EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 503-522
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Hermite Hadamard Type Inequalities Involving (k-p) Fractional Operator with $(\alpha, h-m)-p$ convexity

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Abstract. We establish various fractional convex inequalities of the Hermite-Hadamard type which generalize the previously obtained results in the literature. Various types of such inequalities are obtained and given as corollaries. The main motivation of the paper is to generalize the recently published results in terms of the $(\alpha, h-m)-p$ convexity with k-p Riemann Liouville fractional operator. The application of Hölders inequality is given in tandem with the k-p fractional operator of the convex type.

2020 Mathematics Subject Classifications: 26D10, 26A33

Key Words and Phrases: Hermite-Hadamard inequality, $(\alpha, h-m)-p$-convex function, Hölder inequality, Fractional Inequality

1. Introduction

Convex inequalities in mathematics have been an ongoing topic of research since the introduction of the first convex inequality by Jensen. Many inequalities followed as a consequence of the said inequality , see books [24, 31]. Inequalities have applications in many fields , such as analysis, optimization and the probability theory. For further information, we refer the reader to the papers [8, 9, 12, 17, 25, 26, 35]. The inequality that has attracted the most attention in the math community is the Hermite-Hadamard inequality [16]. The said inequality has been generalized in various forms by many mathematicians throughout the years. The inequality was proved independently by Charles Hermite and Jacques Hadamard. This inequality is stated as follows:
Let $\mathcal{F} : \mathbb{I} \to \mathbb{R}$ be a convex function on $\mathbb{I}$ in $\mathbb{R}$, where $\mathbb{I}$ is a bounded subset of $\mathbb{R}$ and $\rho_1, \rho_2 \in \mathbb{I}$ with $\rho_1 < \rho_2$, then

$$\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) \leqslant \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} \mathcal{F}(t)dt \leqslant \frac{\mathcal{F}(\rho_1)+\mathcal{F}(\rho_2)}{2}.$$

Lately, various types of Hermite-Hadamard type inequalities have been studied and generalized for different types of convex functions under different conditions and parameters.

DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4689

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See the following papers for more information and references therein [2–7, 10, 11, 14, 15, 22, 23, 32–34, 37–39, 41–43].In 1695, l'Hospital sent a letter to Leibniz. In his message an important question about the order of the derivative emerged, what might be a derivative of order $\frac{1}{2}$? That letter sparked the interest of many upcoming mathematicians to investigate further into the matter of fractional derivatives. Then came Fourier in 1822 who suggested an integral representation to define the derivative, and his version can be considered the first definition of the derivative of the arbitrary positive order. Abel in 1826 solved an integral equation associated with tautochrone problem, which was the first application of FC(fractional calculus). After Abel, many mathematicians proceeded to work in the field, some of the names: Riemann, Grünwald and Letnikov, Hadamard, Weyl, and many more. In the late upper half of the 20th century, Caputo formulated a definition, more restrictive than the Riemann-Liouville but more appropriate to discuss problems involving fractional differential equations with initial conditions. Fractional calculus was found to be useful in physics as well, for example Whatcraft and Meerschaert (2008) described a fractional conservation of mass, Fractional Schrödinger equation in quantum theory, and many others. Different types of fractional integrals and derivatives were defined throughout the years, we refer the interested reader to the following books [18, 28, 44] for more information on the matter.
The motivation for this paper comes from the recently published paper by Stojiljković et al.[40] where the authors established some Theorems regarding $k-p$ fractional inequalities. In this paper, we generalize the obtained inequalities.
The goal of this paper is to provide various convex inequalities with the usage of the $(\alpha, h-m)-p$ convexity in addition to the usage of the fractional calculus.
We start by defining various types of convex-inequalities. From Jensen's inequality which was the first inequality of its type to the $(\alpha, h-m)-p$ convexity which will be used in the paper.

Definition 1. *For an interval $\mathcal{I}$ in $\mathbb{R}$, a function $\mathcal{F}:\mathcal{I}\to\mathbb{R}$ is said to be convex on $\mathcal{I}$ if,*

$$\mathcal{F}(\zeta\rho_1+(1-\zeta)\rho_2)\leqslant\zeta\mathcal{F}(\rho_1)+(1-\zeta)\mathcal{F}(\rho_2)$$

for all $\rho_1,\rho_2\in\mathcal{I}$ and $\zeta\in[0,1]$ holds and is said to be a concave function if the inequality is reversed.

Among the first generalizations of the convex function was given by Hudzik and Maligranda , in their paper [19].

Definition 2. *A function $\mathcal{F}:[0,+\infty)\to\mathbb{R}$ is said to be s-convex in the second sense if*

$$\mathcal{F}(tx+(1-t)y)\leqslant t^s\mathcal{F}(x)+(1-t)^s\mathcal{F}(y)$$

holds for all $x,y\in[0,+\infty), t\in[0,1]$ and for some fixed $s\in(0,1]$.

The (s,m) convexity generalized the s convexity, J. Park asserted a new definition given in the following and gave some properties about this class of functions in [30].

Definition 3. *For some fixed $s \in (0,1]$ and $m \in [0,1]$ a mapping $\mathcal{F} : [0,+\infty) \to \mathbb{R}$ is said to be (s,m)-convex in the second sense on $\mathcal{I}$ if*

$$\mathcal{F}(t\rho_1 + m(1-t)\rho_2) \leqslant t^s\mathcal{F}(\rho_1) + m(1-t)^s\mathcal{F}(\rho_2)$$

holds for all $\rho_1, \rho_2 \in \mathcal{I}$ and $t \in [0,1]$.

The following definition was introduced by Zhong Fang which generalizes the p-convexity. More about the property of the class of (p,h) convex functions can be found here [13].

Definition 4. *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function and it is also assumed that $(0,1) \subset J$. We say that $\mathcal{F} : \mathcal{I} \to \mathbb{R}$ is a (p,h)-convex function or that $\mathcal{F}$ belongs to the class $ghx(h,p,\mathcal{I})$, if $\mathcal{F}$ is non-negative and*

$$\mathcal{F}([\alpha\rho_1^p + (1-\alpha)\rho_2^p]^{\frac{1}{p}}) \leqslant h(\alpha)\mathcal{F}(\rho_1) + h(1-\alpha)\mathcal{F}(\rho_2)$$

for all $\rho_1, \rho_2 \in \mathcal{I}$ and $\alpha \in (0,1)$. Similarly, if the inequality is reversed, then $\mathcal{F}$ is said to be a (p,h)-concave function or belong to the class $ghv(h,p,\mathcal{I})$.

The following definition is due to M. Emin Ozdemir et al. [29], it generalizes the definition of h- convex functions.

Definition 5. *Let $J \subset \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. If $\mathcal{F} : [0,b] \to \mathbb{R}$ is a (h-m)-convex function, if $\mathcal{F}$ is non-negative and, for all $\rho_1, \rho_2 \in [0,b], m \in [0,1]$ and $\alpha \in (0,1)$, one has*

$$\mathcal{F}(\alpha\rho_1 + m(1-\alpha)\rho_2) \leqslant h(\alpha)\mathcal{F}(\rho_1) + mh(1-\alpha)\mathcal{F}(\rho_2).$$

For suitable choices of h and m, the class of (h–m)-convex functions is reduced to different known classes of convex and related functions defined on $[0,b]$ given in the following remark.

In the following cases, we fix various parameters in the (h-m)-convexity to obtain various other types of convexity:
1. If $m = 1$, then we get an h-convex function.
2. If $h(\alpha) = \alpha$, then we get an m-convex function.
3. If $h(\alpha) = \alpha$ and m = 1, then we get a convex function.
4. If $h(\alpha) = 1$ and m = 1, then we get a p-function.
5. If $h(\alpha) = \alpha^s$ and m = 1, then we get an s-convex function in the second sense.
6. If $h(\alpha) = \frac{1}{\alpha}$ and $m = 1$, then we get a Godunova–Levin function.
7. If $h(\alpha) = \frac{1}{\alpha^s}$ and $m = 1$, then we get an s–Godunova–Levin function of the second kind.

Motivation behind defining the following class of convex functions comes from the last two defined convex classes, as this one unifies them all.
The following definition given by Jia et al. [20] generalizes all the previously defined types of convex functions.

Definition 6. *Let $J \subset \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0,+\infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\mathcal{F} : I \to \mathbb{R}$ is said to be $(\alpha, h-m)$-p convex, if*

$$\mathcal{F}\left((ta^p + m(1-t)b^p)^{\frac{1}{p}}\right) \leqslant h(t^\alpha)\mathcal{F}(a) + mh(1-t^\alpha)\mathcal{F}(b)$$

holds provided $(ta^p + m(1-t)b^p)^{\frac{1}{p}} \in \mathbb{I}$ for $t \in [0,1]$ and $(\alpha, m) \in [0,1]^2$.

Before we introduce the fractional type integrals, we need the following definitions.

The Pochammer k-symbol $(y)_{m,k}$ is defined as (see [1])

$$(y)_{m,k} = y(y+k)(y+2k)...(y+(m-1)k),$$

where $m \in \mathbb{N} \cup 0, k > 0$.
The $k-$gamma function Γ_k is given by (see [1]).

$$\Gamma_k(y) = \lim_{m \to +\infty} \frac{m!k^m(mk)^{\frac{y}{k}-1}}{(y)_{m,k}}$$

where $k > 0, y \in \mathbb{C} \setminus k\mathbb{Z}^- \cup 0$.
When $k = 1$ the above definitions reduce to the Pochammer symbol $(y)_m$

$$(y)_m = \begin{cases} \prod_{r=1}^{m}(y+r-1), m \in \mathbb{N} \\ 1, m = 0 \end{cases}$$

and Γ function defined as

$$\Gamma(t) = \int_0^{+\infty} e^{-z} z^{t-1} dz.$$

In the following we will introduce the fractional type integrals which will be used throughout the paper.

Definition 7. *The Riemann–Liouville fractional integral is defined by [18, 28, 44] where $\Re(\alpha) > 0$ and $\mathcal{F}$ is locally integrable.*

$${}_aI_t^\alpha \mathcal{F}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} \mathcal{F}(x) dx.$$

The following definition represents the Riemann-Liouville k fractional integral which was defined by Mubeen and Habibullah [27].

Definition 8. *Let $\mathcal{G} \in L_1[a,b]$. Then the k-fractional integrals of order $\alpha, k > 0$ with $a \geqslant 0$ are defined as:*

$$\mathcal{I}_{a+}^{\alpha,k} \mathcal{G}(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} \mathcal{G}(t) dt, x > a$$

and

$$\mathcal{I}_{b-}^{\alpha,k}\mathcal{G}(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(t-x)^{\frac{\alpha}{k}-1}\mathcal{G}(t)dt, x<b$$

where $\Gamma_k(.)$ is the k-Gamma function.

The following definition is due to Udita Katugampola [21] of Katugampola Fractional integrals, which generalizes the Riemann-Liouville fractional integrals.

Definition 9. *Let $[a,b]\subset\mathbb{R}$ be a finite interval. Then, the left-and right-sided Katugampola fractional integrals of order $\alpha>0$ of $\mathcal{F}\in[a,b]$ are defined by*

$$^{p}I_{a^+}^{\alpha}\mathcal{F}(x):=\frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{t^{p-1}}{(x^p-t^p)^{1-\alpha}}\mathcal{F}(t)dt$$

and

$$^{p}I_{b^-}^{\alpha}\mathcal{F}(x):=\frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{t^{p-1}}{(t^p-x^p)^{1-\alpha}}\mathcal{F}(t)dt$$

with $a<x<b$ and $p>0$, if the integrals exist.

The following definition [36] generalizes all the previously defined fractional integrals.

Definition 10. *The $(k-p)$ Riemann-Liouville fractional integral operator ${}_{k}^{p}J_{c}^{\alpha}$ of order $\alpha>0$ for a real valued function $\mathcal{G}(t)$ is defined as*

$${}_{k}^{p}J_{c}^{\alpha}\mathcal{G}(x)=\frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\theta)}\int_c^x[x^{p+1}-t^{p+1}]^{\frac{\alpha}{k}-1}t^p\mathcal{G}(t)dt$$

where $k>0, p\in\mathbb{R}, p\neq-1$.

The left and right sided $(k-p)$ Riemann-Liouville fractional integral operators are given by

$${}_{k}^{p}J_{c^+}^{\alpha}\mathcal{G}(x)=\frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_c^x[x^{p+1}-t^{p+1}]^{\frac{\alpha}{k}-1}t^p\mathcal{G}(t)dt$$

$${}_{k}^{p}J_{d^-}^{\alpha}\mathcal{G}(x)=\frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^d[t^{p+1}-x^{p+1}]^{\frac{\alpha}{k}-1}t^p\mathcal{G}(t)dt$$

Special cases

1. When $p=0$ the $(k-p)$ Riemann-Liouville fractional integral reduces to k-Riemann-Liouville fractional integral.
2. When k=1 the $(k-p)$ Riemann-Liouville fractional integral reduces to Katugampola fractional integral.
3. When $k=1, p=0$ the $(k-p)$ Riemann-Liouville fractional integral reduces to Riemann-Liouville fractional integral.

Recently published paper by Stojiljković. et al [40] proved some inequalities regarding $k-p$ fractional operators. We state two of them for the completeness because the Theorems in this paper generalize the results in the recently published paper.
Let $J \subset \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. If $\mathcal{F} : [a,b] \to \mathbb{R}$ is an (h-m)-convex function, such that the Riemann–Liouville k-fractional integral is defined, $\zeta \in (0,1)$, $\Re(\frac{\alpha}{k}) > 0, \alpha \neq 0$, and in one of the cases, the following inequality holds:

(i) $a > 0, b > a, 0 < m < \frac{a}{b}$

(ii) $a > 0, b < a, 0 < m \leqslant 1$

$$\frac{\mathcal{F}\left(\frac{a+bm}{2}\right)}{h(\frac{1}{2})} \leqslant \alpha\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(a)^-}\mathcal{F}(mb)}{(a-bm)^{\frac{\alpha}{k}}} + \frac{mI^{\alpha,k}_{(b)^+}\mathcal{F}(\frac{a}{m})}{(\frac{a}{m}-b)^{\frac{\alpha}{k}}}\right)$$
$$\leqslant \frac{\alpha(\mathcal{F}(a)+m\mathcal{F}(b))}{k}\int_0^1 t^{\frac{\alpha}{k}-1}h(t)dt + \frac{\alpha(\mathcal{F}(a)+m\mathcal{F}(b))}{k}\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t)dt.$$

Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. Let $\mathcal{F} : [a^p, b^p] \to \mathbb{R}$ be a (p,h)-convex function, $p > 0, \zeta \in (0,1)$. Then, the following inequality holds

$$\frac{\mathcal{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)}{h(\frac{1}{2})} \leqslant \frac{2^{\frac{\alpha}{k}}p^{\frac{\alpha}{k}}\alpha\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}_k^{p-1}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+}\mathcal{F}(b) + {}_k^{p-1}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-}\mathcal{F}(a)\right)$$
$$\leqslant \frac{\alpha p}{k}\left(\mathcal{F}(a)+\mathcal{F}(b)\right)\left(\int_0^1 t^{\frac{\alpha p}{k}-1}\left(h\left(\frac{t^p}{2}\right)+h\left(1-\frac{t^p}{2}\right)\right)dt\right).$$

In our analysis, we will need the integral version of the Hölder's inequality. If $f, g \in C([r,s],\mathbb{R})$ and $\lambda, \alpha \in \mathbb{R}$ with $\lambda > 1$ and $\frac{1}{\lambda}+\frac{1}{\alpha}=1$, then

$$\int_a^b |f(t)g(t)dt| \leqslant \left(\int_a^b |f(t)|^{\lambda}dt\right)^{\frac{1}{\lambda}}\left(\int_a^b |g(t)|^{\alpha}dt\right)^{\frac{1}{\alpha}}.$$

2. Main results

The following Theorem generalizes the Theorem 1 from the recently published paper [40] about $k-p$ fractional inequalities.
Theorem 1. *Let $\mathcal{F} : [a^p, b^p] \to \mathbb{R}$. If $\mathcal{F}$ is $(\alpha, h-m)-p$ convex on $[a^p, b^p]$, then the inequality holds in one of the following cases*
1.$a > 0, b > a, 0 < m \leqslant \frac{a}{b}$
2.$a > 0, b < a, 0 < m \leqslant 1$

$$\mathcal{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right) \leqslant \frac{h\left(\frac{1}{2^{\alpha}}\right)\theta\Gamma_k(\theta)p^{\frac{\theta}{k}}}{(a^p-m^pb^p)^{\frac{\theta}{k}}}{}_k^{p-1}J^{\theta}_{a^-}\mathcal{F}(mb)$$

$$+m^p\frac{\theta\Gamma_k(\theta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)p^{\frac{\theta}{k}}}{(\frac{a^p}{m^p}-b^p)^{\frac{\theta}{k}}}{}_k^{p-1}J^\theta_{(b)^+}\mathcal{F}\left(\frac{a}{m}\right)$$

$$\leqslant\left(\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(a)+m^ph\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}(b)\right)\int_0^1 h(t^{lp})t^{\frac{\theta p}{k}-1}dt\right)\cdot\frac{\theta p}{k}$$

$$+\left(\left(m^ph\left(\frac{1}{2^\alpha}\right)\mathcal{F}(b)+h\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}(a)\right)\int_0^1 h(1-t^{lp})t^{\frac{\theta p}{k}-1}dt\right)\cdot\frac{\theta p}{k}.$$

Proof. Using the definition of a (α,h,m)-p convex function we have

$$\mathcal{F}\left((tx^p+m(1-t)y^p)^{\frac{1}{p}}\right)\leqslant h(t^\alpha)\mathcal{F}(x)+mh(1-t^\alpha)\mathcal{F}(y).$$

Setting $t=\frac{1}{2}$ and $x^p=m^p(1-t^p)b^p+t^pa^p, y^p=(1-t^p)\frac{a^p}{m^p}+b^pt^p$ in the inequality, we get the following

$$\mathcal{F}\left([\frac{a^p+b^pm^p}{2}]^{\frac{1}{p}}\right)\leqslant h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left([m^p(1-t^p)b^p+t^pa^p]^{\frac{1}{p}}\right)$$

$$+m^ph\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}\left([(1-t^p)\frac{a^p}{m^p}+b^pt^p]^{\frac{1}{p}}\right).$$

Multiplying both sides by $t^{\frac{\theta p}{k}-1}$ and integrating with respect to t from 0 to 1 we get

$$\int_0^1\mathcal{F}\left([\frac{a^p+b^pm^p}{2}]^{\frac{1}{p}}\right)t^{\frac{\theta p}{k}-1}dt\leqslant\int_0^1 t^{\frac{\theta p}{k}-1}h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left([m^p(1-t^p)b^p+t^pa^p]^{\frac{1}{p}}\right)dt$$

$$+\int_0^1 t^{\frac{\theta p}{k}-1}m^ph\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}\left([(1-t^p)\frac{a^p}{m^p}+b^pt^p]^{\frac{1}{p}}\right)dt$$

Integrating the left hand side is easy, therefore we focus on the right hand side. In the first integral we introduce a substitution $m^p(1-t^p)b^p+(ta)^p=y^p$. From which we get that

$$\int_0^1 t^{\frac{\theta p}{k}-1}h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left([m^p(1-t^p)b^p+t^pa^p]^{\frac{1}{p}}\right)dt$$

$$=\frac{1}{(a^p-m^pb^p)^{\frac{\theta}{k}}}\int_{mb}^a\mathcal{F}(y)(y^p-m^pb^p)^{\frac{\theta}{k}-1}y^{p-1}dy.$$

Multiplying the integral with the needed constants for the $k-p$ Riemann Liouville fractional integral, we get the following

$$\int_0^1 t^{\frac{\theta p}{k}-1}h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left([m^p(1-t^p)b^p+t^pa^p]^{\frac{1}{p}}\right)dt$$

$$=h\left(\frac{1}{2^\alpha}\right)\frac{k\Gamma_k(\theta)}{p^{1-\frac{\theta}{k}}(a^p-m^pb^p)^{\frac{\theta}{k}}}{}_k^{p-1}J^\theta_{a^-}\mathcal{F}(mb).$$

Similar procedure can be applied to the second integral introducing a substitution $\frac{a^p}{m^p} - \frac{t^p a^p}{m^p} + b^p t^p = y^p$ *while noting that* $\frac{a^p}{m^p} > b^p$. *From which we get that*

$$\int_0^1 t^{\frac{\theta p}{k}-1} m^p h\left(\frac{2^\alpha - 1}{2^\alpha}\right) \mathcal{F}\left([(1-t^p)\frac{a^p}{m^p} + b^p t^p]^{\frac{1}{p}}\right) dt$$

$$= h\left(\frac{2^\alpha - 1}{2^\alpha}\right) \frac{m^{\frac{p\theta}{k}} \Gamma_k(\theta)}{p^{1-\frac{\theta}{k}} (a^p - m^p b^p)^{\frac{\theta}{k}}} {}^{p-1}_{k} J^\theta_{b^+} \mathcal{F}\left(\frac{a}{m}\right).$$

Now we focus on the right hand side inequality

$$h\left(\frac{1}{2^\alpha}\right) \mathcal{F}\left([m^p(1-t^p)b^p + t^p a^p]^{\frac{1}{p}}\right) + m^p h\left(\frac{2^\alpha - 1}{2^\alpha}\right) \mathcal{F}\left([(1-t^p)\frac{a^p}{m^p} + b^p t^p]^{\frac{1}{p}}\right).$$

Using the definition of (α, h, m)*-p convexity and multiplying both sides by* $t^{\frac{\theta p}{k}-1}$ *and integrating with respect to t from 0 to 1 we get that*

$$h\left(\frac{1}{2^\alpha}\right) \frac{k\Gamma_k(\theta)}{p^{1-\frac{\theta}{k}} (a^p - m^p b^p)^{\frac{\theta}{k}}} {}^{p-1}_{k} J^\theta_{a^-} \mathcal{F}(mb)$$

$$+ m^p h\left(\frac{2^\alpha - 1}{2^\alpha}\right) \frac{m^{\frac{p\theta}{k}} \Gamma_k(\theta)}{p^{1-\frac{\theta}{k}} (a^p - m^p b^p)^{\frac{\theta}{k}}} {}^{p-1}_{k} J^\theta_{b^+} \mathcal{F}\left(\frac{a}{m}\right) \leqslant$$

$$\left(\left(h\left(\frac{1}{2^\alpha}\right) \mathcal{F}(a) + m^p h\left(\frac{2^\alpha - 1}{2^\alpha}\right) \mathcal{F}(b)\right) \int_0^1 h(t^{lp}) t^{\frac{\theta p}{k}-1} dt\right) \frac{\theta p}{k}$$

$$+ \left(m^p h\left(\frac{1}{2^\alpha}\right) \mathcal{F}(b) + h\left(\frac{2^\alpha - 1}{2^\alpha}\right) \mathcal{F}(a)\right) \int_0^1 h(1 - t^{lp}) t^{\frac{\theta p}{k}-1} dt\bigg) \frac{\theta p}{k}.$$

Connecting the left hand side inequality with the right hand side inequality, we get the desired inequality.

Corollary 1. *Setting* $p = 1$ *and* $l, \alpha = 1$ *in the previously derived Theorem, we obtain Theorem 1 from the paper [40]*

$$\frac{\mathcal{F}\left(\frac{a+bm}{2}\right)}{h(\frac{1}{2})} \leqslant \alpha\Gamma_k(\alpha)\left(\frac{I^{\alpha,k}_{(a)^-} \mathcal{F}(mb)}{(a - bm)^{\frac{\alpha}{k}}} + \frac{m I^{\alpha,k}_{(b)^+} \mathcal{F}(\frac{a}{m})}{(\frac{a}{m} - b)^{\frac{\alpha}{k}}}\right)$$

$$\leqslant \frac{\alpha(\mathcal{F}(a) + m\mathcal{F}(b))}{k} \int_0^1 t^{\frac{\alpha}{k}-1} h(t) dt + \frac{\alpha(\mathcal{F}(a) + m\mathcal{F}(b))}{k} \int_0^1 t^{\frac{\alpha}{k}-1} h(1-t) dt.$$

Corollary 2. *Setting* $p = 3$ *in the previously derived Theorem, we obtain a new inequality of the* $k - p$ *Riemann Liouville fractional type*

$$\mathcal{F}\left([\frac{a^3 + m^3 b^3}{2}]^{\frac{1}{3}}\right) \leqslant \frac{h\left(\frac{1}{2^\alpha}\right) \theta\Gamma_k(\theta) 3^{\frac{\theta}{k}}}{(a^3 - m^3 b^3)^{\frac{\theta}{k}}} {}^{2}_{k} J^\theta_{a^-} \mathcal{F}(mb)$$

$$+m^3\frac{\theta\Gamma_k(\theta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)3^{\frac{\theta}{k}}}{(\frac{a^3}{m^3}-b^3)^{\frac{\theta}{k}}}\,{}_k^2J^\theta_{(b)^+}\mathcal{F}\left(\frac{a}{m}\right)$$

$$\leqslant\left(\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(a)+m^3h\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}(b)\right)\int_0^1 h(t^{3l})t^{\frac{3\theta}{k}-1}dt\right)\cdot\frac{3\theta}{k}$$

$$+\left(\left(m^3h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(b)+m^3h\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\mathcal{F}(a)}{m^3}\right)\int_0^1 h(1-t^{3l})t^{\frac{3\theta}{k}-1}dt\right)\cdot\frac{3\theta}{k}.$$

Theorem 2. *Let $\mathcal{F}:[a^p,b^p]\to\mathbb{R}$. If $\mathcal{F}$ is $(\alpha,h-m)-p$ convex on $[a^p,b^p]$, then the inequality holds in the following case $a\geqslant 0, b>a, \frac{a}{b}<m\leqslant 1$*

$$\mathcal{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)\leqslant h\left(\frac{1}{2^\alpha}\right)\frac{m^{\frac{p\theta}{k}}\theta\Gamma_k(\theta)2^{\frac{\theta}{k}}}{p^{-\frac{\theta}{k}}(m^pb^p-a^p)^{\frac{\theta}{k}}}\,{}_k^{p-1}J^\theta_{\left((\frac{a^p}{2m^p}+\frac{b^p}{2})^{\frac{1}{p}}\right)^-}\mathcal{F}\left(\frac{a}{m}\right)$$

$$+h\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{2^{\frac{\theta}{k}}\theta\Gamma_k(\theta)}{p^{-\frac{\theta}{k}}(m^pb^p-a^p)^{\frac{\theta}{k}}}\,{}_k^{p-1}J^\theta_{\left((\frac{a^p}{2}+\frac{b^pm^p}{2})^{\frac{1}{p}}\right)^+}\mathcal{F}(mb)$$

$$\leqslant\frac{\theta p}{k}\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(a)+m^ph\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}(b)\right)\int_0^1 h\left(\left(\frac{t^p}{2}\right)^l\right)t^{\frac{\theta p}{k}-1}dt$$

$$+\frac{\theta p}{k}\left(h\left(\frac{1}{2^\alpha}\right)m^p\mathcal{F}(b)+h\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}(a)\right)\int_0^1 h\left(1-\left(\frac{t^p}{2}\right)^l\right)t^{\frac{\theta p}{k}-1}dt.$$

Proof. *Using the definition of a (α,h,m)-p convex function we have*

$$\mathcal{F}\left((tx^p+m(1-t)y^p)^{\frac{1}{p}}\right)\leqslant h(t^\alpha)\mathcal{F}(x)+mh(1-t^\alpha)\mathcal{F}(y).$$

Setting $t=\frac{1}{2}$ and $x^p=\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p, y^p=\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}(\frac{a}{m})^p$ in the inequality, we get the following

$$\mathcal{F}\left([\frac{a^p+b^pm^p}{2}]^{\frac{1}{p}}\right)\leqslant h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left(\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)$$

$$+m^ph\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}(\frac{a}{m})^p]^{\frac{1}{p}}\right).$$

Multiplying the inequality with $t^{\frac{\theta p}{k}-1}$ and integrating with respect to t from 0 to 1 we get

$$\int_0^1 t^{\frac{\theta p}{k}-1}\mathcal{F}\left([\frac{a^p+b^pm^p}{2}]^{\frac{1}{p}}\right)dt\leqslant\int_0^1 t^{\frac{\theta p}{k}-1}h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left(\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)dt$$

$$+\int_0^1 t^{\frac{\theta p}{k}-1}m^ph\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p]^{\frac{1}{p}}\right)dt.$$

Integrating the left hand side is easy. Let us focus on the right hand side. Introducing a substitution $\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p=z^p$ *we get the following equality*

$$\int_0^1 t^{\frac{\theta p}{k}-1}h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left(\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)dt$$

$$=\int_{\left(\frac{a^p}{2}+\frac{m^pb^p}{2}\right)^{\frac{1}{p}}}^{mb}\mathcal{F}(z)(m^pb^p-z^p)^{\frac{\theta}{k}-1}z^{p-1}dz\cdot\frac{2^{\frac{\theta}{k}}}{(m^pb^p-a^p)^{\frac{\theta}{k}}}.$$

Where we used the condition $a\geqslant 0, b>a, \frac{a}{b}<m\leqslant 1$ *to swap the upper and lower boundary. Which clearly can be seen to be of the* $k-p$ *Riemann Liouville fractional integral form, therefore we obtain*

$$\int_{\left(\frac{a^p}{2}+\frac{m^pb^p}{2}\right)^{\frac{1}{p}}}^{mb}\mathcal{F}(z)(m^pb^p-z^p)^{\frac{\theta}{k}-1}z^{p-1}dz$$

$$\cdot\frac{2^{\frac{\theta}{k}}}{(m^pb^p-a^p)^{\frac{\theta}{k}}}=\frac{2^{\frac{\theta}{k}}k\Gamma_k(\theta)}{p^{1-\frac{\theta}{k}}(m^pb^p-a^p)^{\frac{\theta}{k}}}\,{}^{p-1}_{\ \ k}J^{\theta}_{\left(\left(\frac{a^p}{2}+\frac{m^pb^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathcal{F}(mb).$$

Applying the similar technique while using the substitution in the second integral $\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}(\frac{a}{m})^p=y^p$ *we obtain the following equality*

$$\int_0^1 t^{\frac{\theta p}{k}-1}m^ph\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p]^{\frac{1}{p}}dt\right)$$

$$=\int_{\frac{a}{m}}^{\left(\frac{b^p}{2}+\frac{a^p}{2m^p}\right)^{\frac{1}{p}}}\mathcal{F}(y)\left(y^p-\frac{a^p}{m^p}\right)^{\frac{\theta}{k}-1}y^{p-1}dy\cdot\frac{1}{(\frac{b^p}{2}-\frac{a^p}{2m^p})^{\frac{\theta}{k}}}.$$

Which can be seen to be of the form of the $k-p$ *Riemann Liouville fractional integral, therefore we obtain*

$$\int_{\frac{a}{m}}^{\left(\frac{b^p}{2}+\frac{a^p}{2m^p}\right)^{\frac{1}{p}}}\mathcal{F}(y)\left(y^p-\frac{a^p}{m^p}\right)^{\frac{\theta}{k}-1}y^{p-1}dy\cdot\frac{1}{(\frac{b^p}{2}-\frac{a^p}{2m^p})^{\frac{\theta}{k}}}$$

$$=\frac{m^{\frac{p\theta}{k}}k\Gamma_k(\theta)2^{\frac{\theta}{k}}}{p^{1-\frac{\theta}{k}}(m^pb^p-a^p)^{\frac{\theta}{k}}}\,{}^{p-1}_{\ \ k}J^{\theta}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathcal{F}\left(\frac{a}{m}\right).$$

Now we focus on obtaining the right hand side inequality. Using the definition of the $(\alpha,h-m)-p$ *convex function on the following expression, we obtain*

$$h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left(\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)$$

$$+m^p h\left(\frac{2^\alpha-1}{2^\alpha}\right)\mathcal{F}\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}(\frac{a}{m})^p]^{\frac{1}{p}}\right)\leqslant$$

$$\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(a)+m^p h\left(1-\frac{1}{2^\alpha}\right)\mathcal{F}(b)\right)h\left(\left(\frac{t^p}{2}\right)^l\right)$$

$$+\left(h\left(\frac{1}{2^\alpha}\right)m^p\mathcal{F}(b)+h\left(1-\frac{1}{2^\alpha}\right)\mathcal{F}(a)\right)h\left(1-\left(\frac{t^p}{2}\right)^l\right).$$

Multiplying the inequality with $t^{\frac{\theta p}{k}-1}$ *and integrating with respect to* t *from* 0 *to* 1 *we obtain*

$$\frac{k\Gamma_k(\theta)2^{\frac{\theta}{k}}}{p^{1-\frac{\theta}{k}}(m^pb^p-a^p)^{\frac{\theta}{k}}}\left(m^{\frac{p\theta}{k}}\,{}^{p-1}_{k}J^{\theta}_{\left((\frac{a^p}{2m^p}+\frac{b^p}{2})^{\frac{1}{p}}\right)^-}\mathcal{F}\left(\frac{a}{m}\right)+{}^{p-1}_{k}J^{\theta}_{\left((\frac{a^p}{2}+\frac{m^pb^p}{2})^{\frac{1}{p}}\right)^+}\mathcal{F}(mb)\right)$$

$$\leqslant$$

$$\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(a)+m^p h\left(1-\frac{1}{2^\alpha}\right)\mathcal{F}(b)\right)\int_0^1 t^{\frac{\theta p}{k}-1}h\left(\left(\frac{t^p}{2}\right)^l\right)dt$$

$$+\left(h\left(\frac{1}{2^\alpha}\right)m^p\mathcal{F}(b)+h\left(1-\frac{1}{2^\alpha}\right)\mathcal{F}(a)\right)\int_0^1 t^{\frac{\theta p}{k}-1}h\left(1-\left(\frac{t^p}{2}\right)^l\right)dt.$$

Connecting the left and right hand side inequality and multiplying everything with the constant from the left hand side, we obtain the desired inequality.

Corollary 3. *Setting* $\alpha, l, m = 1$ *in the previously derived inequality, we obtain Theorem 4 from the paper [40] , namely we obtain*

$$\frac{\mathcal{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)}{h(\frac{1}{2})}\leqslant\frac{2^{\frac{\alpha}{k}}p^{\frac{\alpha}{k}}\alpha\Gamma_k(\alpha)}{(b^p-a^p)^{\frac{\alpha}{k}}}\left({}^{p-1}_{k}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+}\mathcal{F}(b)+{}^{p-1}_{k}J^{\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-}\mathcal{F}(a)\right)$$

$$\leqslant\frac{\alpha p}{k}\left(\mathcal{F}(a)+\mathcal{F}(b)\right)\left(\int_0^1 t^{\frac{\alpha p}{k}-1}\left(h\left(\frac{t^p}{2}\right)+h\left(1-\frac{t^p}{2}\right)\right)dt\right).$$

Corollary 4. *Setting* $p=3$ *in the previously derived inequality, we obtain the new inequality of the fractional* $k-p$ *Riemann Liouville type*

$$\mathcal{F}\left([\frac{a^3+m^3b^3}{2}]^{\frac{1}{3}}\right)\leqslant h\left(\frac{1}{2^\alpha}\right)\frac{m^{\frac{3\theta}{k}}\theta\Gamma_k(\theta)2^{\frac{\theta}{k}}}{3^{-\frac{\theta}{k}}(m^3b^3-a^3)^{\frac{\theta}{k}}}\,{}^{2}_{k}J^{\theta}_{\left((\frac{a^3}{2m^3}+\frac{b^3}{2})^{\frac{1}{3}}\right)^-}\mathcal{F}\left(\frac{a}{m}\right)$$

$$+h\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{2^{\frac{\theta}{k}}\theta\Gamma_k(\theta)}{3^{-\frac{\theta}{k}}(m^3b^3-a^3)^{\frac{\theta}{k}}}\,{}^{2}_{k}J^{\theta}_{\left((\frac{a^3}{2}+\frac{b^3m^3}{2})^{\frac{1}{3}}\right)^+}\mathcal{F}(mb)$$

$$\leqslant \frac{3\theta}{k}\left(h\left(\frac{1}{2^{\alpha}}\right)\mathcal{F}(a)+m^{3}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\mathcal{F}(b)\right)\int_{0}^{1}h\left(\left(\frac{t^{3}}{2}\right)^{l}\right)t^{\frac{3\theta}{k}-1}dt$$

$$+\frac{3\theta}{k}\left(h\left(\frac{1}{2^{\alpha}}\right)m^{3}\mathcal{F}(b)+h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\mathcal{F}(a)\right)\int_{0}^{1}h\left(1-\left(\frac{t^{3}}{2}\right)^{l}\right)t^{\frac{3\theta}{k}-1}dt.$$

In the following we present a new Theorem of the generalized q type.

Theorem 3. *Let $\mathcal{F}:[a^{p},b^{p}]\rightarrow\mathbb{R}$. If f is $(\alpha,h-m)-p$ convex on $[a^{p},b^{p}]$ and $a\geqslant 0, b> a, \frac{a}{b}<m\leqslant 1$, then the following inequality holds*

$$\mathcal{F}\left([\frac{a^{p}+m^{p}b^{p}}{2}]^{\frac{1}{p}}\right)\leqslant$$

$$\frac{h\left(\frac{1}{2^{\alpha}}\right)2^{\frac{\theta}{k}}p^{\frac{\theta}{k}}\theta\Gamma_{k}(\theta)}{(b^{p}m^{p}-a^{p})^{\frac{\theta}{k}}}{}_{k}^{p-1}J^{\theta}_{\left(\left(\frac{q}{2}(a^{p}-m^{p}b^{p})+\frac{a^{p}}{2}+\frac{b^{p}m^{p}}{2}\right)^{\frac{1}{p}}\right)^{+}}\mathcal{F}\left(\left(\frac{q}{2}(a^{p}-b^{p}m^{p})+m^{p}b^{p}\right)^{\frac{1}{p}}\right)$$

$$+\frac{h\left(1-\frac{1}{2^{\alpha}}\right)m^{p+\frac{\theta p}{k}}2^{\frac{\theta}{k}}p^{\frac{\theta}{k}}\theta\Gamma_{k}(\theta)}{(b^{p}m^{p}-a^{p})^{\frac{\theta}{k}}}{}_{k}^{p-1}J_{\left(\left(\frac{q}{2}(b^{p}-\frac{a^{p}}{m^{p}})+\frac{a^{p}}{2m^{p}}+\frac{b^{p}}{2}\right)^{\frac{1}{p}}\right)^{-}}\mathcal{F}\left(\left(\frac{q}{2}(b^{p}-\frac{a^{p}}{m^{p}})+\frac{a^{p}}{m^{p}}\right)^{\frac{1}{p}}\right)$$

$$\leqslant\left(\left(h\left(\frac{1}{2^{\alpha}}\right)\mathcal{F}(a)+h\left(1-\frac{1}{2^{\alpha}}\right)m^{p}\mathcal{F}(b)\right)\int_{0}^{1}h\left(\left(\frac{q+t^{p}}{2}\right)^{l}\right)t^{\frac{\theta p}{k}-1}dt\right)\frac{\theta p}{k}$$

$$+\left(\left(h\left(\frac{1}{2^{\alpha}}\right)\mathcal{F}(b)m^{p}+h\left(1-\frac{1}{2^{\alpha}}\right)\mathcal{F}(a)\right)\int_{0}^{1}h\left(1-\left(\frac{q+t^{p}}{2}\right)^{l}\right)t^{\frac{\theta p}{k}-1}dt\right)\frac{\theta p}{k}.$$

Proof. *Since $\mathcal{F}$ is $(\alpha,h-m)-p$ convex, we have the following inequality*

$$\mathcal{F}\left((ta^{p}+m(1-t)b^{p})^{\frac{1}{p}}\right)\leqslant h(t^{\alpha})\mathcal{F}(a)+mh(1-t^{\alpha})\mathcal{F}(b).$$

Setting $t=\frac{1}{2}$ and $x^{p}=\frac{q+t^{p}}{2}a^{p}+\left(1-\frac{q+t^{p}}{2}\right)b^{p}m^{p}, y^{p}=\frac{q+t^{p}}{2}b^{p}+\left(1-\frac{q+t^{p}}{2}\right)\frac{a^{p}}{m^{p}}$ in the inequality, we obtain

$$\mathcal{F}\left([\frac{a^{p}+m^{p}b^{p}}{2}]^{\frac{1}{p}}\right)\leqslant h\left(\frac{1}{2^{\alpha}}\right)\mathcal{F}\left((\frac{q+t^{p}}{2}a^{p}+\left(1-\frac{q+t^{p}}{2}\right)b^{p}m^{p})^{\frac{1}{p}}\right)$$

$$+h\left(1-\frac{1}{2^{\alpha}}\right)m^{p}\mathcal{F}\left((\frac{q+t^{p}}{2}b^{p}+\left(1-\frac{q+t^{p}}{2}\right)\frac{a^{p}}{m^{p}})^{\frac{1}{p}}\right).$$

Multiplying the inequality with $t^{\frac{\theta p}{k}-p}t^{p-1}$ and integrating with respect to t from 0 to 1 we get

$$\int_{0}^{1}t^{\frac{\theta p}{k}-1}\mathcal{F}\left([\frac{a^{p}+m^{p}b^{p}}{2}]^{\frac{1}{p}}\right)dt\leqslant$$

$$\int_0^1 t^{\frac{\theta p}{k}-1}h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left((\frac{q+t^p}{2}a^p+\left(1-\frac{q+t^p}{2}\right)b^pm^p)^{\frac{1}{p}}\right)dt$$
$$+\int_0^1 t^{\frac{\theta p}{k}-1}h\left(1-\frac{1}{2^\alpha}\right)m^p\mathcal{F}\left((\frac{q+t^p}{2}b^p+\left(1-\frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right)dt.$$

The left hand side is easy to integrate, therefore we focus our attention to the other two integrals. Introducing a substitution $x^p = \frac{q+t^p}{2}a^p + \left(1-\frac{q+t^p}{2}\right)b^pm^p$ *while noting that* $a \geqslant 0, b > a, \frac{a}{b} < m \leqslant 1$, *we obtain*

$$\int_0^1 t^{\frac{\theta p}{k}-1}h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left((\frac{q+t^p}{2}a^p+\left(1-\frac{q+t^p}{2}\right)b^pm^p)^{\frac{1}{p}}\right)dt$$
$$=\int_{\left(\frac{q}{2}(a^p-b^pm^p)+\frac{a^p}{2}+\frac{m^pb^p}{2}\right)^{\frac{1}{p}}}^{\left(\frac{q}{2}(a^p-m^pb^p)+b^pm^p\right)^{\frac{1}{p}}}\mathcal{F}(z)(b^pm^p+\frac{q}{2}(a^p-b^pm^p)-z^p)^{\frac{\theta}{k}-1}z^{p-1}dz\cdot\frac{2^{\frac{\theta}{k}}}{(b^pm^p-a^p)^{\frac{\theta}{k}}}.$$

Which can be seen to be of the $k-p$ *Riemann Liouville integral form, therefore we obtain the following equality*

$$\int_{\left(\frac{q}{2}(a^p-b^pm^p)+\frac{a^p}{2}+\frac{m^pb^p}{2}\right)^{\frac{1}{p}}}^{\left(\frac{q}{2}(a^p-m^pb^p)+b^pm^p\right)^{\frac{1}{p}}}\mathcal{F}(z)(b^pm^p+\frac{q}{2}(a^p-b^pm^p)-z^p)^{\frac{\theta}{k}-1}z^{p-1}dz\cdot\frac{2^{\frac{\theta}{k}}}{(b^pm^p-a^p)^{\frac{\theta}{k}}}$$
$$=\frac{h\left(\frac{1}{2^\alpha}\right)2^{\frac{\theta}{k}}k\Gamma_k(\theta)}{p^{1-\frac{\theta}{k}}(b^pm^p-a^p)^{\frac{\theta}{k}}}\,{}^{p-1}_{\ \ k}J^{\theta}_{\left(\left(\frac{q}{2}(a^p-m^pb^p)+\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathcal{F}\left(\left(\frac{q}{2}(a^p-b^pm^p)+m^pb^p\right)^{\frac{1}{p}}\right).$$

Using the similar technique on the other integral, using the substitution

$$\frac{q+t^p}{2}b^p+\left(1-\frac{q+t^p}{2}\right)\frac{a^p}{m^p}=z^p$$

we get

$$\int_0^1 t^{\frac{\theta p}{k}-1}h\left(1-\frac{1}{2^\alpha}\right)m^p\mathcal{F}\left((\frac{q+t^p}{2}b^p+\left(1-\frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right)dt$$
$$=\int_{\left(\frac{q}{2}\left(b^p-\frac{a^p}{m^p}\right)+\frac{a^p}{m^p}\right)^{\frac{1}{p}}}^{\left(\frac{q}{2}\left(b^p-\frac{a^p}{m^p}\right)+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}}\mathcal{F}(z)\left(z^p-\frac{q}{2}(b^p-\frac{a^p}{m^p})-\frac{a^p}{m^p}\right)^{\frac{\theta}{k}}\cdot\frac{2^{\frac{\theta}{k}}m^{\frac{\theta p}{k}}}{(m^pb^p-a^p)^{\frac{\theta}{k}}}.$$

Which can be seen to be of the $k-p$ *Riemann Liouville integral form, therefore we obtain*

$$\int_{\left(\frac{q}{2}\left(b^p-\frac{a^p}{m^p}\right)+\frac{a^p}{m^p}\right)^{\frac{1}{p}}}^{\left(\frac{q}{2}\left(b^p-\frac{a^p}{m^p}\right)+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}}\mathcal{F}(z)\left(z^p-\frac{q}{2}(b^p-\frac{a^p}{m^p})-\frac{a^p}{m^p}\right)^{\frac{\theta}{k}}\cdot\frac{2^{\frac{\theta}{k}}m^{\frac{\theta p}{k}}}{(m^pb^p-a^p)^{\frac{\theta}{k}}}$$
$$=\frac{2^{\frac{\theta}{k}}m^{\frac{p\theta}{k}}k\Gamma_k(\theta)}{p^{1-\frac{\theta}{k}}(b^pm^p-a^p)^{\frac{\theta}{k}}}\,{}^{p-1}_{\ \ k}J^{\theta}_{\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathcal{F}\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right).$$

In order to obtain the right hand side inequality, we use the definition of the $(\alpha, h-m)-p$ *convex function and deduce the following*

$$h\left(\frac{1}{2^\alpha}\right)\mathcal{F}\left(\left(\frac{q+t^p}{2}a^p+\left(1-\frac{q+t^p}{2}\right)b^pm^p\right)^{\frac{1}{p}}\right)$$

$$+h\left(1-\frac{1}{2^\alpha}\right)m^p\mathcal{F}\left(\left(\frac{q+t^p}{2}b^p+\left(1-\frac{q+t^p}{2}\right)\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right)\leqslant$$

$$\left(\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(a)+h\left(1-\frac{1}{2^\alpha}\right)m^p\mathcal{F}(b)\right)h\left(\left(\frac{q+t^p}{2}\right)^l\right)t^{\frac{\theta p}{k}-1}\right)$$

$$+\left(\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(b)m^p+h\left(1-\frac{1}{2^\alpha}\right)\mathcal{F}(a)\right)h\left(1-\left(\frac{q+t^p}{2}\right)^l\right)t^{\frac{\theta p}{k}-1}\right).$$

Multiplying the inequality with $t^{\frac{\theta p}{k}-1}$ *and integrating with respect to* t *from* 0 *to* 1*, while also multiplying everything with the constant from the left hand side, we obtain the original inequality*

$$\mathcal{F}\left(\left[\frac{a^p+m^pb^p}{2}\right]^{\frac{1}{p}}\right)\leqslant$$

$$\frac{h\left(\frac{1}{2^\alpha}\right)2^{\frac{\theta}{k}}p^{\frac{\theta}{k}}\theta\Gamma_k(\theta)}{(b^pm^p-a^p)^{\frac{\theta}{k}}}\,{}_k^{p-1}J^{\theta}_{\left(\left(\frac{q}{2}(a^p-m^pb^p)+\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathcal{F}\left(\left(\frac{q}{2}(a^p-b^pm^p)+m^pb^p\right)^{\frac{1}{p}}\right)$$

$$+\frac{h\left(1-\frac{1}{2^\alpha}\right)m^{p+\frac{\theta p}{k}}2^{\frac{\theta}{k}}p^{\frac{\theta}{k}}\theta\Gamma_k(\theta)}{(b^pm^p-a^p)^{\frac{\theta}{k}}}\,{}_k^{p-1}J_{\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathcal{F}\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right)$$

$$\leqslant\left(\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(a)+h\left(1-\frac{1}{2^\alpha}\right)m^p\mathcal{F}(b)\right)\int_0^1 h\left(\left(\frac{q+t^p}{2}\right)^l\right)t^{\frac{\theta p}{k}-1}dt\right)\frac{\theta p}{k}$$

$$+\left(\left(h\left(\frac{1}{2^\alpha}\right)\mathcal{F}(b)m^p+h\left(1-\frac{1}{2^\alpha}\right)\mathcal{F}(a)\right)\int_0^1 h\left(1-\left(\frac{q+t^p}{2}\right)^l\right)t^{\frac{\theta p}{k}-1}dt\right)\frac{\theta p}{k}.$$

Corollary 5. *Setting* $p=2$ *in the previously derived Theorem, we obtain the following new inequality*

$$\mathcal{F}\left(\left[\frac{a^2+m^2b^2}{2}\right]^{\frac{1}{2}}\right)\leqslant$$

$$\frac{h\left(\frac{1}{2^\alpha}\right)2^{\frac{\theta}{k}}2^{\frac{\theta}{k}}\theta\Gamma_k(\theta)}{(b^2m^2-a^2)^{\frac{\theta}{k}}}\,{}_k^{1}J^{\theta}_{\left(\left(\frac{q}{2}(a^2-m^2b^2)+\frac{a^2}{2}+\frac{b^2m^2}{2}\right)^{\frac{1}{2}}\right)^+}\mathcal{F}\left(\left(\frac{q}{2}(a^2-b^2m^2)+m^2b^2\right)^{\frac{1}{2}}\right)$$

$$+\frac{h\left(1-\frac{1}{2^\alpha}\right)m^{2+\frac{2\theta}{k}}2^{\frac{\theta}{k}}2^{\frac{\theta}{k}}\theta\Gamma_k(\theta)}{(b^2m^2-a^2)^{\frac{\theta}{k}}}\,{}_k^{1}J_{\left(\left(\frac{q}{2}(b^2-\frac{a^2}{m^2})+\frac{a^2}{2m^2}+\frac{b^2}{2}\right)^{\frac{1}{2}}\right)^-}\mathcal{F}\left(\left(\frac{q}{2}(b^2-\frac{a^2}{m^2})+\frac{a^2}{m^2}\right)^{\frac{1}{2}}\right)$$

$$\leqslant \left(\left(h\left(\frac{1}{2^{\alpha}}\right)\mathcal{F}(a)+h\left(1-\frac{1}{2^{\alpha}}\right)m^{2}\mathcal{F}(b)\right)\int_{0}^{1}h\left(\left(\frac{q+t^{2}}{2}\right)^{l}\right)t^{\frac{2\theta}{k}-1}dt\right)\frac{2\theta}{k}$$

$$+\left(\left(h\left(\frac{1}{2^{\alpha}}\right)\mathcal{F}(b)m^{2}+h\left(1-\frac{1}{2^{\alpha}}\right)\mathcal{F}(a)\right)\int_{0}^{1}h\left(1-\left(\frac{q+t^{2}}{2}\right)^{l}\right)t^{\frac{2\theta}{k}-1}dt\right)\frac{2\theta}{k}.$$

Corollary 6. *Setting* $p=7,l,\alpha=3$ *in the previously derived Theorem, we obtain a new inequality as a consequence*

$$\mathcal{F}\left([\frac{a^{7}+m^{7}b^{7}}{2}]^{\frac{1}{7}}\right)\leqslant$$

$$\frac{h\left(\frac{1}{2^{3}}\right)2^{\frac{\theta}{k}}7^{\frac{\theta}{k}}\theta\Gamma_{k}(\theta)}{(b^{7}m^{7}-a^{7})^{\frac{\theta}{k}}}{}_{k}^{6}J^{\theta}_{\left(\left(\frac{q}{2}(a^{7}-m^{7}b^{7})+\frac{a^{7}}{2}+\frac{b^{7}m^{7}}{2}\right)^{\frac{1}{7}}\right)^{+}}\mathcal{F}\left(\left(\frac{q}{2}(a^{7}-b^{7}m^{7})+m^{7}b^{7}\right)^{\frac{1}{7}}\right)$$

$$+\frac{h\left(1-\frac{1}{2^{3}}\right)m^{7+\frac{7\theta}{k}}2^{\frac{\theta}{k}}7^{\frac{\theta}{k}}\theta\Gamma_{k}(\theta)}{(b^{7}m^{7}-a^{7})^{\frac{\theta}{k}}}{}_{k}^{6}J_{\left(\left(\frac{q}{2}(b^{7}-\frac{a^{7}}{m^{7}})+\frac{a^{7}}{2m^{7}}+\frac{b^{7}}{2}\right)^{\frac{1}{7}}\right)^{-}}\mathcal{F}\left(\left(\frac{q}{2}(b^{7}-\frac{a^{7}}{m^{7}})+\frac{a^{7}}{m^{7}}\right)^{\frac{1}{7}}\right)$$

$$\leqslant \left(\left(h\left(\frac{1}{2^{3}}\right)\mathcal{F}(a)+h\left(1-\frac{1}{2^{3}}\right)m^{7}\mathcal{F}(b)\right)\int_{0}^{1}h\left(\left(\frac{q+t^{7}}{2}\right)^{\alpha}\right)t^{\frac{7\theta}{k}-1}dt\right)\frac{7\theta}{k}$$

$$+\left(\left(h\left(\frac{1}{2^{3}}\right)\mathcal{F}(b)m^{7}+h\left(1-\frac{1}{2^{3}}\right)\mathcal{F}(a)\right)\int_{0}^{1}h\left(1-\left(\frac{q+t^{7}}{2}\right)^{3}\right)t^{\frac{7\theta}{k}-1}dt\right)\frac{7\theta}{k}.$$

Theorem 4. *Let* $\mathcal{F}:[x^{p},y^{p}]\to\mathbb{R}$. *If* f *is* $(\alpha,h-m)-p$ *convex on* $[x^{p},y^{p}]$ *and if the following condition holds, then*
$0<m<1, x>0, x<y<\frac{x}{m}$,

$$\frac{k\Gamma_{k}(\theta)}{p^{1-\frac{\theta}{k}}}\left(\frac{{}_{k}^{p-1}J^{\theta}_{x^{-}}\mathcal{F}(my)}{(x^{p}-m^{p}y^{p})^{\frac{\theta}{k}}}+\frac{{}_{k}^{p-1}J^{\theta}_{y^{-}}\mathcal{F}(mx)}{(y^{p}-m^{p}x^{p})^{\frac{\theta}{k}}}\right)$$

$$\leqslant (f(x)+f(y))\int_{0}^{1}h(t^{\alpha p})t^{\frac{\theta p}{k}-1}dt+m^{2p}\left(f\left(\frac{y}{m^{p}}\right)+f\left(\frac{x}{m^{p}}\right)\right)\int_{0}^{1}h(1-t^{\alpha p})t^{\frac{\theta p}{k}-1}dt.$$

$$\leqslant (f(x)+f(y))\left(\int_{0}^{1}(h(t^{\alpha p}))^{l}dt\right)^{\frac{1}{l}}\left(\int_{0}^{1}(t^{\frac{\theta p}{k}-1})^{q}dt\right)^{\frac{1}{q}}$$

$$+m^{2p}\left(f\left(\frac{y}{m^{p}}\right)+f\left(\frac{x}{m^{p}}\right)\right)\left(\int_{0}^{1}(h(1-t^{\alpha p}))^{l}dt\right)^{\frac{1}{l}}\left(\int_{0}^{1}(t^{\frac{\theta p}{k}-1})^{q}dt\right)^{\frac{1}{q}}.$$

Proof. *Since* $\mathcal{F}$ *is* $(\alpha,h-m)-p$ *convex, we have the following inequalities*

$$\mathcal{F}\left([t^{p}x^{p}+m^{2p}(1-t^{p})\frac{y^{p}}{m^{p}}]^{\frac{1}{p}}\right)\leqslant h(t^{\alpha p})f(x)+m^{2p}h(1-t^{\alpha p})f\left(\frac{y}{m^{p}}\right),$$

$$\mathcal{F}\left([t^p y^p + m^{2p}(1-t^p)\frac{x^p}{m^p}]^{\frac{1}{p}}\right) \leqslant h(t^{\alpha p})f(y) + m^{2p}h(1-t^{\alpha p})f\left(\frac{x}{m^p}\right).$$

Adding both inequalities and multiplying with $t^{\frac{\theta p}{k}-1}$ *and integrating with respect to* t *we get*

$$\int_0^1 t^{\frac{\theta p}{k}-1}\mathcal{F}\left([t^p x^p + m^{2p}(1-t^p)\frac{y^p}{m^p}]^{\frac{1}{p}}\right)dt + \int_0^1 t^{\frac{\theta p}{k}-1}\mathcal{F}\left([t^p y^p + m^{2p}(1-t^p)\frac{x^p}{m^p}]^{\frac{1}{p}}\right)dt$$

$$\leqslant (f(x)+f(y))\int_0^1 h(t^{\alpha p})t^{\frac{\theta p}{k}-1}dt + m^{2p}\left(f\left(\frac{y}{m^p}\right)+f\left(\frac{x}{m^p}\right)\right)\int_0^1 h(1-t^{\alpha p})t^{\frac{\theta p}{k}-1}dt.$$

Which when identified in terms of the $k-p$ *fractional operator, we get*

$$\frac{k\Gamma_k(\theta)}{p^{1-\frac{\theta}{k}}}\left(\frac{{}^{p-1}_{k}J^{\theta}_{x^-}\mathcal{F}(my)}{(x^p - m^p y^p)^{\frac{\theta}{k}}} + \frac{{}^{p-1}_{k}J^{\theta}_{y^-}\mathcal{F}(mx)}{(y^p - m^p x^p)^{\frac{\theta}{k}}}\right)$$

$$\leqslant (f(x)+f(y))\int_0^1 h(t^{\alpha p})t^{\frac{\theta p}{k}-1}dt + m^{2p}\left(f\left(\frac{y}{m^p}\right)+f\left(\frac{x}{m^p}\right)\right)\int_0^1 h(1-t^{\alpha p})t^{\frac{\theta p}{k}-1}dt.$$

Now applying the Hölders inequality on the integrals, we get

$$(f(x)+f(y))\int_0^1 h(t^{\alpha p})t^{\frac{\theta p}{k}-1}dt + m^{2p}\left(f\left(\frac{y}{m^p}\right)+f\left(\frac{x}{m^p}\right)\right)\int_0^1 h(1-t^{\alpha p})t^{\frac{\theta p}{k}-1}dt$$

$$\leqslant (f(x)+f(y))\left(\int_0^1 (h(t^{\alpha p}))^l dt\right)^{\frac{1}{l}}\left(\int_0^1 (t^{\frac{\theta p}{k}-1})^q dt\right)^{\frac{1}{q}}$$

$$+m^{2p}\left(f\left(\frac{y}{m^p}\right)+f\left(\frac{x}{m^p}\right)\right)\left(\int_0^1 (h(1-t^{\alpha p}))^l dt\right)^{\frac{1}{l}}\left(\int_0^1 (t^{\frac{\theta p}{k}-1})^q dt\right)^{\frac{1}{q}}.$$

Connecting the left and right hand side, we obtain the inequality.

Corollary 7. *Setting* $l, q = \frac{1}{2}$ *we get a new* $(\alpha, h-m)-p$ $k-p$ *fractional inequality*

$$\frac{k\Gamma_k(\theta)}{p^{1-\frac{\theta}{k}}}\left(\frac{{}^{p-1}_{k}J^{\theta}_{x^-}\mathcal{F}(my)}{(x^p - m^p y^p)^{\frac{\theta}{k}}} + \frac{{}^{p-1}_{k}J^{\theta}_{y^-}\mathcal{F}(mx)}{(y^p - m^p x^p)^{\frac{\theta}{k}}}\right)$$

$$\leqslant (f(x)+f(y))\int_0^1 h(t^{\alpha p})t^{\frac{\theta p}{k}-1}dt + m^{2p}\left(f\left(\frac{y}{m^p}\right)+f\left(\frac{x}{m^p}\right)\right)\int_0^1 h(1-t^{\alpha p})t^{\frac{\theta p}{k}-1}dt.$$

$$\leqslant (f(x)+f(y))\left(\int_0^1 (h(t^{\alpha p}))^{\frac{1}{2}} dt\right)^{2}\left(\int_0^1 (t^{\frac{\theta p}{k}-1})^{\frac{1}{2}} dt\right)^{2}$$

$$+m^{2p}\left(f\left(\frac{y}{m^p}\right)+f\left(\frac{x}{m^p}\right)\right)\left(\int_0^1 (h(1-t^{\alpha p}))^{\frac{1}{2}} dt\right)^{2}\left(\int_0^1 (t^{\frac{\theta p}{k}-1})^{\frac{1}{2}} dt\right)^{2}.$$

3. Conclusions and Outlook

In this paper new fractional variations of the Hermite-Hadamard inequality have been obtained, as well as an application of the Hölder's inequality in the fractional setting. In the generalizations, $k-p$ fractional operator has been utilized in tandem with $(\alpha, h-m)-p$ convexity to produce the results. Recently reported results in the literature have been given as corollaries. Questions arise whether further generalizations of the obtained convex-fractional inequalities are obtainable. A possible open problem for further investigation is whether $k-p$ Riemann Liouville fractional operator can be paired up with Raina's function to produce more fractional convex inequalities. Another interesting problem is whether interval valued analysis can be used to generalize the obtained inequalities.

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