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First and Third Isomorphism Theorems for the Dual B-Algebra

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Abstract. In this paper, some properties of the dual B-homomorphism are provided, along with the natural dual B-homomorphism and the fundamental theorem of dual B-homomorphisms for dual B-algebras. The first and third isomorphism theorems for the dual B-algebra are also presented in the paper.

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1. Introduction

K. Belleza and J.P. Vilela in their paper in 2019 [2] introduced the dual B-Algebra, its relationship with other algebras, and its characteristics. More studies were then conducted on the said topic. One of the recent papers published by K. Belleza and J.R. Albaracin in 2022 [1] discussed about dual B-filters and dual B-subalgebras in a topological dual B-algebra, wherein the researchers first constructed a congruence relation on a dual B-algebra which is necessary in creating a natural homomorphism from one dual B-algebra onto another; an important first step in this study.

While many other algebraic structures prepared different approaches in constructing isomorphism to their respective algebras (see [6], [3], [5]), J. Neggers and H.S. Kim in particular, presented a fundamental theorem of B-homomorphism for B-algebras and using the said theorem created the 1st and 3rd isomorphism theorems for the B-algebra in 2002 [7]. Later in 2015, J.C. Endam and J.P. Vilela also provided more insights on the properties of normal subsets of B-algebra and B-homomorphism, and presented proof for the 2nd isomorphism theorem for the B-algebras [4].

This, in turn warrants a need for investigation of the dual B-algebra as to whether the isomorphism theorems can be constructed within the dual B-algebra since there exists a close relationship between the B-algebra and the dual B-algebra [2].

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2. Preliminaries

Definition 1. [2] A *Dual B-Algebra*, (or dB-algebra), X is a triple $(X, \cdot, 1)$ where X is a non-empty set with a binary operation " $\cdot$ " and a constant 1 satisfying the following axioms for all x, y, z in X:

$$\text{(DB1) } x \cdot x = 1; \quad \text{(DB2) } 1 \cdot x = x; \quad \text{(DB3) } x \cdot (y \cdot z) = ((y \cdot 1) \cdot x) \cdot z.$$

Example 1. [1] Let $X = \{1, a, b, c\}$ with the binary operation $\cdot$ as defined in the table:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $(X, \cdot, 1)$ is a dB-algebra.

Lemma 1. [2] Let $(X, \cdot, 1)$ be a dB-algebra, then for any $x, y \in X$, $x \cdot y = 1$ implies $x = y$

Definition 2. [1] Let X be a dB-algebra and S a nonempty subset of X. Then S is called a *dual B-subalgebra*, (or dB-subalgebra), of X if S itself is a dB-algebra with binary operation of X on S.

Remark 1. [1] If S is a dB-subalgebra of X, then $1 \in S$.

Theorem 1. [1] S is a dB-subalgebra if and only if for any $x, y \in S$, $x \cdot y \in S$.

Example 2. Consider the dB-algebra $X = \{1, a, b, c\}$ with the binary operation $\cdot$ as defined in the table:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

By Remark 2.6, $A = (1, a)$ is a dB-subalgebra of X but $B = (1, a, c)$ is not a dB-subalgebra of X since $a \cdot c = b \notin B$.

Definition 3. [1] Let X be a dB-algebra. A subset F of X is called a *dual B-filter*, (or dB-filter), if it satisfies the following:

(i.) $1 \in F$;

(ii.) for each $x, y \in X$, $x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Proposition 1. [1] If F is a dB-filter of a dB-algebra X, then F is a dB-subalgebra of X.

Definition 4. [1] Let X be a dB-algebra and N a nonempty subset of X. Then N is a *normal* subset of X if for any $a \cdot b, x \cdot y \in N, (a \cdot x) \cdot (b \cdot y) \in N$. A dB-filter F of a dB-algebra X is called a *normal dB-filter* if F is a normal subset of X. A dB-subalgebra S of a dB-algebra X is called a *normal dB-subalgebra* if S is a normal subset of X.

Example 3. [1] Let $X = \{1, a, b, c, d, e\}$ with the binary operation $\cdot$ as defined in the table:

$\cdot$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	b	1	a	d	e	c
b	a	b	1	e	c	d
c	c	d	e	1	a	b
d	d	e	c	b	1	a
e	e	c	d	a	b	1

Then $(X, \cdot, 1)$ is a dB-algebra. So,

(a) The set A=$\{1, a, e\}$ is not a dB-filter since $\exists e \cdot c = a$ but $c \notin A$

(b) The set B=$\{1, c\}$ is a dB-filter but is not normal since $\exists c \cdot 1 = a \cdot e = c \in B$ but $(c \cdot a) \cdot (1 \cdot e) = d \cdot e = a \notin B$

(c) The set C=$\{1, a, b\}$ is a normal dB-filter.

Theorem 2. [1] Let $(X, \cdot, 1)$ be a dB-algebra and S a normal dB-subalgebra of X. The relation defined by $x \sim y$ if and only if $x \cdot y, y \cdot x \in S$ is a *congruence relation* on X for any $x, y \in X$.

Definition 5. [1] Let $(X, \cdot, 1)$ be a dB-algebra and S a normal dB-subalgebra of X. Define a *congruence class* $[x]_S$ by $[x]_S = \{y \in X | y \sim x\}$ and define X/S to be the set of all congruence classes of X, that is $X/S = \big\{[x]_S | x \in X\big\}$.

3. Results

Lemma 2. Let S be a normal dB-subalgebra of a dB-algebra $(X, \cdot, 1)$ and $x, y \in X$. Then $[x]_S = [y]_S$ if and only if $x \sim y$.

Proof. Suppose $[x]_S = [y]_S$. Then $z \in [x]_S$ implies that $z \in [y]_S$. We have that $z \sim x, z \sim y$ and since $\sim$ is symmetric and transitive by Theorem 2, then $z \sim x, z \sim y$ implies $x \sim z, z \sim y$ which implies that $x \sim y$.

Now, suppose $x \sim y$. Then $x \cdot y, y \cdot x \in S$. Let $z \in [x]_S$, then $z \sim x$. We have that $z \sim x, x \sim y$ implies $z \sim y$ which implies that $z \in [y]_S$. Hence, $[x]_S \subseteq [y]_S$. Similarly, let $a \in [y]_S$, then $a \sim y$. We have that $a \sim y, x \sim y$ implies $a \sim y, y \sim x$ which implies that $a \sim x$ and so $a \in [x]_S$. Thus $[y]_S \subseteq [x]_S$ and it follows that $[x]_S = [y]_S$. $\square$

Theorem 3. Let S be a normal dB-subalgebra of a dB-algebra $(X, \cdot, 1_X)$. Then $(X/S, *, [1]_S)$ with the binary operation $*$ on X/S defined by

$$[x]_S * [y]_S = [x \cdot y]_S \text{ for all } x, y \in X$$

is a dB-algebra. X/S is called the *quotient dB-algebra* of X by S.

Proof. Let $x_1, x_2, y_1, y_2 \in X$ such that $[x_1]_S = [x_2]_S$ and $[y_1]_S = [y_2]_S$. Then $x_1 \sim x_2$ and $y_1 \sim y_2$. Since $\sim$ is a congruence relation, we have that $x_1 \cdot y_1 \sim x_2 \cdot y_2$ and by Lemma 2, $[x_1 \cdot y_1]_S = [x_2 \cdot y_2]_S$ which implies that $[x_1]_S * [y_1]_S = [x_2]_S * [y_2]_S$. Hence $*$ is well-defined.

Now, for all $x, y, z \in X$,

$$[x]_S * [x]_S = [x \cdot x]_S = [1]_S \tag{DB1}$$
$$[1]_S * [x]_S = [1 \cdot x]_S = [x]_S \tag{DB2}$$
$$\begin{aligned} [x]_S * \big([y]_S * [z]_S\big) &= [x]_S * \big([y \cdot z]_S\big) = [x \cdot (y \cdot z)]_S = [((y \cdot 1) \cdot x) \cdot z]_S \\ &= [(y \cdot 1) \cdot x]_S * [z]_S = \big([y \cdot 1]_S * [x]_S\big) * [z]_S \\ &= \big(([y]_S * [1]_S) * [x]_S\big) * [z]_S \end{aligned} \tag{DB3}$$

Hence, $(X/S, *, [1]_S)$ is a dB-algebra. □

Proposition 2. Let $(X, \cdot, 1)$ be a dB-algebra and S be a subset of X. Then S is a normal dB-subalgebra of X if and only if S is a normal dB-filter of X.

Proof. Suppose S is a normal dB-filter of X. It follows from Proposition 1 and S as a normal subset of X that S is a normal dB-subalgebra of X.

Now, suppose S is a normal dB-subalgebra of X. Let $x, y \in X$ such that $x \cdot y \in S$ and $x \in S$. Since S is a dB-subalgebra, then $1 \in S$. Since $1, x \in S$ and S is closed by Theorem 1, we have that $x \cdot 1 \in S$ and since $x \cdot y \in S$, it follows that $(x \cdot x) \cdot (1 \cdot y) \in S$ since S is normal. Then, $y = 1 \cdot y = 1 \cdot (1 \cdot y) = (x \cdot x) \cdot (1 \cdot y) \in S$. Hence, S is a normal dB-filter. □

Definition 6. Let $(X, \cdot, 1_X)$ and $(Y, *, 1_Y)$ be dB-algebras. A mapping $\Phi : X \to Y$ is called a *dual B-homomorphism* (or dB-homomorphism), from X into Y if

$$\Phi(x \cdot y) = \Phi(x) * \Phi(y)$$

for any $x, y \in X$.

A dB-homomorphism Φ is called *dB-monomorphism*, *dB-epimorphism*, or *dB-isomorphism* (denoted by $X \cong Y$), if Φ is one-to-one, onto, or a bijection, respectively. An isomorphism $\Phi : X \to X$ is called *dB-automorphism*.

The *kernel* of the dB-homomorphism Φ, denoted by $\ker \Phi$, is the set whose elements of X are mapped to 1_Y.

Example 4. Let $(\mathbb{R}^+, \cdot, 1)$ be a dB-algebra with the binary operator $\cdot$ be defined as $x \cdot y = \frac{y}{x}$ for all x, y in $\mathbb{R}^+$.
Define $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Phi(x) = x^2$ for all $x \in \mathbb{R}^+$.
For all $x, y \in \mathbb{R}^+$, $x = y$ implies $x^2 = y^2$ which implies that $\Phi(x) = \Phi(y)$. Hence, Φ is well-defined.
Now,

$$\Phi(x \cdot y) = \Phi\left(\frac{y}{x}\right) = \frac{y^2}{x^2} = x^2 \cdot y^2 = \Phi(x) \cdot \Phi(y)$$

for all $x, y \in \mathbb{R}^+$. Hence, Φ is a dB-homomorphism.
Suppose that $\Phi(x) = \Phi(y)$ for all $x, y \in \mathbb{R}^+$, then $x^2 = y^2$ implies that $x = y$ which means Φ is one-to-one. Now, for all $y \in \mathbb{R}^+$, $\exists x \in \mathbb{R}^+$ such that $x = \sqrt{y}$ implies $x^2 = y$ which implies that $\Phi(x) = y$ and so Φ is onto. Consequently, Φ is a dB-automorphism.
The kernel of this dB-automorphism is

$$\begin{aligned} \ker \Phi &= \{x \in \mathbb{R}^+ | \Phi(x) = 1\} \\ &= \{x \in \mathbb{R}^+ | x^2 = 1\} \\ &= \{x \in \mathbb{R}^+ | x = 1\} \\ &= \{1\} \end{aligned}$$

The next corollary, which is needed for the following results, is immediate from Lemma 1 and DB1.

Corollary 1. Let $(X, \cdot, 1)$ be a dB-algebra, then for any $x, y \in X$, $x = y$ implies that $x \cdot y = 1$.

Theorem 4. Let $\Phi : X \to Y$ be a dB-homomorphism, $(X, \cdot, 1_X)$, $(Y, *, 1_Y)$ be dB-algebras, and $S \subseteq X$, then

(i) $\Phi(1_X) = 1_Y$

(ii) Φ is a dB-monomorphism, if and only if $\ker \Phi = \{1_X\}$

(iii) $Im(\Phi)$ is a dB-subalgebra of Y.

(iv) $\ker \Phi$ is a dB-filter of X and consequently a dB-subalgebra of X.

(v) If S is a dB-filter of X, then $\Phi(S)$ is a dB-filter of Y and consequently a dB-subalgebra of Y.

Proof. Suppose $\Phi : X \to Y$ be a dB-homomorphism and $S \subseteq X$,

(i) Since Φ is a dB-homomorphism and by DB1,

$$\Phi(1_X) = \Phi(1_X \cdot 1_X) = \Phi(1_X) * \Phi(1_X) = 1_Y.$$

(ii) Suppose Φ is a dB-monomorphism. It follows from (i.) that $1_X \in \ker\Phi$. Let $x \in \ker\Phi$. then $\Phi(x) = 1_Y = \Phi(1_X)$. Since Φ is one-to-one, $\Phi(x) = \Phi(1_X)$ implies $x = 1_X$. Hence, $\ker\Phi = \{1_X\}$. Conversely, suppose $\ker\Phi = \{1_X\}$ and $x, y \in X$ such that $\Phi(x) = \Phi(y)$. By Corollary 1, $\Phi(x) * \Phi(y) = 1_Y = \Phi(x \cdot y)$. Then $x \cdot y \in \ker\Phi$. Since $\ker\Phi = \{1_X\}$, $x \cdot y = 1_X$ and it follows that $x = y$ by Lemma 1. Hence, Φ is one-to-one i.e. Φ is a dB-monomorphism.

(iii) Let $x, y \in Im(\Phi)$. Then there exists $a, b \in X$ such that $x = \Phi(a)$, $y = \Phi(b)$. This implies that $x * y = \Phi(a) * \Phi(b) = \Phi(a \cdot b) \in Im(\Phi)$ since $a \cdot b \in X$. Thus, $Im(\Phi)$ is a dB-subalgebra of Y.

(iv) By Definition 6, $\ker\Phi \subseteq X$ and by (i.), $1_X \in \ker\Phi$ which also implies that $\ker\Phi \neq \varnothing$. Let $x \cdot y \in \ker\Phi$ and $x \in \ker\Phi$. Then for all $y \in X$,

$$\Phi(y) = 1_Y * \Phi(y) = \Phi(x) * \Phi(y) = \Phi(x \cdot y) = 1_Y.$$

Hence, $y \in \ker\Phi$ and it follows that $\ker\Phi$ is a dB-filter of X. Consequently, by Theorem 1, $\ker\Phi$ is a dB-subalgebra of X.

(v) Let S be a dB-filter of X, then $1_X \in S$ and by (i.), $\Phi(1_X) = 1_Y \in \Phi(S)$. Now, for all $x, y \in X$ such that $x \in S$ and $x \cdot y \in S$ implies that $\Phi(x) \in \Phi(S)$ and $\Phi(x) * \Phi(y) = \Phi(x \cdot y) \in S$. Since S is a dB-filter of X, then $y \in S$ also implies that $\Phi(y) \in \Phi(S)$. Hence, $\Phi(S)$ is a dB-filter of Y. Consequently, by Theorem 1, $\Phi(S)$ is a dB-subalgebra of Y. □

Theorem 5. Let S be a normal dB-subalgebra (normal dB-filter) of a dB-algebra $(X, \cdot, 1)$. Then the mapping $\Phi : (X, \cdot, 1) \to (X/S, *, [1]_S)$ given by $\Phi(x) = [x]_S$ for all $x \in X$ is a dB-epimorphism and $\ker\Phi = S$. The mapping Φ in this case is called the *natural dB-homomorphism* of X onto X/S.

Proof. Let $x, y \in X$ such that $x = y$ which by Corollary 1, $x \cdot y = 1 \in S$ and $y \cdot x = 1 \in S$. Then $x \sim y$ implies $[x]_S = [y]_S$ which implies that $\Phi(x) = \Phi(y)$. Hence, Φ is well-defined.

Now, let $a, b \in X$. Then $\Phi(a \cdot b) = [a \cdot b]_S = [a]_S * [b]_S = \Phi(a) * \Phi(b)$. This shows that Φ is a dB-homomorphism. Since $\Phi(X) = \{\Phi(a) : a \in X\} = \{[a]_S : a \in X\} = X/S$, it shows that Φ is onto and so Φ is a dB-epimorphism.

To show that $\ker\Phi = S$, let $x \in \ker\Phi$. Then $[x]_S = \Phi(x) = [1]_S$ and so $x \sim 1$. It follows that $x \cdot 1 \in S$ and $1 \cdot x \in S$. Since $1 \in S$ and S is also a dB-filter by Proposition 2, then $x \in S$ and so $\ker\Phi \subseteq S$. Now, let $x \in S$. By Remark 1, $1 \in S$, and since S is closed by Theorem 1, $1 \cdot x \in S$ and $x \cdot 1 \in S$. Then $x \sim 1$, and so $[x]_S = [1]_S$. Since $\Phi(x) = [x]_S = [1]_S$, then $x \in ker\Phi$. This implies that $S \subseteq \ker\Phi$ and it follows that $\ker\Phi = S$. □

Lemma 3. Let $f : (X, \cdot, 1_X) \to (Y, *, 1_Y)$ and $g : (Y, *, 1_Y) \to (Z, *', 1_Z)$ be dB-homomorphisms, then $g \circ f : (X, \cdot, 1_X) \to (Z, *', 1_Z)$ is also a dB-homomorphism ($\circ$ is the usual composition of functions).

Proof. Let $x, y \in X$. Since f and g are dB-homomorphisms, then

$$(g \circ f)(x \cdot y) = g\big(f(x \cdot y)\big) = g\big(f(x) * f(y)\big) = g\big(f(x)\big) *' g\big(f(y)\big) = (g \circ f)(x) *' (g \circ f)(y).$$

Hence, $g \circ f$ is a dB-homomorphism. □

Theorem 6. *Fundamental Theorem of dB-homomorphism for dB-Algebras*

Let Φ be a dB-homomorphism of a dB-algebra $(X, \cdot, 1_X)$ onto a dB-algebra $(Y, *, 1_Y)$, $S \subseteq \ker \Phi$ be a normal dB-subalgebra (normal dB-filter) of X, and g be the natural dB-homomorphism of X onto $(X/S, \theta, [1]_S)$. Then there exists a unique dB-homomorphism h of X/S onto Y such that $\Phi = h \circ g$. Furthermore, h is one-to-one if and only if $S = \ker \Phi$.

Proof. Define the map $h : X/S \to Y$ by $h\big([x]_S\big) = \Phi(x)$ for all $[x]_S \in X/S$.
Let $[x]_S, [y]_S \in X/S$ such that $[x]_S = [y]_S$. Then $x \sim y$, so $x \cdot y \in S$ and $y \cdot x \in S$. Since $S \subseteq \ker \Phi$, $x \cdot y \in \ker \Phi$ and $y \cdot x \in \ker \Phi$. Thus $\Phi(x) * \Phi(y) = \Phi(x \cdot y) = 1_Y$ and $\Phi(y) * \Phi(x) = \Phi(y \cdot x) = 1_Y$. By Lemma 1, $\Phi(x) = \Phi(y)$ and so $h\big([x]_S\big) = h\big([y]_S\big)$. Hence, h is well-defined.

Let $[x]_S, [y]_S \in X/S$. Then

$$h\big([x]_S \theta [y]_S\big) = h\big([x \cdot y]_S\big) = \Phi(x \cdot y) = \Phi(x) * \Phi(y) = h\big([x]_S\big) * h\big([y]_S\big).$$

Thus, h is a dB-homomorphism.

Since Φ is onto, for all $y \in Y$ there exists $x \in X$ such that $\Phi(x) = y$. As $h\big([x]_S\big) = \Phi(x)$ for all $[x]_S \in X/S$, it follows that there exists $[x]_S \in X/S$ such that $h\big([x]_S\big) = y$ for all $y \in Y$. Hence, h is onto.

Suppose $h' : X/S \to Y$ is another function such that $\Phi = h' \circ g$. Let $[x]_S \in X/S$, then $h'\big([x]_S\big) = h'\big(g(x)\big) = (h' \circ g)(x) = \Phi(x) = h\big([x]_S\big)$. Thus, $h'\big([x]_S\big) = h\big([x]_S\big)$ for all $[a]_S \in X/S$, i.e. h is unique.

Now, to show that h is one-to-one if and only if $S = \ker \Phi$, suppose h is one-to-one and $x \in \ker \Phi$. Then $h\big([x]_S\big) = \Phi(x) = 1_Y = h\big([1]_S\big)$ and since h is one-to-one, $[x]_S = [1]_S$. It follows that $x \sim 1_X$, and so $x \cdot 1_X \in S$ and $1_X \cdot x \in S$. Since $1_X \in S$ and S is a dB-filter, $x \in S$. Thus, $\ker \Phi \subseteq S$ and since $S \subseteq \ker \Phi$ by hypothesis, $\ker \Phi = S$.

Suppose that $\ker \Phi = S$ and $[x]_S, [y]_S \in X/S$ such that $h\big([x]_S\big) = h\big([y]_S\big)$. Then $\Phi(x) = \Phi(y)$. By Corollary 1, $1_Y = \Phi(x) * \Phi(y) = \Phi(x \cdot y)$ which implies that $x \cdot y \in \ker \Phi = S$. Similarly, $1_Y = \Phi(y) * \Phi(x) = \Phi(y \cdot x)$ implies that $y \cdot x \in S$. Hence, $x \sim y$ and it follows that $[x]_S = [y]_S$, showing that h is one-to-one. □

Theorem 7. *First Isomorphism Theorem for the dB-Algebra*

Let Φ be a dB-homomorphism of a dB-algebra $(X, \cdot, 1_X)$ into a dB-algebra $(Y, *, 1_Y)$, then $(X/\ker \Phi, \theta, [1]_{ker\Phi}) \cong \Phi(X)$.

Proof. Let $S = \ker \Phi$, g be the natural dB-homomorphism from X onto X/S, the mapping $f : X/S \to \Phi(X)$ be defined by $f\big([x]_S\big) = \Phi(x)$ for all $[x]_S \in X/S$, and recall that $\Phi(X)$ is a dB-subalgebra of Y by Theorem 4 (iii) which implies that $\Phi(X)$ has the same binary operator as Y.

Let $[x]_S, [y]_S \in X/S$ such that $[x]_S = [y]_S$. Then $x \sim y$ and it follows that $x \cdot y \in S$ and $y \cdot x \in S$. Since $S = ker\Phi$, $\Phi(x \cdot y) = \Phi(x) * \Phi(y) = 1_Y = \Phi(y) * \Phi(x) = \Phi(y \cdot x)$ and by Lemma 1, $\Phi(x) = \Phi(y)$ which is $f\big([x]_S\big) = f\big([y]_S\big)$. Hence, f is well-defined.

Let $[x]_S, [y]_S \in X/S$. Then

$$f\big([x]_S \theta [y]_S\big) = f\big([x \cdot y]_S\big) = \Phi(x \cdot y) = \Phi(x) * \Phi(y) = f\big([x]_S\big) * f\big([y]_S\big)$$

Thus, f is a dB-homomorphism.

Let $[x]_S, [y]_S \in X/S$ such that $f\big([x]_S\big) = f\big([y]_S\big)$. Then $\Phi(x) = \Phi(y)$. It follows that $1_Y = \Phi(x) * \Phi(y) = \Phi(x \cdot y)$ which implies that $x \cdot y \in \ker \Phi = S$. Similarly, $1_Y = \Phi(y) * \Phi(x) = \Phi(y \cdot x)$ implies that $y \cdot x \in S$. Thus, $x \sim y$ which implies that $[x]_S = [y]_S$, so f is one-to-one.

Let $y \in \Phi(X)$, then there exists $x \in X$ such that $y = \Phi(x)$ and $[x]_S \in X/S$. Then $f\big([x]_S\big) = \Phi(x) = y$. Hence, f is onto and consequently, f is a dB-isomorphism. □.

Proposition 3. Suppose $f : (G, \cdot, 1_G) \to (G/H_1, *, [1]_{H_1})$ is a dB-epimorphism of dB-algebras. If H_2 is a normal dB-subalgebra of G, then $f(H_2)$ is a normal dB-subalgebra of G/H_1.

Proof. It follows from Theorem 4 (iii) that $f(H_2)$ is a dB-subalgebra of G/H_1. Now to show that $f(H_2)$ is normal, let $[x]_{H_1} * [y]_{H_1}, [a]_{H_1} * [b]_{H_1} \in f(H_2)$ for any $[x]_{H_1}, [y]_{H_1}, [a]_{H_1}$, and $[b]_{H_1} \in G/H_1$. Since f is onto, then there exists $j, k, l, m \in G$ such that $f(j) = [x]_{H_1}$, $f(k) = [y]_{H_1}$, $f(l) = [a]_{H_1}$, $f(m) = [b]_{H_1}$. Suppose $j \cdot k, l \cdot m \in H_2$. Then $(j \cdot l) \cdot (k \cdot m) \in H_2$ since H_2 is normal, which then implies that $f\big((j \cdot l) \cdot (k \cdot m)\big) \in f(H_2)$. It follows that

$$\begin{aligned} f\big((j \cdot l) \cdot (k \cdot m)\big) = f(j \cdot l) * f(k \cdot m) &= \big(f(j) * f(l)\big) * \big(f(k) * f(m)\big) \\ &= \big([x]_{H_1} * [a]_{H_1}\big) * \big([y]_{H_1} * [b]_{H_1}\big) \end{aligned}$$

Thus, $f(H_2)$ is normal and consequently, $f(H_2)$ is a normal dB-subalgebra of G/H_1. □

Theorem 8. *Third Isomorphism Theorem for the dB-Algebra*

Let f be a natural dB-homomorphism of a dB-algebra $(G, \cdot, 1_G)$ onto a dB-algebra $(G/H_1, *, [1]_{H_1})$, H_2 be a normal dB-subalgebra of G such that $\ker f = H_1 \subseteq H_2$, and g, g' be the natural dB-homomorphisms of G onto $(G/H_2, \cdot', [1]_{H_2})$ and G/H_1 onto $((G/H_1)/(H_2/H_1), *', [1]_{H_2/H_1})$, respectively. Then there exists a unique dB-isomorphism h of G/H_2 onto $(G/H_1)/(H_2/H_1)$, that is $G/H_2 \cong (G/H_1)/(H_2/H_1)$, where $g' \circ f = h \circ g$.

Proof. Since $f(H_2)$ is a normal dB-subalgebra of G/H_1 by Proposition 3, we have that $f(H_2) = \ker g'$ by Theorem 5.

Suppose $a \in H_2$, then $f(a) \in f(H_2)$ which implies that $f(a) \in \ker g'$. Then

$$(g' \circ f)(a) = g'\big(f(a)\big) = g'\big([1]_{H_1}\big) = \big[[1]_{H_1}\big]_{H_2/H_1}$$

by Theorem 4 (i). This implies that $a \in \ker(g' \circ f)$ and so, $H_2 \subseteq \ker(g' \circ f)$.

Conversely, suppose $a \in \ker(g' \circ f)$, then

$$(g' \circ f)(a) = g'\big(f(a)\big) = g'\big([a]_{H_1}\big) = \big[[a]_{H_1}\big]_{H_2/H_1} = [1]_{H_2/H_1}.$$

By Theorem 4 (i), we have that

$$g'\big([1]_{H_1}\big) = \big[[1]_{H_1}\big]_{H_2/H_1} = [1]_{H_2/H_1}.$$

Then $[1]_{H_1} \sim [a]_{H_1}$. This implies that $[a]_{H_1} *' [1]_{H_1} \in H_2/H_1$ and $[1]_{H_1} *' [a]_{H_1} \in H_2/H_1$ which by DB2 implies that $[a]_{H_1} \in H_2/H_1$. It follows that $f(a) \in H_2/H_1 = \ker g'$ by Theorem 5 and so, $f(a) \in f(H_2)$. Since f is onto, there exists $x \in H_2$ such that $f(x) = f(a)$ implies $[x]_{H_1} = [a]_{H_1}$ which implies that $x \sim a$. So, $x \cdot a \in H_1$ and $a \cdot x \in H_1$. Since $H_1 \subseteq H_2$, $x \cdot a \in H_2$ and $a \cdot x \in H_2$ and it follows that since H_2 is also a normal dB-filter by Proposition 2, $a \in H_2$. This implies that $\ker(g' \circ f) \subseteq H_2$, and consequently $\ker(g' \circ f) = H_2$.

By Theorem 6, there exists a unique dB-isomorphism h of G/H_2 onto $(G/H_1)/(H_2/H_1)$ such that $g' \circ f = h \circ g$. □

4. Conclusion

In this paper, it is shown that the necessary and sufficient condition for a db-filter to be a db-subalgebra and vice versa is normality. Using the quotient dB-algebra, along with some properties (such as normality) of the dB-filter, dB-subalgebra, and dB-homomorphism presented in the paper, the natural dB-homomorphism is determined; this then led to the creation of the fundamental theorem of dB-homomorphisms for dB-algebras. Following the aforementioned theorem, the first and third isomorphism theorems for the dB-algebra are constructed.

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