EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 440-453
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Hop Differentiating Hop Dominating Sets in Graphs

Sergio R. Canoy, Jr.[1,*], Chrisley Jade C. Saromines[1]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Premier Research Institute of Science and Mathematics, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. A subset S of $V(G)$, where G is a simple undirected graph, is hop dominating if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v,w) = 2$ and it is hop differentiating if $N_G^2[u] \cap S \neq N_G^2[v] \cap S$ for any two distinct vertices $u, v \in V(G)$. A set $S \subseteq V(G)$ is hop differentiating hop dominating if it is both hop differentiating and hop dominating in G. The minimum cardinality of a hop differentiating hop dominating set in G, denoted by $\gamma_{dh}(G)$, is called the hop differentiating hop domination number of G. In this paper, we investigate some properties of this newly defined parameter. In particular, we characterize the hop differentiating hop dominating sets in graphs under some binary operations.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop domination, hop differentiating, join, corona, lexicographic product

1. Introduction

Differentiating-domination in a graph, a variation of the standard domination, was defined by Gimbel et al. in [6]. A differentiating set in a given network can be viewed as a set of sensitive monitors used to safeguard a given facility, that is, to identify the exact location of an intruder (e.g. a burglar, a fire, etc.) whenever a problem in a facility arises. The requirement that the set have to be dominating would mean that every vertex where there is no monitor on it is connected to at least one monitoring device. Moreover, finding the differentiating-domination number of a graph is equivalent to finding the least number of monitors that can do the certain task in a given network. In other studies, a differentiating dominating set is also referred to as an identifying code (see [14]). Differentiating-domination and some related concepts had been studied in [3], [4], [9], [10], [13], [15], [17], and [18].

In 2015, Natarajan et al. (see [16]) introduced hop domination and made an initial investigation of the concept. The study has led other researchers to investigate it further

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4673
Email addresses: sergio.canoy@g.msuiit.edu.ph (S. Canoy), chrisley.saromines@g.msuiit.edu.ph (C. Saromines)

and define some of its variants. In fact, a number of variations of hop domination had already been investigated (see [1], [2], [7], [8], [11], [12], [19], [20], [21], and [22]).

In this paper, we define and do an initial study of the concept of hop differentiating hop dominating set in a graph. It must be pointed out that a hop differentiating set is 'almost' a hop dominating set because it may allow at most a vertex outside the set to be 'hop undominated'. A result that deals with the concept for disconnected graphs would show that the condition 'hop differentiating hop dominating' cannot always be replaced by 'hop differentiating'. This makes 'hop differentiating hop dominating' an interesting concept to consider. This present study is motivated by the introduction of hop domination and differentiating-domination concepts. The new parameter, just like differentiating-domination, can also be used to model the problem of determining the location of monitoring devices so as to identify the exact location of an intruder in a certain facility.

2. Terminology and Notation

Let $G = V(G), E(G))$ be an undirected graph. For any two vertices u and v of G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u,v)$ is called a *u-v geodesic*. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. The *open neighborhood* of $X \subseteq V(G)$ is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$. The *minimum degree* of G, denoted by $\delta(G)$, is given by $\delta(G) = \min\{deg_G(u) : u \in V(G)\}$, where $deg_G(u) = |N_G(u)|$.

A set $D \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if for every $v \in V(G) \setminus D$ (resp. $v \in V(G)$), there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$ (resp. $N_G(D) = V(G)$). The *domination number* (resp. *total domination number*) of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the minimum cardinality of a dominating (resp. total dominating) set in G. Any dominating (resp. total dominating) set in G with cardinality $\gamma(G)$ (resp. $\gamma_t(G)$), is called a γ-set (resp. γ_t-set) in G. If $\gamma(G) = 1$ and $\{v\}$ is a dominating set in G, then we call v a *dominating vertex* in G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u,v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v,u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The *closed hop neighborhood* of X is the set $N_G^2[X] = N_G^2(X) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* in G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality among all hop dominating sets in G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A set $S \subseteq V(G)$ is *differentiating* in G if for any two distinct vertices $v, w \in V(G)$, $N_G[v] \cap S \neq N_G[w] \cap S$. A differentiating set S is *differentiating-dominating* in G if

$N_G(v) \cap S \neq \varnothing$ for each $v \in V(G) \setminus S$. The smallest cardinality of a differentiating (resp. differentiating-dominating) set in G is denoted by $dn(G)$ (resp. $\gamma_D(G)$). Any differentiating (resp. differentiating-dominating) set in G with cardinality $dn(G)$ (resp. $\gamma_D(G)$) is called a dn-set (resp. γ_D-set). A set $S \subseteq V(G)$ is *hop differentiating* in G if $N^2_G[u] \cap S \neq N^2_G[v] \cap S$ for every two distinct vertices u and v of $V(G)$. A hop differentiating set in G which is also hop dominating is called a *hop differentiating hop dominating* set. The minimum cardinality of a hop differentiating (resp. hop differentiating hop dominating) set in G, denoted by $hdn(G)$ (resp. $\gamma_{dh}(G)$), is called the *hop differentiating number* (resp. *hop differentiating hop domination number*) of G. Any hop differentiating (resp. hop differentiating hop dominating) set in G with cardinality $hdn(G)$ (resp. $\gamma_{dh}(G)$) is called an hdn-set (resp. γ_{dh}-set). Suppose G is a non-trivial connected graph and suppose that there exist distinct vertices u and v of G such that $N^2_G[u] = N^2_G[v]$. Then $N^2_G[u] \cap S = N^2_G[v] \cap S$ for any set $S \subseteq V(G)$. This implies that G does not admit a hop differentiating set.

A connected graph G is *point determining* if distinct vertices have distinct open neighborhoods, that is, $N_G(a) \neq N_G(b)$ for distinct vertices $a, b \in V(G)$. Graph G is said to be *point distinguishing* if distinct vertices have distinct closed neighborhoods, that is, $N_G[a] \neq N_G[b]$ whenever $a, b \in V(G)$ and $a \neq b$ (see [5] and [23]). Graph G is *distance-two point determining* (resp. *distance-two point distinguishing*) if $N^2_G(x) \neq N^2_G(y)$ (resp. $N^2_G[x] \neq N^2_G[y]$) for any distinct vertices $x, y \in V(G)$. It is *totally distance-two point determining* if $N^2_G(x) \neq N^2_G(y)$ and $N^2_G[x] \neq N^2_G[y]$ for any distinct vertices $x, y \in V(G)$. G is *complement point distinguishing* if $V(G) \setminus N_G(x) \neq V(G) \setminus N_G(y)$ for any distinct vertices $x, y \in V(G)$. In other words, G is *complement point distinguishing* if $\overline{G}$ is point distinguishing.

A set $S \subseteq V(G)$ is *pointwise non-dominating* if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $v \notin N_G(u)$, i.e., $[V(G) \setminus N_G(v)] \cap S \neq \varnothing$. The minimum cardinality of a pointwise non-dominating set in G, denoted by $pnd(G)$, is called a *pointwise non-domination number* of G. Let G be a complement point distinguishing graph. A set $S \subseteq V(G)$ is *complement differentiating* in G (or differentiating in $\overline{G}$) if for any two distinct vertices $v, w \in V(G)$, $N_{\overline{G}}[v] \cap S = [V(G) \setminus N_G(v)] \cap S \neq [V(G) \setminus N_G(w)] \cap S = N_{\overline{G}}[w] \cap S$. A complement differentiating set S in G is called *complement differentiating-dominating* (or *complement differentiating and pointwise non-dominating* or *differentiating-dominating in* $\overline{G}$) if for each $v \in V(G) \setminus S$, $[V(G) \setminus N_G[v]] \cap S = N_{\overline{G}}(v) \cap S \neq \varnothing$. The smallest cardinality of a complement differentiating (resp. complement differentiating-dominating) set in G is denoted by $cdn(G)$ (resp. $cdpnd(G)$). Any complement-differentiating (resp. complement differentiating-dominating) set in G with cardinality $cdn(G)$ (resp. $cdpnd(G)$) is called a cdn-set (resp. a $cdpnd$-set) in G. Clearly, $cdn(G) = dn(\overline{G})$ and $cdpnd(G) = \gamma_D(\overline{G})$.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle\{v\}\rangle + H^v$. The *lexicographic product* $G[H]$ is the graph with vertex set

$V(G[H]) = V(G) \times V(H)$ and $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. Any non-empty set $C \subseteq V(G) \times V(H)$ can be expressed as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Specifically, $T_x = \{a \in V(H) : (x,a) \in C\}$ for each $x \in S$.

3. Results

Throughout, a graph is understood to be distance-two point distinguishing whenever a hop differentiating set is assumed (or mentioned) in it.

Lemma 1. *Let G be a graph on n vertices. Then*

$$\gamma_{dh}(G) \geq \lceil \frac{lnn + ln2}{ln2} \rceil.$$

Proof. Let S be a hop differentiating hop dominating set of G. Since S is a hop dominating set, $N_G^2[v] \cap S \neq \varnothing$ for every $v \in V(G)$. Moreover, because it is hop differentiating, it follows that $2^{|S|} > n$. Hence, $|S| \geq \lceil \frac{lnn+ln2}{ln2} \rceil$. In particular, if S is a γ_{dh}-set of G, then $\gamma_{dh}(G) \geq \lceil \frac{lnn+ln2}{ln2} \rceil$. □

Theorem 1. *Let $G_1, G_2, \ldots, G_k$ be the distinct (distance-two point distinguishing) components of G, where $k \geq 2$. Then S is a hop differentiating hop dominating set in G if and only if $S_j = S \cap V(G_j)$ is a hop differentiating hop dominating set in G_j for each $j \in \{1, 2, \ldots, k\}$.*

Proof. Suppose S is a hop differentiating hop dominating set in G and let $j \in \{1, 2, \ldots, k\}$. Let $v \in V(G_j) \setminus S_j$. Since $v \notin S$ and S is a hop dominating set, there exists $w \in S$ such that $v \in N_G^2(w)$. This implies that $w \in S_j$ and $v \in N_{G_j}^2(w)$. This shows that S_j is a hop dominating set in G_j. Next, let $a, b \in V(G_j)$ where $a \neq b$. Since S is a hop differentiating set

$$N_{G_j}^2[a] \cap S_j = N_G^2[a] \cap S \neq N_G^2[b] \cap S = N_{G_j}^2[b] \cap S_j.$$

Thus, S_j is a hop differentiating hop dominating set in G_j for each $j \in \{1, 2, \ldots, k\}$.

For the converse, suppose that $S_j = S \cap V(G_j)$ is a hop differentiating hop dominating set in G_j for each $j \in \{1, 2, \ldots, k\}$. Then clearly, S is a hop dominating set in G. Let $v, w \in V(G)$ with $v \neq w$ and let G_i and G_j be the components of G with $v \in V(G_i)$ and $w \in V(G_j)$. If $i \neq j$, then

$$N_G^2[v] \cap S = N_{G_i}^2[v] \cap S_i \neq N_{G_j}^2[w] \cap S_j = N_G^2[w] \cap S.$$

If $i = j$, then

$$N_G^2[v] \cap S = N_{G_i}^2[v] \cap S_i \neq N_{G_i}^2[w] \cap S_i = N_G^2[w] \cap S$$

since S_i is a hop differentiating set in G_i. Therefore, S is a hop differentiating hop dominating set in G. □

It is worth mentioning that Theorem 1 does not hold if 'hop differentiating hop dominating' is replaced by 'hop differentiating'. Indeed, if there are two distinct hop differentiating sets S_j and S_k which have each a single vertex in $V(G_j) \setminus S_j$ and $V(G_k) \setminus S_k$, respectively, such that these vertices are not hop-dominated in the respective components, then the set S cannot be a hop differentiating set in G.

The next result follows from Theorem 1.

Corollary 1. *Let $G_1, G_2, \ldots, G_k$ be the distinct components of G. Then $\gamma_{dh}(G) = \sum_{j=1}^{k} \gamma_{dh}(G_j)$.*

Corollary 2. *Let $G_1, G_2, \ldots, G_k$ be the distinct components of G. If each of these components is complete, then $\gamma_{dh}(G) = |V(G)|$. In particular, $\gamma_{dh}(K_n) = \gamma_{dh}(\overline{K}_n) = n$ for all $n \geq 1$.*

Proposition 1. *Let G be a graph on $n \geq 3$ vertices. Then $3 \leq \gamma_{dh}(G) \leq n$. Moreover, the following hold:*

(i) If $n = 3$ and $\gamma_{dh}(G) = 3$, then $G \in \{K_3, \overline{K}_3, K_1 \cup K_2\}$.

(ii) If $n = 4$, then $\gamma_{dh}(G) = 3$ if and only if G is a graph obtained from K_3 by attaching a pendant vertex to one of the vertices of K_3.

Proof. Suppose S is a γ_{dh}-set of G. Clearly, $\gamma_{dh}(G) \leq n$. Now, by Lemma 1,

$$\gamma_{dh}(G) \geq \lceil \frac{ln n + ln 2}{ln 2} \rceil \geq \lceil \frac{ln 3 + ln 2}{ln 2} \rceil = 3.$$

Next, suppose $n = 3$ and $\gamma_{dh}(G) = 3$. By Corollary 2, $G \in \{K_3, \overline{K}_3, K_1 \cup K_2\}$, showing that (i) holds.

Suppose now that $n = 4$ and $\gamma_{dh}(G) = 3$. Let $S = \{a, b, c\}$ be a γ_{dh}-set of G and let $v \in V(G) \setminus S$. Since

$$\gamma_{dh}(K_1 \cup K_3) = \gamma_{dh}(K_2 \cup K_2) = \gamma_{dh}(\overline{K}_2 \cup K_2) = \gamma_{dh}(\overline{K}_4) = \gamma_{dh}(K_4) = 4$$

by Corollary 2, and because G is distance-two point distinguishing,

$$G \notin \{K_1 \cup K_3, K_2 \cup K_2, \overline{K}_2 \cup K_2, \overline{K}_4, K_1 \cup P_3, K_4, P_4, C_4, K_{1,3}, H\},$$

where H is obtained from C_4 by adding an edge connecting the non-adjacent vertices of C_4. Since there are only eleven (11) non-isomorphic graphs of order four (4), it follows that G is a graph obtained from K_3 by attaching a pendant vertex to one of the vertices of K_3.

For the converse, suppose that G is a graph obtained from K_3 by attaching a pendant vertex to one of the vertices of K_3. Let $V(G) = \{a, b, c, v\}$ such that $\langle\{a, b, c\}\rangle = K_3$ and

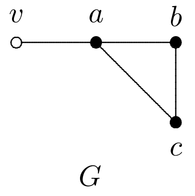

Figure 1: Graph G on 4 vertices and $\gamma_{dh}(G) = 3$

$va \in E(G)$ (see Figure 1). Let $S = \{a, b, c\}$. Since $d_G(v, b) = 2$, S is a hop dominating set in G. Moreover, since $N_G^2[v] \cap S = \{b, c\}$, $N_G^2[a] \cap S = \{a\}$, $N_G^2[b] \cap S = \{b\}$, and $N_G^2[c] \cap S = \{c\}$ are all distinct, S is a hop differentiating set. Thus, by the first part, $\gamma_{dh}(G) = 3$. This completes the proof of (ii). □

The next result is found in [11].

Theorem 2. *Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is hop dominating in $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating in G and H, respectively.*

Theorem 3. *Let G and H be any two (complement distance-two point distinguishing) graphs. Then $S \subseteq V(G+H)$ is hop differentiating hop dominating in $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are complement differentiating and pointwise non-dominating sets in G and H (differentiating-dominating in $\overline{G}$ and $\overline{H}$), respectively.*

Proof. Suppose that S is a hop differentiating hop dominating set in $G+H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. Since S is a hop dominating set in $G+H$, $S_G \neq \varnothing$ and $S_H \neq \varnothing$. By Theorem 2, S_G and S_H are pointwise non-dominating sets in G and H, respectively. If $S_G = V(G)$, then it is complement differentiating in G. Next, let $x, y \in V(G)$ where $x \neq y$. Since S is a hop differentiating set, $[V(G) \setminus N_G(x)] \cap S_G = N_{G+H}^2[x] \cap S \neq N_{G+H}^2[y] \cap S = [V(G) \setminus N_G(y)] \cap S_G$, showing that S_G is complement differentiating in G. Thus, S_G is a complement differentiating and pointwise non-dominating set in G. Similarly, S_H is a complement differentiating and pointwise non-dominating set in H.

For the converse, suppose that $S = S_G \cup S_H$ where S_G and S_H are complement differentiating and pointwise non-dominating sets in G and H, respectively. Then S is a hop dominating set in $G+H$ by Theorem 2. Next, let $a, b \in V(G+H)$ where $a \neq b$. Suppose that $a, b \in V(G)$. Since S_G is complement differentiating in G, $N_{G+H}^2[a] \cap S = [V(G) \setminus N_G(a)] \cap S_G \neq [V(G) \setminus N_G(b)] \cap S_G = N_{G+H}^2[b] \cap S$. Similarly, $N_{G+H}^2[a] \cap S = [V(H) \setminus N_H(a)] \cap S_H \neq [V(H) \setminus N_H(b)] \cap S_H = N_{G+H}^2[b] \cap S$ if $a, b \in V(H)$. Suppose now that $a \in V(G)$ and $b \in V(H)$. If $a \in S_G$, then $a \in (N_{G+H}^2[a] \cap S) \setminus (N_{G+H}^2[b] \cap S)$. If $a \notin S_G$, then there exists $d \in S_G \setminus N_G(a)$ because S_G is pointwise non-dominating in G. Hence, $d \in (N_{G+H}^2[a] \cap S) \setminus (N_{G+H}^2[b] \cap S)$. In either case, we have $N_{G+H}^2[a] \cap S \neq N_{G+H}^2[b] \cap S$. Therefore, S is a hop differentiating hop dominating set in $G+H$. □

Corollary 3. *Let G be a graph and let n be a positive integer. Then $S \subseteq V(K_n + G)$ is a hop differentiating hop dominating set in $K_n + G$ if and only if $S = V(K_n) \cup S_G$, where S_G is complement differentiating and pointwise non-dominating set in G.*

Proof. The only pointwise non-dominating set in K_n is $V(K_n)$. Thus, by Theorem 3, the result follows. □

The next results follow directly from Theorem 3 and Corollary 3.

Corollary 4. *Let G and H be any two graphs. Then*

$$\gamma_{dh}(G + H) = cdpnd(G) + cdpnd(H) = \gamma_D(\overline{G}) + \gamma_D(\overline{H}).$$

Corollary 5. *Let G be a graph and let n be a positive integer. Then $\gamma_{dh}(K_n + G) = n + cdpnd(G) = n + \gamma_D(\overline{G})$.*

The next result is a restatement of the one in [11].

Theorem 4. *Let G and H be any two graphs. A set $C \subseteq V(G)$ is a hop dominating set in $G \circ H$ if and only if $C = A \cup (\cup_{v \in V(G)} C_v)$, where $A \subseteq V(G)$ and $C_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:*

- (i) *For each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in N_G(w)$ with $C_y \neq \varnothing$.*
- (ii) *C_w is a pointwise non-dominating set in H^w for each $w \in V(G) \setminus N_G(A)$.*

Theorem 5. *Let G and H be non-trivial connected graphs such that H is complement point distinguishing. Then $S \subseteq V(G \circ H)$ is hop differentiating hop dominating in $G \circ H$ if and only if $S = A \cup [\cup_{v \in V(G)} D_v]$ and satisfies the following conditions:*

- (i) *D_w is a pointwise non-dominating set in H^w for each $w \in V(G) \setminus N_G(A)$.*
- (ii) *D_v is complement differentiating in H^v for each $v \in V(G)$.*
- (iii) *For any two distinct vertices $v, w \in V(G)$, $N_G(v) \neq N_G(w)$ or $N^2_G[v] \cap A \neq N^2_G[w] \cap A$.*
- (iv) *D_w is a total dominating set in H^w whenever $N_G(v) = \{w\}$ for some $v \in V(G)$.*
- (v) *If D_v and D_w, where $v \neq w$, are not pointwise non-dominating in H^v and H^w, respectively, then $N_G(v) \cap A \neq N_G(w) \cap A$.*

Proof. Suppose S is a hop differentiating hop dominating set in $G \circ H$. Let $A = S \cap V(G)$ and let $D_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = A \cup [\cup_{v \in V(G)} D_v]$ and, by Theorem 4, (i) holds. Let $v \in V(G)$ and let $a, b \in V(H^v)$ with $a \neq b$. Since S is a hop differentiating set,

$$([V(H^v) \setminus N_{H^v}(a)] \cap D_v) \cup [N_G(v) \cap A] = N^2_{G \circ H}[a] \cap S$$
$$\neq N^2_{G \circ H}[b] \cap S = ([V(H^v) \setminus N_{H^v}(b)] \cap D_v) \cup [N_G(v) \cap A].$$

Hence,

$$[V(H^v) \setminus N_{H^v}(a)] \cap D_v \neq [V(H^v) \setminus N_{H^v}(b)] \cap D_v,$$

showing that D_v is a complement differentiating set in H^v. Thus, (ii) holds. Next, let $v, w \in V(G)$ with $v \neq w$. Since S is a hop differentiating set,

$$\begin{aligned}
[N_G^2[v] \cap A] \cup [\cup_{x \in N_G(v)} D_x] &= N_{G\circ H}^2[v] \cap S \\
&\neq N_{G\circ H}^2[w] \cap S \\
&= [N_G^2[w] \cap A] \cup [\cup_{y \in N_G(w)} D_y].
\end{aligned}$$

This implies that $N_G^2[v] \cap A \neq N_G^2[w] \cap A$ or $N_G(v) \neq N_G(w)$, showing that (iii) holds. To show (iv), let $w \in V(G)$ such that $N_G(v) = \{w\}$ for some $v \in V(G)$. Suppose D_w is not a total dominating set in H^w. Then there exists $p \in V(H^w)$ such that $p \notin N_{H^w}(D^w)$. It follows that

$$N_{G\circ H}^2[p] \cap S = (N_G(w) \cap A) \cup [(V(H^w) \setminus N_{H^w}(p)) \cap D_w] = (N_G(w) \cap A) \cup D_w = N_{G\circ H}^2[v] \cap S,$$

a contradiction to the assumption that S is a hop differentiating set. Therefore, D_w is a total dominating set in H^w, showing that (iv) holds. Finally, suppose D_v and D_w, where $v \neq w$, are not pointwise non-dominating sets in H^v. Then there exist $p \in V(H^v) \setminus D_v$ and $q \in V(H^w) \setminus D_w$ such that $(V(H^v) \setminus N_{H^v}(p)) \cap D_v = \varnothing$ and $(V(H^w) \setminus N_{H^w}(q)) \cap D_w = \varnothing$. Since S is hop differentiating, $N_G(v) \cap A \neq N_G(w) \cap A$. This shows that (v) holds.

For the converse, suppose that S is as described and satisfies properties (i)-(v). Let $v \in V(G) \setminus A$ and choose any $u \in N_G(v)$. By (ii), D_u is complement differentiating and so $D_u \neq \varnothing$. Thus, S satisfies (i) and (ii) of Theorem 4, showing that it is a hop dominating set in $G \circ H$. Now let $a, b \in V(G \circ H)$ with $a \neq b$ and let $v, w \in V(G)$ such that $a \in V(v + H^v)$ and $b \in V(w + H^w)$. Consider the following cases:

Case 1: $v = w$

Suppose $a, b \in V(H^v)$. Since D_v is a complement differentiating set in H^v (by (ii)), $N_{G\circ H}^2[a] \cap S \neq N_{G\circ H}^2[b] \cap S$. Suppose $a = v$ and $b \in V(H^v)$. Pick any $z \in N_G(v)$. Since $D_z \subseteq N_{G\circ H}^2[a] \setminus N_{G\circ H}^2[b]$, it follows that $N_{G\circ H}^2[a] \cap S \neq N_{G\circ H}^2[b] \cap S$.

Case 2: $v \neq w$

Suppose $a = v$ and $b = w$. Then $v, w \in V(G)$. By property (iii), $N_G(v) \neq N_G(w)$ or $N_G^2[v] \cap A \neq N_G^2[w] \cap A$. If $N_G^2[v] \cap A \neq N_G^2[w] \cap A$, then $N_{G\circ H}^2[a] \cap S \neq N_{G\circ H}^2[b] \cap S$. Suppose $N_G(v) \neq N_G(w)$. We may assume that there exists $p \in N_G(v) \setminus N_G(w)$. Then $D_p \subseteq N_{G\circ H}^2[a] \setminus N_{G\circ H}^2[b]$. Hence, $N_{G\circ H}^2[a] \cap S \neq N_{G\circ H}^2[b] \cap S$.

Next, suppose that $a = v$ and $b \in V(H^w)$ (or $b = w$ and $a \in V(H^v)$). If $|N_G(v)| > 1$ or $vw \notin E(G)$, pick any $z \in N_G(v) \setminus \{w\}$. Then $D_z \subseteq N_{G\circ H}^2[a] \setminus N_{G\circ H}^2[b]$. It follows that $N_{G\circ H}^2[a] \cap S \neq N_{G\circ H}^2[b] \cap S$. Suppose that $N_G(v) = \{w\}$. By (iv), D_w is a total dominating set of H^w. Hence, $(V(H^w) \setminus N_{H^w}(b)) \cap D_w \neq D_w$. This would imply that $N_{G\circ H}^2[a] \cap S \neq N_{G\circ H}^2[b] \cap S$. Finally, suppose that $a \in V(H^v)$ and $b \in V(H^w)$. If $[V(H^v) \setminus N_{H^w}(a)] \cap D_v \neq \varnothing$ or $[V(H^w) \setminus N_{H^w}(b)] \cap D_w \neq \varnothing$, then $N_{G\circ H}^2[a] \cap S \neq N_{G\circ H}^2[b] \cap S$. Suppose both sets are empty. Then $N_G(v) \cap A \neq N_G(w) \cap A$ by (v). It follows that $N_{G\circ H}^2[a] \cap S \neq N_{G\circ H}^2[b] \cap S$.

Accordingly, S is a hop differentiating hop dominating set of $G \circ H$. □

Corollary 6. *Let G and H be non-trivial connected graphs such that $\delta(G) \geq 2$ and G and H are point determining and complement point distinguishing, respectively. Then*

$$\gamma_{dh}(G \circ H) \leq cdpnd(H)|V(G)|.$$

Proof. Let $A = \varnothing$ and let D_v be a *cdpnd*-set of H for each $v \in V(G)$. Then $S = A \cup [\cup_{v \in V(G)} D_v] = \cup_{v \in V(G)} D_v$ is a hop differentiating hop dominating set in $G \circ H$ by Theorem 5. Thus,

$$\gamma_{dh}(G \circ H) \leq |C| = cdpnd(H)|V(G)|.$$

This proves the assertion. □

We note that the bound given in Corollary 6 is tight. Indeed, if $G = H = K_2$, then $cdpnd(H) = 2$ and $\gamma_{dh}(G \circ H) = 4 = cdpnd(H)|V(G)|$.

The next result is found in [11].

Theorem 6. *Let G and H be connected non-trivial graphs. A subset $C = \bigcup_{x \in S}[\{x\} \times T_x]$ of $V(G[H]$ is a hop dominating set in $G[H]$ if and only if the following conditions hold.*

(*i*) *S is a hop dominating set in G.*

(*ii*) *T_x is a pointwise non-dominating set in H for each $x \in S \setminus N_G^2(S)$.*

Theorem 7. *Let G and H be non-trivial connected graphs such that G and H are, respectively, distance-two point distinguishing and complement point distinguishing. Then $C = \bigcup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a hop differentiating hop dominating set in $G[H]$ if and only if the following conditions hold:*

(*i*) *$S = V(G)$*

(*ii*) *T_x is a pointwise non-dominating set in H for each $x \in S \setminus N_G^2(S)$.*

(*iii*) *T_x is a complement differentiating set in H for all $x \in S$.*

(*iv*) *If $N_G^2(x) = N_G^2(y)$ for distinct vertices x and y, then T_x or T_y is pointwise non-dominating in H.*

Proof. Suppose C is a hop differentiating hop dominating set in $G[H]$. Then, by Theorem 6, (*ii*) holds. Suppose there exists $z \in V(G) \setminus S$. Pick distinct vertices $a, b \in V(H)$. Then $(z, a), (z, b) \in V(G[H]) \setminus C$ and so

$$N_{G[H]}^2[(z,a)] \cap C = \bigcup_{x \in N_G^2(z) \cap S} [\{x\} \times T_x] = N_{G[H]}^2[(z,b)] \cap C.$$

This implies that C is not a hop differentiating set, contrary to our assumption. Thus, $S = V(G)$, showing that (i) holds. Now let $x \in S$ and $p, q \in V(H)$ with $p \neq q$. Then $(x,p),(x,q) \in V(G[H])$ and

$$N^2_{G[H]}[(x,p)] \cap C = [\{x\} \times [(V(H) \setminus N_H(p)) \cap T_x]] \cup [\cup_{w \in N^2_G(x) \cap S}(\{w\} \times T_w)]$$

and

$$N^2_{G[H]}[(x,q)] \cap C = [\{x\} \times [(V(H) \setminus N_H(q)) \cap T_x]] \cup [\cup_{w \in N^2_G(x) \cap S}(\{w\} \times T_w)].$$

Since C is a hop differentiating set,

$$(V(H) \setminus N_H(p)) \cap T_x \neq (V(H) \setminus N_H(q)) \cap T_x.$$

Hence, T_x is a complement-differentiating set in H, showing that (iii) holds. Next, suppose that x and y are distinct vertices of G with $N^2_G(x) = N^2_G(y)$. Suppose T_x and T_y are not pointwise non-dominating sets. Then there exist $p \in V(H) \setminus T_x$ and $q \in V(H) \setminus T_y$ such that $[V(H) \setminus N_H(p)] \cap T_x = \varnothing$ and $[V(H) \setminus N_H(q)] \cap T_y = \varnothing$. Since $N^2_G(x) = N^2_G(y)$, it follows that $N^2_{G[H]}[(x,p)] \cap C = N^2_{G[H]}[(y,q)] \cap C$, contradicting the assumption that C is a hop differentiating set in $G[H]$. Thus, T_x or T_y is pointwise non-dominating in H, showing that (iv) holds.

For the converse, suppose that C satisfies properties (i)-(iv). Since (i) and (ii) hold, C is a hop dominating set by Theorem 6. Next, let $(v,q),(w,s) \in V(G[H])$ with $(v,q) \neq (w,s)$. Then

$$N^2_{G[H]}[(v,q)] \cap C = [\{v\} \times [(V(H) \setminus N_H(q)) \cap T_v] \cup [\bigcup_{z \in N^2_G(v)} [\{z\} \times T_z],$$

and

$$N^2_{G[H]}[(w,s)] \cap C = [\{w\} \times (V(H) \setminus N_H(s)) \cap T_w] \cup [\bigcup_{y \in N^2_G(w)} [\{y\} \times T_y].$$

Consider the following cases:

Case 1: $v = w$

Then $q, s \in V(H)$ with $q \neq s$. By (iii), T_v is a complement-differentiating set; hence, $[V(H) \setminus N_H(q)] \cap T_v \neq [V(H) \setminus N_H(s)] \cap T_v$. It follows that $N^2_{G[H]}[(v,q)] \cap C \neq N^2_{G[H]}[(v,s)] \cap C$.

Case 2: $v \neq w$

Suppose first that $d_G(v,w) \neq 2$. If $N^2_G(v) \neq N^2_G(w)$, then clearly, $N^2_{G[H]}[(v,q)] \cap C \neq N^2_{G[H]}[(w,s)] \cap C$. If $N^2_G(v) = N^2_G(w)$, then T_v or T_w is pointwise non-dominating in H by (iv). Hence, $N^2_{G[H]}[(v,q)] \cap C \neq N^2_{G[H]}[(w,s)] \cap C$. Next, suppose that $d_G(v,w) = 2$. Since G is distance-two point distinguishing, $N^2_G[v] \neq N^2_G[w]$. It follows that $N^2_{G[H]}[(v,q)] \cap C \neq N^2_{G[H]}[(w,s)] \cap C$.

Accordingly, C is a hop differentiating hop dominating set in $G[H]$. □

Corollary 7. *Let G and H be non-trivial connected graphs such that G and H are, respectively, distance-two point distinguishing and complement point distinguishing. Then*

$$\gamma_{dh}(G[H]) \leq |V(G)|cdpnd(H) = |V(G)|\gamma_D(\overline{H}).$$

If, in addition, G is also distance-two point determining and $\gamma(G) \neq 1$, then

$$\gamma_{dh}(G[H]) = |V(G)|cdn(H) = |V(G)|dn(\overline{H}).$$

Proof. Let $S = V(G)$ and let T_x be a *cdpnd*-set in H for each $x \in V(G)$. By Theorem 7, $C = \bigcup_{x\in S}[\{x\} \times T_x]$ is a hop differentiating hop dominating set in $G[H]$. It follows that $\gamma_{dh}(G[H]) \leq |C| = |V(G)|cdpnd(H)$.

Next, suppose that $\gamma(G) \neq 1$. Let $S' = V(G)$ and let R_x be a *cdn*-set in H for each $x \in S'$. Since $\gamma(G) \neq 1$, $x \in N^2_G(S')$ for each $x \in S'$. Thus, by Theorem 7, $C = \bigcup_{x\in S'}[\{x\} \times R_x]$ is a hop differentiating hop dominating set in $G[H]$. It follows that $\gamma_{dh}(G[H]) \leq |C| = |V(G)|cdn(H)$. Now, if $C_0 = \bigcup_{x\in S_0}[\{x\} \times T_x]$ is a γ_{dh}-set in $G[H]$, then $S_0 = V(G)$ and T_x is a complement-differentiating set in H for each $x \in V(G)$, by Theorem 7. Hence, $\gamma_{dh}(G[H]) = |C_0| = \sum_{x\in S_0}|T_x| \geq |V(G)|cdn(H)$. Therefore, $\gamma_{lh}(G[H]) = |V(G)|cdn(H)$. □

Corollary 8. *Let G and H be non-trivial connected graphs such that G and H are, respectively, totally distance-two point determining and complement point distinguishing. If $\gamma(G) = 1$, then $\gamma_{dh}(G[H]) = cdpnd(H) + (|V(G)| - 1)cdn(H)$.*

Proof. Let $D_G = \{v \in V(G) : \{v\}$ is a dominating set of $G\}$. Since G is distance-two point distinguishing, it follows that $|D_G| = 1$. Set $S = V(G)$. Let T_v be a *cdpnd*-set in H for $v \in D_G$ and let T_x be a *cdn*-set in H for each $x \in V(G) \setminus \{v\}$. Then, by Theorem 7, $C = [\bigcup_{x\in S\setminus\{v\}}(\{x\} \times T_x)] \cup (\{v\} \times T_v)$ is a hop differentiating hop dominating set in $G[H]$. Hence,

$$\gamma_{dh}(G[H]) \leq |C| = cdpnd(H) + (|V(G)| - 1)cdn(H).$$

Suppose now that $C^* = [\bigcup_{x\in S^*}(\{x\} \times R_x)]$ is a γ_{dh}-set in $G[H]$ and let $D_G = \{v\}$. By Theorem 7, $S^* = V(G)$, R_v is complement-differentiating and pointwise non-dominating and R_x is complement-differentiating in H for each $x \in V(G) \setminus \{v\}$. Thus,

$$\gamma_{dh}(G[H]) = |C^*| = |R_v| + \sum_{x\in S^*\setminus\{v\}} |R_x| \geq cdpnd(H) + (|V(G)| - 1)cdn(H).$$

Therefore, $\gamma_{dh}(G[H]) = cdpnd(H) + (|V(G)| - 1)cdn(H)$ as asserted. □

Corollary 9. *Let G be a non-trivial connected totally distance-two point determining graph and let $p \geq 2$ be a positive integer. Then*

$$\gamma_{dh}(G[K_p]) = \begin{cases} (p-1)|V(G)| & if\ \gamma(G) \neq 1 \\ (p-1)|V(G)| + 1 & if\ \gamma(G) = 1. \end{cases}$$

Proof. Suppose first that $\gamma(G) \neq 1$. By Corollary 7 and the fact that $cdn(K_p) = dn(\overline{K}_p) = p - 1$, it follows that $\gamma_{dh}(G[K_p]) = (p-1)|V(G)|$.

Next, suppose that $\gamma(G) = 1$. By Corollary 8 and the fact that $cdpnd(K_p) = \gamma_D(\overline{K}_p) = p$, we have $\gamma_{dh}(G[K_p]) = p + (p-1)(|V(G)| - 1) = (p-1)|V(G)| + 1$. □

Corollary 10. *Let H be a non-trivial connected complement point distinguishing graph and let $p \geq 2$ be a positive integer. Then $\gamma_{dh}(K_p[H]) = p[cdpnd(H)]$.*

Proof. Let $G = K_p$. Then v is a dominating vertex of G for each $v \in V(G)$. Thus, if $C_0 = \bigcup_{z \in S_0}[\{z\} \times T_z]$ is a γ_{dh}-set of $G[H]$, then $S_0 = V(G)$ and each T_z is a $cdpnd$-set of H by Theorem 7. Consequently, $\gamma_{dh}(K_p[H]) = p[cdpnd(H)]$. □

4. Conclusion

Hop differentiating hop domination is introduced and studied for some graphs. In particular, characterizations of the hop differentiating hop dominating sets in the join, corona, and lexicographic product of two graphs are given. These characterizations are used to obtain either an upper bound or the exact value of the hop differentiating hop domination number of the graph. The concept can be studied further for other interesting graphs and the complexity of the hop differentiating hop dominating decision problem can likewise be investigated.

5. Acknowledgements

The authors would like to thank the referees for the invaluable assistance they gave us through their comments and suggestions which contributed to the improvement of the paper. The authors would like to thank the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP)-Philippines and MSU-Iligan Institute of Technology for funding this research.

References

[1] S. Ayyaswamy, B. Krishnakumari, B. Natarjan, and Y. Venkatakrishnan. Bounds on the hop domination number of a tree. *Proceedings-Mathematical Sciences.*, 125(4):449–455, 2015.

[2] S. Ayyaswamy, C. Natarajan, and G. Sathiamoorphy. A note on hop domination number of some special families of graphs. *International Journal of Pure and Applied Mathematics.*, 119(12):11465–14171, 2018.

[3] C. Colbourn, P. Slater, and L. Stewart. Locating-dominating sets in seriesparallel networks. *Congr. Numer.*, 56:135–162, 1987.

[4] A. Finbow and B. Hartnell. Locating-dominating sets in seriesparallel networks. *Congr. Numer.*, 65:191–200, 1988.

[5] D. Geoffrey. Nuclei for totally point determining graphs. *Discrete Mathematics*, 21:145–162, 1978.

[6] J. Gimbel, B. van Gorden, M. Nicolescu, C. Umstead, and N. Vaianna. Location with dominating sets. *Congress Numer.*, 151:129–144, 2001.

[7] J. Hassan and S. Canoy Jr. Hop independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 15(4):1783–1796, 2022.

[8] M. Henning and N. Rad. On 2-step and hop dominating sets in graphs. *Graphs and Combinatorics.*, 33(4):913–927, 2017.

[9] S. Canoy Jr. and G. Malacas. Determining the intruder's location in a given network: Locating-dominating sets in a graph. *NRCP Research Journal*, 13(1):1–8, 2013.

[10] S. Canoy Jr. and G. Malacas. Differentiating-dominating sets in graphs under binary operations. *Tamkang Journal of Mathematics*, 46(1):51–60, 2015.

[11] S. Canoy Jr., R. Mollejon, and J. G. Canoy. Hop dominating sets in graphs under binary operations. *Eur. J. Pure Appl. Math.*, 12(4):1455–1463, 2019.

[12] S. Canoy Jr. and G. Salasalan. A variant of hop dominationin graphs. *Eur. J. Pure Appl. Math.*, 15(2):342–353, 2021.

[13] S. Canoy Jr. and G. Salasalan. Locating-hop domination in graphs. *Kyungpook Mathematical Journal*, 62:193–204, 2022.

[14] M. Karpovsky, K. Chakrabarty, and L. Levitin. On a new class of codes for identifying vertices in graphs. *IEEE Trans. Inform. Theory*, 44(2):599–611, 1998.

[15] S. Canoy Jr., G. Malacas and D. Tarepe. Locating-dominating sets in graphs. *Applied Mathematical Sciences*, 8:4381–4388, 2014.

[16] C. Natarajan and S. Ayyaswamy. Hop domination in graphs ii. *Versita*, 23(2):187–199, 2015.

[17] B. Omamalin, S. Canoy Jr., and H Rara. Locating total dominating sets in the join, corona and composition of graphs. *Applied Mathematical Sciences*, 8:2363–2374, 2014.

[18] S. Omega and S. Canoy Jr. Locating sets in a graph. *Applied Mathematical Sciences*, 9:2957–2964, 2015.

[19] Y. Pabilona and H. Rara. Connected hop domination in graphs under some binary operations. *Asian-Eur. J. Math.*, 11(5):1850075–1–1850075–11, 2018.

[20] R. Rakim, H. Rara, and C.J. Saromines. Perfect hop domination in graphs. *Applied Mathematical Sciences*, 1(13):635–649, 2018.

[21] G. Salasalan and S. Canoy Jr. Global hop domination numbers of graphs. *Eur. J. Pure Appl. Math.*, 14(1):112–125, 2021.

[22] G. Salasalan and S. Canoy Jr. Revisiting domination, hop domination, and global hop domination in graphs. *Eur. J. Pure Appl. Math.*, 14(4):1415–1428, 2021.

[23] D.P. Summer. Point determination in graphs*. *Discrete Mathematics, North-Holland Publishing Company*, pages 179–187, 1973.