_x)$. Hence, T_x is a 1-movable locating set of H for each $x \in V(G)$ or $T_x \setminus \{a\}$ is locating and (*ii*) holds. Suppose (*iii*) does not hold. Then there exist $p \in V(H) \setminus (T_x \setminus \{a\})$ and $q \in V(H) \setminus T_y$ such that $N_H(p) \cap (T_x \setminus \{a\}) = T_x \setminus \{a\}$ and $N_H(q) \cap T_y = T_y$ for some adjacent vertices x and y of G with $N_G[x] = N_G[y]$ and for some $a \in T_x$. Hence, both $W \setminus \{(x, a)\}$ and $(W \setminus \{(x, a)\}) \cup \{(y, b)\}$ are not resolving sets, a contradiction. Thus, (*iii*) holds.

Statement (*iv*) is proved similarly. If (*v*) does not hold, then $W \setminus \{(x, a)\}$ and $(W \setminus \{(x, a)\} \cup \{(y, b)\})$ are not hop dominating sets of $G[H]$ for all $y \in N_G(x)$ and $b \in V(H) \setminus T_x$ or $x = y$ and $b \in N_H(a)$. This is a contradiction to W being a 1-movable resolving hop dominating set of $G[H]$. Hence, (*v*) holds.

For the converse, suppose that W satisfies properties (*i*) to (*v*). By Theorem 5, W is a resolving hop dominating set of $G[H]$. Let $x \in V(G)$ and $a \in T_x$. Then $(x, a) \in W$ and

$$W \setminus \{(x,a)\} = \left[\bigcup_{v \in S \setminus \{x\}} (\{v\} \times T_v) \right] \cup [\{x\} \times (T_x \setminus \{a\})]$$

and for some $b \in N_H(a) \cap (V(H) \setminus T_x)$,

$$(W \setminus \{(x,a)\}) \cup \{(x,b)\} = \left[\bigcup_{z \in S \setminus \{x\}} (\{z\} \times T_z)\right] \cup [\{x\} \times ((T_x \setminus \{a\}) \cup \{b\})]$$

and

$$(W \setminus \{(x,a)\}) \cup \{(y,q)\} = \left[\bigcup_{p \in S \setminus \{(x,y)\}} (\{p\} \times T_p)\right] \cup [\{x\} \times (T_x \setminus \{a\})]$$
$$\cup [\{y\} \times (T_y \cup \{q\})]$$

for some $y \in V(G) \cap N_G(x)$ and $q \in V(H) \setminus T_y$.

By (i) to (v) and Theorem 5, for every $(x,a) \in W$ either $W \setminus \{(x,a)\}$ is a resolving hop dominating set of $G[H]$ or there exists $(y,b) \in N_{G[H]}((x,a)) \cap (V(G[H]) \setminus W)$ such that $(W \setminus \{(x,a)\}) \cup \{(y,b)\}$ is a resolving hop dominating set of $G[H]$. Therefore, W is a 1-movable resolving hop dominating set of $G[H]$. □

Corollary 5. *Let G be a nontrivial connected totally point determining graph with $\gamma(G) \neq 1$ and H be a nontrivial connected graph with $\triangle(H) \leq |V(H)| - 2$. Then*

$$\gamma^1_{mRh}(G[H]) = |V(G)|mln(H).$$

Proof: Let $S = V(G)$ and let R_x be an mln-set of H for each $x \in S$. Since $\gamma_G \neq 1$, $x \in N_G(S,2)$ for each $x \in S$. By Theorem 6, $W = \bigcup_{x \in S} [\{x\} \times R_x]$ is a 1-movable resolving hop dominating set of $G[H]$. Thus,

$$\gamma^1_{mRh}(G[H]) \leq |W| = |V(G)||R_x| = |V(G)|mln(H).$$

Now, if $W_0 = \bigcup_{x \in S_0} (\{x\} \times T_x)$ is a γ^1_{mRh}-set of $G[H]$ then $S_0 = V(G)$ and T_x is a 1-movable locating set of H for each $x \in V(G)$ by Theorem 6. Hence,

$$\gamma^1_{mRh}(G[H]) = |W_0| = |V(G)||T_x| \geq |V(G)|mln(H).$$

Therefore, $\gamma^1_{mRh}(G[H]) = |V(G)|mln(H)$. □

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