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Some Properties and Realization Problems Involving Connected Outer-Hop Independent Hop Domination in Graphs

Javier A. Hassan[1,*], Abdurajan B. Lintasan[1], Nurijam Hanna M. Mohammad[1]

[1] *Mathematics and Sciences Department, College of Arts and Sciences, MSU Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*

Abstract. In this paper, we construct a realization problems involving connected outer-hop independent hop domination and we determine its connections with other known parameters in graph theory. In particular, given two positive integers a and b with $2 \leq a \leq b$ are realizable as the connected hop domination, connected outer-hop independent hop domination, and connected outer-independent hop domination numbers, respectively, of a connected graph. In addition, we characterize the connected outer-hop independent hop dominating sets in some families of graphs, join and corona of two graphs, and we use these results to derive formulas for the parameters of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop independent set, connected outer-hop independent hop dominating set, connected outer-hop independent hop domination number

1. Introduction

Hop domination has been one of the widely studied topics of research in graph theory. Several mathematicians have investigated this concept and introduced variants because of its nice application to different fields and in networks. Some newly defined variations are studied in many classes of graphs (see [3–5, 7–12, 14]).

In 2021, Nanding et al. [12] introduced and studied the concept called connected outer-independent hop domination in a graph. They characterized this newly defined sets on graphs under some binary operations and obtained some nice formulas and bounds.

Recently, Hassan et al. [6] introduced the concept of hop independent set in a graph and defined the parameter called hop independence number. The authors have shown that the hop independence number is incomparable with the standard independence number

*Corresponding author.
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Email addresses: javierhassan@msutawi-tawi.edu.ph (J. Hassan),
abdurajanlintasan@msutawi-tawi.edu.ph (A. Lintasan),
hannamohammad@msutawi-tawi.edu.ph (N.H. Mohammad)

of a graph. In fact, the authors have shown that the absolute difference between the hop independence number and the independence number of a graph can be made arbitrarily large.

Motivated by the aforementioned studies, the concept of connected outer-hop independent hop domination in a graph will be introduced and investigated in this study. Some properties and realization results involving this parameter will be formulated. Moreover, exact values or bounds for the parameter will be given for some families of graphs, join, and corona of two graphs. Just like hop domination, we believe that this new parameter will yield significant results in the topic of domination and can lead to other interesting research directions in the future.

2. Terminology and Notation

Let G be a simple graph. Then $S \subseteq V(G)$ is a *clique* if the subgraph $\langle S \rangle$ induced by S is complete. The maximum cardinality of a clique set in G, denoted by $\omega(G)$, is called a clique number of G. Any clique set with cardinality equal to $\omega(G)$ is called a ω-set.

A subset D of $V(G)$ is called a *pointwise non-dominating* set of G if for each $v \in V(G) \setminus D$, there exists $u \in D$ such that $v \notin N_G(u)$.

A subset D of $V(G)$ is *independent* if for every pair of distinct vertices $v, w \in D$, we have $d_G(v, w) \neq 1$. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the independence number of G. Any independent set with cardinality equal to $\alpha(G)$ is called an α-set.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The *closed hop neighborhood* of X in G is the set $N_G^2[X] = N_G^2(X) \cup X$.

A subset S of $V(G)$ is a *hop dominating* of G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A hop dominating set $D \subseteq V(G)$ is called a connected hop dominating if the subgraph $\langle D \rangle$ induced by D is connected. The minimum cardinality among all connected hop dominating sets of G, denoted by $\gamma_{ch}(G)$, is called the *connected hop domination number* of G. Any connected hop dominating set with cardinality equal to $\gamma_{ch}(G)$ is called a γ_{ch}-set.

A connected hop dominating set $C \subseteq V(G)$ is called a *connected outer-independent hop dominating* if $V(G) \setminus C$ is an independent set in G. The minimum cardinality of a connected outer-independent hop dominating set in G, denoted by $\gamma_{ch}^{oi}(G)$, is called the *connected outer-independent hop domination number* of G. Any connected outer-independent hop dominating set with cardinality equal to $\gamma_{ch}^{oi}(G)$ is called a γ_{ch}^{oi}-set.

A subset D of $V(G)$ is *hop independent* if for every pair of distinct vertices $v, w \in D$, we have $d_G(v,w) \neq 2$. The maximum cardinality of a hop independent set in G, denoted by $\alpha_h(G)$, is called the *hop independence number* of G. Any hop independent set with cardinality equal to $\alpha_h(G)$ is called an α_h-set.

Let G and H be two graphs. The *join* of G and H, denoted by $G+H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* G and H, denoted by $G \circ H$, the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} + H^v \rangle$.

3. Results

We begin this section by introducing the concept called connected outer-hop independent hop domination in a graph.

Definition 1. Let G be a connected graph. A subset C of $V(G)$ is called a *connected outer-hop independent hop dominating* if C is a connected hop dominating set and $V(G) \setminus C$ is a hop independent set in G. The minimum cardinality of a connected outer-hop independent hop dominating set in G, denoted by, $\gamma_{ch}^{ohi}(G)$ is called the *connected outer-hop independent hop domination number* of G. Any connected outer-hop independent hop dominating set with cardinality equal to $\gamma_{ch}^{ohi}(G)$ is called a γ_{ch}^{ohi}-set of G.

Proposition 1. *Let G be a connected graph. Then $\gamma_{ch}(G) \leq \gamma_{ch}^{ohi}(G)$, and this bound is sharp.*

Proof. Let C be a γ_{ch}^{ohi}-set of G. Then C is a connected hop dominating set in G (by definition). Since $\gamma_{ch}(G)$ is the minimum cardinality among all connected hop dominating sets in G, it follows that $\gamma_{ch}^{ohi}(G) = |C| \geq \gamma_{ch}(G)$.

To see that the bound is sharp, consider $G = P_7 = [v_1, v_2, \ldots, v_7]$. Let $S = \{v_3, v_4, v_5\}$. Observe that $\langle S \rangle$ is connected and $N_G^2[S] = V(G)$. This means that S is a connected hop dominating in G. Since any connected hop dominating set in G contains S, S is the minimum connected hop dominating set in G. Hence, $\gamma_{ch}(P_7) = 3$. Moreover, since $d_G(a,b) \neq 2$ for every $a, b \in V(G) \setminus S$, it follows that S is the minimum connected outer-hop independent hop dominating set in G. Consequently, $\gamma_{ch}(P_7) = 3 = \gamma_{ch}^{ohi}(P_7)$. □

Theorem 1. *Let G be a connected graph with $|V(G)| = n \geq 1$. Then each of the following is true.*

(i) $1 \leq \gamma_{ch}^{ohi}(G) \leq n$.

(ii) $\gamma_{ch}^{ohi}(G) = 1$ *if and only if G is trivial.*

(iii) $\gamma_{ch}^{ohi}(G) = n$ *if and only if G is complete.*

Proof. (i) Since an empty set cannot be a connected outer-hop independent hop dominating set in G, we have $\gamma_{ch}^{ohi}(G) \geq 1$. Moreover, since any connected outer-hop independent hop dominating set in G is contained in $V(G)$, we have $\gamma_{ch}^{ohi}(G) \leq n$. Consequently, $1 \leq \gamma_{ch}^{ohi}(G) \leq n$.

(ii) Suppose $\gamma_{ch}^{ohi}(G) = 1$. Suppose that G is non-trivial. Then G is either connected or disconnected. If G is connected, then there exist $a, b \in V(G)$ such that $d_G(a,b) = 1$. This means that $a \notin N_G^2[b]$ and $b \notin N_G^2[a]$, showing that a singleton set is not the minimum connected outer-hop independent hop dominating set in G. Thus, $\gamma_{ch}^{ohi}(G) > 1$, a contradiction. Suppose that $G_1, \ldots, G_k, k \geq 2$ are the components of G and let S be a connected outer-hop independent hop dominating set of G. Then $S = S_1 \cup \ldots \cup S_k$, where S_i is a connected outer-hop independent hop dominating set of G_i for each $i \in \{1, \ldots, k\}$. Since $k \geq 2$, we have $\gamma_{ch}^{ohi}(G) \geq 2$, a contradiction. Therefore, G must be a trivial graph.

The converse is clear.

(iii) Let $\gamma_{ch}^{ohi}(G) = n$. Suppose further that G is non-complete. Let $x, y \in V(G)$ such that $d_G(x,y) = \Delta(G)$, $\Delta(G)$ is the maximum degree of G. This means that x, y are non-cut vertices of G. Moreover, since G is non-complete, $d_G(x,y) \geq 2$. Now, let $S' = \{V(G) \backslash \{x\}$. Then S' is a connected outer-hop independent hop dominating set in G. Thus, $\gamma_{ch}^{ohi}(G) \leq n-1$, a contradiction. Therefore, G is complete.

Conversely, suppose G is complete. Then $N_G^2[a] = \{a\}$ for every $a \in V(G)$. Let $S = V(G) = \{a_1, a_2, \ldots, a_n\}$. Then S is the minimum connected outer-hop independent hop dominating set of G. Thus, $\gamma_{ch}^{ohi}(G) = n$. □

The next result follows from Theorem 1.

Corollary 1. *Let G be a connected graph on $n \geq 2$ vertices. Then*

(i) $\gamma_{ch}^{ohi}(G) = n$ *if and only if* $G = K_n$.

(ii) $2 \leq \gamma_{ch}^{ohi}(G) \leq n-1$ *if and only if G is non-complete graph.*

(iii) $4 \leq \gamma_{ch}^{ohi}(G) + \gamma_{ch}^{ohi}(G') \leq 2n-2$ *if and only if G and G' are two non-complete graphs.*

(iv) $4 \leq \gamma_{ch}^{ohi}(H) \cdot \gamma_{ch}^{ohi}(J) \leq n^2 - 2n + 1$ *if and only if H and J are two non-complete graphs.*

Theorem 2. *Let G be a connected graph. Then $\gamma_{ch}^{ohi}(G) = \gamma_{ch}(G)$ if and only if G has a γ_{ch}-set D such that $V(G) \setminus D$ is a hop independent set in G.*

Proof. Suppose $\gamma_{ch}^{ohi}(G) = \gamma_{ch}(G)$. Let D be a γ_{ch}^{ohi}-set of G. Then $V(G) \setminus D$ is a hop independent set in G. Since D is a connected hop dominating set and

$$\gamma_{ch}^{ohi}(G) = \gamma_{ch}(G) = |D|,$$

it follows that D is a γ_{ch}-set of G.

Conversely, suppose G has a γ_{ch}-set D such that $V(G) \setminus D$ is a hop independent set in G. Then D is a connected outer-hop independent hop dominating set in G. Hence, $\gamma_{ch}^{ohi}(G) \leq |D| = \gamma_{ch}(G)$. By Proposition 1, $\gamma_{ch}^{ohi}(G) = |D| = \gamma_{ch}(G)$. □

The next result is a realization problem involving connected outer-hop independent hop domination and connected hop domination.

Theorem 3. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_{ch}(G) = a$ and $\gamma_{ch}^{ohi}(G) = b$.*

Proof. For $a = b$, consider the following two cases:
Case 1: $a = 2$

Consider K_a. Then by Corollary 1, $\gamma_{ch}^{ohi}(K_a) = a$. Since $\gamma_{ch}(K_a) = a$, we have $\gamma_{ch}^{ohi}(K_a) = a = \gamma_{ch}(K_a)$.

Case 2: $a \geq 3$

Consider the graph G in Figure 1. Let $D = \{d_1, d_2, \ldots, d_a\}$. Then D is both connected hop dominating and connected outer-hop independent hop dominating of G. Observe that every connected hop dominating (resp. connected outer-hop independent hop dominating) set of G contains D. This follows that D is both a γ_{ch}-set and a γ_{ch}^{ohi}-set of G. Thus, $\gamma_{ch}(G) = a = \gamma_{ch}^{ohi}(G)$.

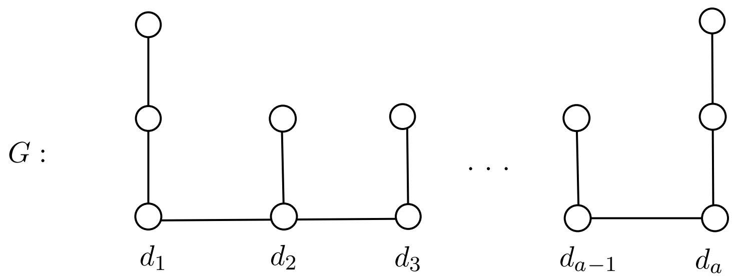

Figure 1: A graph G with $\gamma_{ch}(G) = \gamma_{ch}^{ohi}(G)$

Suppose $a < b$. Let $m = b - a$ and consider the graph G' given in Figure 2. Let $D_1 = \{x_1, x_2, \ldots, x_a\}$ and $D_2 = \{x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_m\}$. Then $N_G'^2[D_1] = V(G')$ and $N_G'^2[D_2] = V(G')$. Since $\langle D_1 \rangle$ and $\langle D_2 \rangle$ are connected, it follows that D_1 and D_2 are both connected hop dominating sets of G'. Moreover, since any connected hop dominating (resp. connected outer-hop independent hop dominating) set D contains D_1 (resp. D_2), D_1 and D_2 are γ_{ch}-set and γ_{ch}^{ohi}-set of G, respectively. Consequently, $\gamma_{ch}(G') = a$ and $\gamma_{ch}^{ohi}(G') = m + a = b$, that is $\gamma_{ch}(G') = a < b = \gamma_{ch}^{ohi}(G')$. □

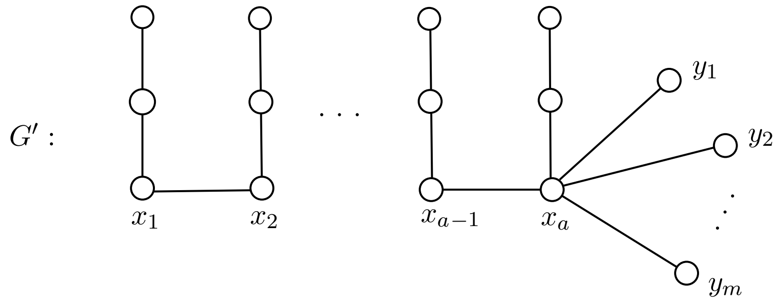

Figure 2: A graph G' with $\gamma_{ch}(G') < \gamma_{ch}^{ohi}(G')$

Corollary 2. *Let n be a positive integer. Then there exists a connected graph G such that $\gamma_{ch}^{ohi}(G) - \gamma_{ch}(G) = n$. In other words, $\gamma_{ch}^{ohi}(G) - \gamma_{ch}(G)$ can be made arbitrarily large.*

The next result is a realization problem involving connected outer-independent hop domination and connected outer-hop independent hop domination.

Theorem 4. *Let a and b be positive integers such that $2 \leq a \leq b$. Then*

(i) there exists a connected graph G such that $\gamma_{ch}^{ohi}(G) = a$ and $\gamma_{ch}^{oi}(G) = b$.

(ii) there exists a connected graph G such that $\gamma_{ch}^{oi}(G) = a$ and $\gamma_{ch}^{ohi}(G) = b$.

Proof. (i) For $a = b$, consider $G = K_a$. Then $\gamma_{ch}^{oi}(G) = a = \gamma_{ch}^{ohi}$.
Suppose $a < b$. Let $m = b - a$ and consider the graph J in Figure 3. Let $S_1 = \{v_1, v_2, \ldots, v_a\}$ and $S_2 = \{v_1, v_2, \ldots, v_a, y_1, y_2, \ldots, y_m\}$. Then S_1 and S_2 are γ_{ch}^{ohi}-set and γ_{ch}^{oi}-set of J, respectively. Hence, $\gamma_{ch}^{ohi}(J) = a$ and $\gamma_{ch}^{oi}(J) = m + a = b$.

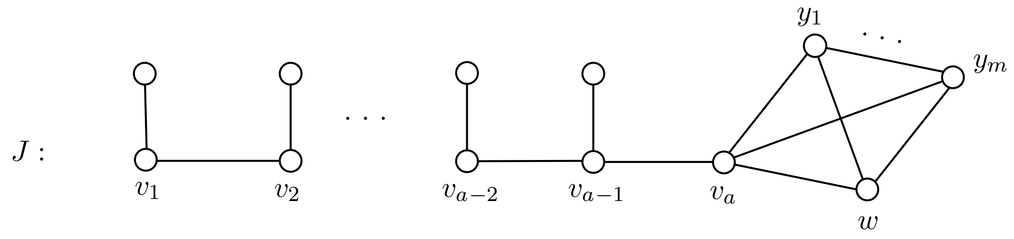

Figure 3: A graph J with $\gamma_{ch}^{ohi}(J) < \gamma_{ch}^{oi}(J)$

(*ii*) For $a = b$, consider $G = K_a$. Then $\gamma_{ch}^{ohi}(G) = a = \gamma_{ch}^{oi}(G)$.
Suppose $a < b$. Let $m = b - a$ and consider the graph H in Figure 4. Let $D_1 = \{x_1, x_2, \ldots, x_a\}$ and $D_2 = \{x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_m\}$. Then D_1 and D_2 are γ_{ch}^{oi}-set and γ_{ch}^{ohi}-set of H, respectively. Hence, $\gamma_{ch}^{oi}(H) = a$ and $\gamma_{ch}^{ohi}(H) = m + a = b$. □

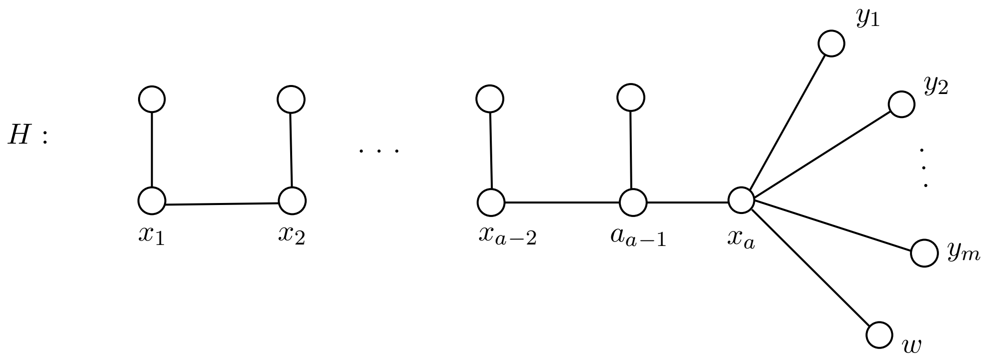

Figure 4: A graph H with $\gamma_{ch}^{oi}(H) < \gamma_{ch}^{ohi}(H)$

Corollary 3. *Let n be a positive integer. Then each of the following statements holds.*

(*i*) *There exists a connected graph G such that $\gamma_{ch}^{oi}(G) - \gamma_{ch}^{ohi}(G) = n$.*

(*ii*) *There exists a connected graph G such that $\gamma_{ch}^{ohi}(G) - \gamma_{ch}^{oi}(G) = n$.*

In other words, the absolute difference $|\gamma_{ch}^{oi}(G) - \gamma_{ch}^{ohi}(G)|$ can be made arbitrarily large.

Proposition 2. *For any positive integer $n \geq 1$,*

$$\gamma_{ch}^{ohi}(P_n) = \begin{cases} 1 \text{ if } n = 1 \\ 2 \text{ if } n = 2, 3, 4, 5 \\ n-4 \text{ if } n \geq 6 \end{cases}$$

Proof. Clearly, $\gamma_{ch}^{ohi}(P_1) = 1$ and $\gamma_{ch}^{ohi}(P_n) = 2$ for $n = 3, 4, 5$. Suppose that $n \geq 6$. Let $P_n = [v_1, v_2, \ldots, v_n]$ and $D = \{v_3, v_4 \cdots, v_{n-3}, v_{n-2}\}$. Then $N^2_{P_n}[D] = V(P_n)$ and $\langle D \rangle$ is connected. Thus, D is a connected hop dominating set of P_n. Since $n \geq 6$, it follows that $d_{P_n}(a, b) \neq 2$ for every $a, b \in V(P_n) \setminus D$. Hence, $V(P_n) \setminus D$ is a hop independent set of P_n, showing that D is a connected outer-hop independent hop dominating set of P_n, and so $\gamma_{ch}^{ohi}(P_n) \leq n - 4$ for all $n \geq 6$. On the other hand, observe that any connected outer-hop independent hop dominating set S in P_n contains D. Therefore, $\gamma_{ch}^{ohi}(P_n) = n - 4$ for all $n \geq 6$.

□

Proposition 3. *For any positive integer $n \geq 3$,*

$$\gamma_{ch}^{ohi}(C_n) = \begin{cases} 3 \text{ if } n = 3 \\ n-2 \text{ if } n \geq 4 \end{cases}$$

Proof. Clearly, $\gamma_{ch}^{ohi}(C_3) = 3$. Suppose $n \geq 4$. Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ and consider $D^* = \{v_1, v_2, \ldots, v_{n-2}\}$. Then $N^2_{C_n}[D^*] = V(C_n)$ and $\langle D^* \rangle$ is connected. Thus, D^* is a connected hop dominating set of C_n. Since $d_{C_n}(v_{n-1}, v_n) = 1$, it follows that D^* is a connected outer-hop independent hop dominating set of C_n. Since the maximum connected hop independent set in C_n is of cardinality 2, it follows that D^* is a γ_{ch}^{ohi}-set of C_n. Therefore, $\gamma_{ch}^{ohi}(C_n) = n - 2$ for all $n \geq 4$. □

Theorem 5. *Let G be any connected graph of order $n \geq 1$. Then $\gamma_{ch}^{ohi}(G) \geq n - \alpha_h(G)$.*

Proof. Let D be a γ_{ch}^{ohi}-set of G. Then $\gamma_{ch}^{ohi}(G) = |D|$ and $V(G) \setminus D$ is a hop independent set in G (by definition). It follows that $\alpha_h(G) \geq |V(G) \setminus D|$. Hence,

$$\begin{aligned} n - \alpha_h(G) \leq n - |V(G) \setminus D| &= n - n + |D| \\ &= |D| \\ &= \gamma_{ch}^{ohi}(G). \end{aligned}$$ □

Remark 1. *The bound in Theorem 5 is sharp. Moreover, strict inequality can be attained.*

For sharpness, consider the graph G in Figure 5. Let $S = \{c, d, e\}$. Then S is a minimum connected hop dominating set of G. Notice that $V(G) \setminus S$ is a hop independent set. It follows that S is a minimum connected outer-hop independent hop dominating set of G. Thus, $\gamma_{ch}^{ohi}(G) = 3$. Next, let $S' = \{b, c, f, g, h\}$. Then S is the maximum hop independent set of G. Consequently, $|V(G)| - \alpha_h(G) = 8 - 5 = 3 = \gamma_{ch}^{ohi}(G)$.

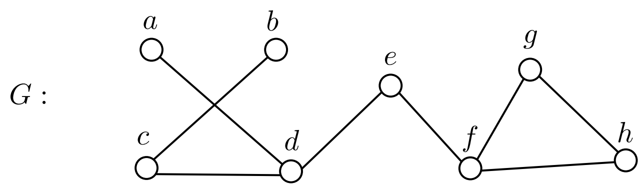

Figure 5: A graph G with $\gamma_{ch}^{ohi}(G) = |V(G)| - \alpha_h(G)$

For strict inequality, consider K_7. Then $\gamma_{ch}^{ohi}(K_7) = 7 = \alpha_h(K_7)$. Thus,

$$\gamma_{ch}^{ohi}(K_7) = 7 > (7 - \alpha_h(K_7)) = 7 - 7 = 0.$$

The following concept will be used in characterizing the connected outer-hop independent hop dominating sets in the join of two graphs.

Definition 2. Let G be a non-complete graph. Then $D \subseteq V(G)$ is called an outer-clique pointwise non-dominating set in G if D is pointwise non-dominating set and $V(G) \setminus D$ is clique set in G. The smallest cardinality of an outer-clique pointwise non-dominating set of G, denoted by $ocpnd(G)$, is called the *outer-clique pointwise non-domination number* of G. Any outer-clique pointwise non-dominating set D of G with $|D| = ocpnd(G)$, is called an *ocpnd-set* of G.

Example 1. Consider the graph G in Figure 6. Let $O = \{a_1, a_2, a_5, a_6\}$. Then O is a pointwise non-dominating set of G. Since $\langle V(G) \setminus O \rangle \cong K_4$, it follows that $V(G) \setminus O$ is clique in G. Thus, O is an outer-clique pointwise non-dominating set of G. Next, let $O' = \{a_1, a_2, a_6\}$. Then O' is a pointwise non-dominating of G. However, O' is not an outer-clique pointwise non-dominating set of G since $V(G) \setminus O' = \{a_3, a_4, a_5, a_7, a_8\}$ is not clique in G. Moreover, since $B = \{a_3, a_4, a_7, a_8\}$ is the maximum clique set in G, it follows that $ocpnd(G) = 4$.

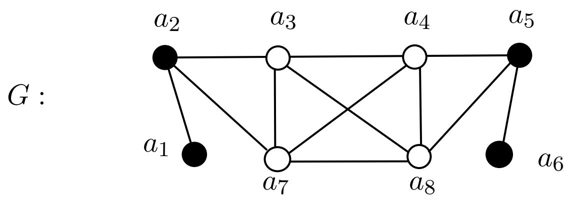

Figure 6: A graph G with $ocpnd(G) = 4$

Theorem 6. *Let G and H be two non-complete graphs. Then $C \subseteq V(G+H)$ is a connected outer-hop independent hop dominating of $G + H$ if and only if $C = C_G \cup C_H$, where C_G and C_H are outer-clique pointwise non-dominating sets of G and H, respectively.*

Proof. Suppose $C \subseteq V(G+H)$ be a connected outer-hop independent hop dominating set of $G + H$. Let $C_G = V(G) \cap C$ and $C_H = V(H) \cap C$. Assume that $C_G = \varnothing$. Then $C = C_H$. Observe that $V(G) \subseteq N_{G+H}(C)$. It follows that $V(G) \notin N^2_{G+H}[C]$, a contradiction. Hence, $C_G \neq \varnothing$. Similarly, $C_H \neq \varnothing$. Now, let $a \in V(G) \setminus C_G$. Since C is a hop dominating set, there exists $b \in C$ such that $d_{G+H}(a, b) = 2$. Thus, $b \in C_G$ and $a \notin N_G(b)$. This means that C_G is a pointwise non-dominating set of G. Since $V(G+H) \setminus C$ is a hop independent set of $G+H$, it follows that $V(G) \setminus C_G$ is a clique set of G. Hence, C_G is an outer-clique pointwise non-dominating set of G. Similarly, C_H is an outer-clique pointwise non-dominating set of H.

Conversely, suppose $C = C_G \cup C_H$, where C_G and C_H are outer-clique pointwise non-dominating sets in G and H, respectively. Clearly, $\langle C \rangle$ is connected and $V(G + H) \setminus C$ is a hop independent set. Now, let $a \in V(G + H) \setminus C$. Suppose $a \in V(G)$. Since C_G is a pointwise non-dominating set of G, there exists $b \in C_G \setminus N_G(a)$. Thus, $d_{G+H}(a, b) = 2$. Similarly, when $a \in V(H)$. Therefore, C is a hop dominating set of $G+H$. Consequently, S is a connected outer-hop independent hop dominating set of $G + H$. □

Corollary 4. *Let G and H be two non-complete graphs. Then*

$$\gamma^{ohi}_{ch}(G + H) = ocpnd(G) + ocpnd(H).$$

Proof. Suppose $C \subseteq V(G+H)$ is a γ^{ohi}_{ch}-set of $G+H$. Then by Theorem 6, $C = C_G \cup C_H$, where C_G and C_H are outer-clique pointwise non-dominating sets of G and H, respectively. Thus,

$$\gamma^{ohi}_{ch}(G + H) = |C| = |C_G| + |C_H| \geq ocpnd(G) + ocpnd(H).$$

On the other hand, let C_G and C_H be $ocpnd$-sets of G and H, respectively. Then by

Theorem 6, $C = C_G \cup C_H$ is a connected outer-hop independent hop dominating set of $G + H$. Hence,

$$ocpnd(G) + ocpnd(H) = |C_G| + |C_H| = |C| \geq \gamma_{ch}^{ohi}(G + H).$$

Therefore,

$$\gamma_{ch}^{ohi}(G + H) = ocpnd(G) + ocpnd(H).$$

□

Theorem 7. *Let G be a complete graph and H be non-complete graph. Then $T \subseteq V(G + H)$ is a connected outer-hop independent hop dominating set in $G + H$ if and only if $T = V(G) \cup T_H$, where T_H is an outer-clique pointwise non-dominating set of H.*

Proof. Suppose that $T = V(G) \cup T_H$ is a connected outer-hop independent hop dominating set in $G + H$. Since G is complete, it follows that $T = V(G) \cup T_H$, where $T_H \neq \varnothing$. Since T is a hop dominating, T_H must be a pointwise non-dominating set in H. If $V(H) \setminus T_H$ is not clique in H, then there exist $a, b \in V(H) \setminus T_H \subseteq V(G + H) \setminus T$ such that $d_H(a, b) \geq 2$. It follows that $d_{G+H}(a, b) = 2$, a contradiction to the fact that $V(G + H) \setminus T$ is a hop independent in $G + H$. Thus, T_H is an outer-clique pointwise non-dominating set of H.

Conversely, assume that $T = V(G) \cup T_H$, where T_H is an outer-clique pointwise non-dominating set of H. Then T is connected outer-hop independent hop dominating set in $G + H$ by Theorem 6. □

Corollary 5. *Let G be a complete graph and H be any non-complete graph. Then*

$$\gamma_{ch}^{ohi}(G + H) = |V(G)| + ocpnd(H).$$

Proof. Suppose $T \subseteq V(G + H)$ is a γ_{ch}^{ohi}-set of $G + H$. Then by Theorem 7, $T = V(G) \cup T_H$, where T_H is an outer-clique pointwise non-dominating set of H. Thus,

$$\gamma_{ch}^{ohi}(G + H) = |T| = |V(G)| + |T_H| \geq |V(G)| + ocpnd(H).$$

On the other hand, let $T = V(G) \cup T_H$, where T_H is an *ocpnd*-set of H. Then by Theorem 7, $T = V(G) \cup T_H$ is a connected outer-hop independent hop dominating set of $G + H$. Hence,

$$|V(G)| + ocpnd(H) = |V(G)| + |T_H| = |T| \geq \gamma_{ch}^{ohi}(G + H).$$

Consequently,

$$\gamma_{ch}^{ohi}(G + H) = |V(G)| + ocpnd(H).$$

□

Theorem 8. *Let G be a non-trivial connected graph and H be any non-complete graph. A set $C \subseteq V(G \circ H)$ is a connected outer-hop independent hop dominating set of $G \circ H$ if and only if $C = V(G) \cup (\bigcup_{a \in V(G)} C_a)$, where $C_a \subseteq V(H^a)$ and $V(H^a) \setminus C_a$ is clique in H^a for each $a \in V(G)$.*

Proof. Suppose $C \subseteq V(G \circ H)$ is a connected outer-hop independent hop dominating set of $G \circ H$ and let $C_a = V(H^a) \cap C$ for each $a \in V(G)$. Since $\langle C \rangle$ is connected, it follows that $C = V(G) \cup (\bigcup_{a \in V(G)} C_a)$. Since $V(G \circ H) \setminus C = \bigcup_{a \in V(G)} (V(H^a) \setminus C_a)$ is a hop independent set of $G \circ H$, it follows that $V(H^a) \setminus C_a$ is a hop independent set of H^a for each $a \in V(G)$. Suppose $V(H^a) \setminus C_a$ is not a clique in H^a for some $a \in V(G)$. Then there exists $u, v \in V(H^a) \setminus C_a \subseteq V(G \circ H) \setminus C$ such that $d_{H^a}(u, v) = d_{G \circ H}(u, v) = 2$ for some $a \in V(G)$, a contradiction to the fact that C is a connected outer-hop independent hop dominating set of $G \circ H$. Therefore, $V(H^a) \setminus C_a$ is clique in H^a for every $a \in V(G)$.

Conversely, suppose $C = V(G) \cup (\bigcup_{a \in V(G)} C_a)$, where $C_a \subseteq V(H^a)$ and $V(H^a) \setminus C_a$ is clique in H^a for each $a \in V(G)$. Clearly, C is a connected hop dominating set of $G \circ H$. Since $V(H^a) \setminus C_a$ is clique in H^a for each $a \in V(G)$, it follows that

$$V(G \circ H) \setminus C = \bigcup_{a \in V(G)} (V(H^a) \setminus C_a)$$

is a hop independent set of $G \circ H$. Therefore, C is a connected outer-hop independent hop dominating set of $G \circ H$. □

Corollary 6. *Let G be a non-trivial connected graph with $|V(G)| = n$ and H be any non-complete graph with $|V(H)| = m$. Then $\gamma_{ch}^{ohi}(G \circ H) = n + n(m - \omega(H))$. In particular, we have*

(i) $\gamma_{ch}^{ohi}(G \circ H) = n + n(m - 2)$ *if* $H = P_m, K_{1,m}$ *for all* $m \geq 3$,

(ii) $\gamma_{ch}^{ohi}(G \circ H) = n + n(m - 2)$ *if* $H = C_m$ *for all* $m \geq 4$,

(iii) $\gamma_{ch}^{ohi}(G \circ W_m) = n + n(m - 3)$ *for all* $m \geq 4$,

(iv) $\gamma_{ch}^{ohi}(G \circ F_m) = n + n(m - 3)$ *for all* $m \geq 3$,

Proof. Let C be a γ_{ch}^{ohi}-set of $G \circ H$. Then $C = V(G) \cup (\bigcup_{v \in V(G)} C_v)$, where $C_v \subseteq V(H^v)$ and $V(H^v) \setminus C_v$ is clique in H^v for each $v \in V(G)$ by Theorem 8. Hence,

$$\begin{aligned}
\gamma_{ch}^{ohi}(G \circ H) = |C| = |V(G)| + |\bigcup_{v \in V(G)} C_v| &= V(G) + \sum_{v \in V(G)} |C_v| \\
&= |V(G)| + \sum_{v \in V(G)} (|V(H^v)| - |V(H^v) \setminus C_v|) \\
&\geq |V(G)| + |V(G)|(m - \omega(H)) \\
&= n + n(m - \omega(H)).
\end{aligned}$$

Therefore, $\gamma_{ch}^{ohi}(G \circ H) \geq n + n(m - \omega(H))$.

On the other hand, for each $v \in V(G)$, let $C_v \subseteq V(H^v)$ such that $V(H^v) \setminus C_v$ is a maximum clique of H^v. Then by Theorem 8, $C = V(G) \cup (\bigcup_{v \in V(G)} C_v)$ is a connected outer-hop independent hop dominating set of $G \circ H$. Thus,

$$\begin{aligned}\gamma_{ch}^{ohi}(G \circ H) \leq |C| = |V(G)| + |\bigcup_{v \in V(G)} C_v| &= V(G) + \sum_{v \in V(G)} |C_v| \\ &= |V(G)| + \sum_{v \in V(G)} (|V(H^v)| - |V(H^v) \setminus C_v|) \\ &= |V(G)| + |V(G)|(|V(H^v)| - \omega(H)) \\ &= n + n(m - \omega(H)).\end{aligned}$$

Consequently, $\gamma_{ch}^{ohi}(G \circ H) = n + n(m - \omega(H))$.

Since $\omega(P_m) = \omega(K_{1,m}) = 2$ for all $m \geq 3$ and $\omega(C_n) = 2$ for all $m \geq 4$, statements (i) and (ii) hold. Also, since $\omega(W_m) = 3$ for all $m \geq 4$ and $\omega(F_m) = 3$ for all $m \geq 3$, statements (iii) and (iv) hold. □

4. Conclusion

This study has initiated the study of the concept called connected outer-hop independent hop domination in a graph. It was shown that the connected outer-hop independent hop domination number is at least equal to the connected hop domination number of a graph. This study gave some lower or upper bounds on the parameter of some graphs. In addition, exact values of the parameter have been obtained for some special graphs and graphs under some binary operations. Realization results involving connected outer-hop independent hop domination were presented. Interested researchers may consider and investigate this newly defined parameter for some products of graphs which were not considered in this study. They may also consider and study the complexity of this parameter.

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