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Grundy Dominating and Grundy Hop Dominating Sequences in Graphs: Relationships and Some Structural Properties

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Abstract. In this paper, we revisit the concepts of Grundy domination and Grundy hop domination in graphs and give some realization results involving these parameters. We show that the Grundy domination number and Grundy hop domination number of a graph G are generally incomparable (one is not always less than or equal the other). It is shown that their absolute difference can be made arbitrarily large. Moreover, the Grundy domination numbers of some graphs are determined.

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Key Words and Phrases: Grundy dominating sequence, Grundy hop dominating sequence, Grundy domination number, Grundy hop domination number

1. Introduction

The concept of Grundy domination in a graph was introduced and initially studied by Bresar et al. [6]. This concept was also considered in other previous studies (see [3], [4], [5], [7]). In [5], exact formulas for Grundy domination numbers of Sierpinski graphs were generated and a linear algorithm for determining these numbers in arbitrary interval graphs was given. Grundy domination number was studied for Kneser graphs in [7] and graph products in [3] and [14].

Recently, Hassan and Canoy [8] introduced and investigated the concept of Grundy hop domination in a graph. They showed that the parameter is at least equal to the hop domination number of a graph G. The authors also characterized the Grundy hop

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dominating sequences in graphs under some binary operations. Previous studies on hop domination can be found in [1], [2], [9], [10], [11], [12], [13], [15], and [16].

This paper revisits the concepts of Grundy domination and Grundy hop domination in a graph and shows, as a particular case of a realization result, that the absolute difference of the Grundy domination and Grundy hop domination numbers can be made arbitrarily large. Moreover, the Grundy domination numbers of some graphs are given.

2. Terminology and Notation

Let G be a simple undirected graph. A set $D \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. Any dominating set with cardinality equal to $\gamma(G)$ is called a γ-set.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of a graph G and let $\hat{S} = \{v_1, v_2, \cdots, v_k\}$ be its corresponding set. Then S is a *legal closed neighborhood sequence* if $N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \varnothing$ for every $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a dominating set of G, then S is called a *Grundy dominating sequence.* The maximum length of a Grundy dominating sequence in a graph G, denoted by $\gamma_{gr}(G)$, is called the *Grundy domination number* of G. We say that vertex v_i *footprints* the vertices from $N_G[v_i] \setminus \cup_{j=1}^{i} N_G[v_j]$, and that v_i is their *footprinter.* Any Grundy dominating sequence S with $|\hat{S}| = \gamma_{gr}(G)$ is called a maximum Grundy dominating sequence or a γ_{gr}-sequence of G. In this case, we call $\hat{S}$ a γ_{gr}-set of G. A legal closed neighborhood sequence $S = (v_1, v_2, \cdots, v_k)$ of G is a *maximum legal closed neighborhood sequence* if for any legal closed neighborhood sequence $(w_1, w_2, \cdots, w_t)$ of G, we have $t \leq k$. A legal closed neighborhood sequence S is *non-dominating* if $\hat{S}$ is a non-dominating set of G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The *closed hop neighborhood* of X in G is the set $N_G^2[X] = N_G^2(X) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of G and let $\hat{S} = \{v_1, \cdots, v_k\}$ be the set *induced* by S. Then S is a *legal closed hop neighborhood sequence* of G if $N_G^2[v_i] \setminus \cup_{j=1}^{i-1} N_G^2[v_j] \neq \varnothing$ for each $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a hop dominating set of G, then S is called a *Grundy hop dominating sequence.* The maximum length of a Grundy hop dominating sequence in a graph G, denoted by $\gamma_{gr}^h(G)$, is called the *Grundy hop domination number* of G. We say that vertex v_i *hop-footprints* the vertices from $N_G^2[v_i] \setminus \cup_{j=1}^{i} N_G^2[v_j]$, and that v_i is their *hop-footprinter.* Any Grundy hop dominating sequence S with $|\hat{S}| = \gamma_{gr}^h(G)$ is called a maximum Grundy hop dominating sequence or a γ_{gr}^h-sequence of G. In this case, we call $\hat{S}$ a γ_{gr}^h-set of G.

A set $S \subseteq V(G)$ is an *independent* set of G if for any two distinct vertices v and w of S, $d_G(v,w) \neq 1$. The maximum cardinality of an independent set of G, denoted by $\alpha(G)$, is called the *independence number* of G. Any independent set with cardinality $\alpha(G)$ is referred to as a *maximum independent* set or α-set of G.

Let $S_1 = (v_1, \cdots, v_n)$ and $S_2 = (u_1, \cdots, u_m)$, $n, m \geq 1$ be two sequences of distinct vertices of G. The *concatenation* of S_1 and S_2, denoted by $S_1 \oplus S_2$, is the sequence given by

$$S_1 \oplus S_2 = (v_1, \cdots, v_n, u_1, \cdots, u_m).$$

Let G and H be any two graphs. The *join* of G and H, denoted by $G+H$, is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

3. Main Results

Theorem 1. *Let a and b be positive integers such that $3 \leq a \leq b$. Then*

(i) *there exists a connected graph G such that $\gamma_{gr}(G) = a$ and $\gamma_{gr}^h(G) = b$, and*

(ii) *there exists a connected graph G' such that $\gamma_{gr}^h(G') = a$ and $\gamma_{gr}(G') = b$.*

Proof. For $a = b$, consider the graph G given in Figure 1. Let $S = (u_1, u_2, \ldots, u_{a-1}, u_a)$. Then S is both a γ_{gr}-sequence and a γ_{gr}^h-sequence of G. Hence, $\gamma_{gr}(G) = a = \gamma_{gr}^h(G)$.

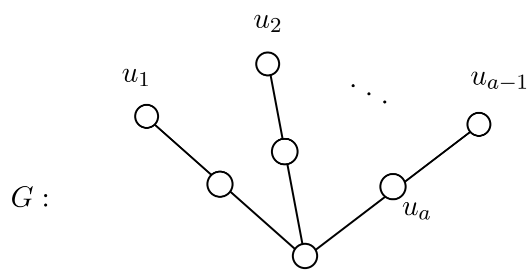

Figure 1: A graph G with $\gamma_{gr}(G) = \gamma_{gr}^h(G)$

Next, suppose $a < b$ and let $m = b - a$. For (i), consider the graph G given in Figure 2, where the graph $G[\{x_a, z_1, z_2, \ldots, z_m\}]$ induced by $\{x_a, z_1, z_2, \ldots, z_m\}$ is complete. It can easily be verified that $S_1 = (x_1, x_2, \cdots, x_a)$ and $S_2 = (u, v, x_1, \ldots, x_{a-2}, z_1, \cdots, z_m)$ are γ_{gr}-sequence and γ_{gr}^h-sequence of G, respectively. Therefore, $\gamma_{gr}(G) = a$ and $\gamma_{gr}^h(G) = m + a = b$.

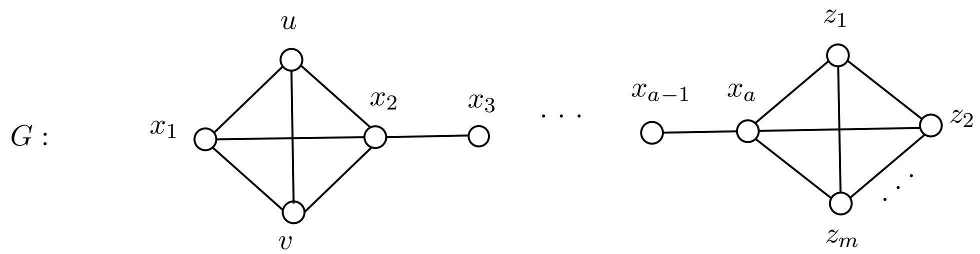

Figure 2: A graph G with $\gamma_{gr}(G) = a < \gamma_{gr}^h(G) = b$

For (ii), consider the graph G' given in Figure 3. Let $S_1 = (s_1, s_2, \cdots, s_a)$ and $S_2 = (s_1, s_2, \ldots, s_a, u_1, u_2, \cdots, u_m)$. Then S_1 and S_2 are γ_{gr}^h-sequence and γ_{gr}-sequence of G', respectively. Thus, $\gamma_{gr}^h(G') = a$ and $\gamma_{gr}(G') = m + a = b$.

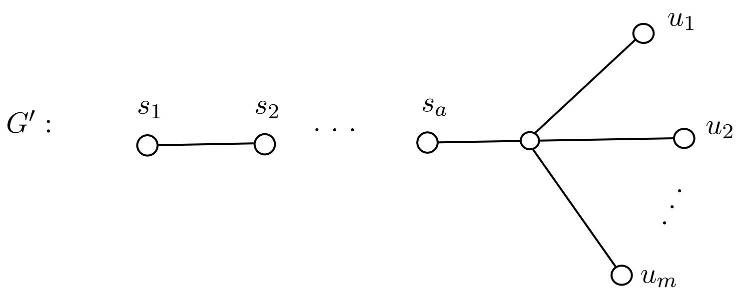

Figure 3: A graph G' with $\gamma_{gr}^h(G') = a < \gamma_{gr}(G') = b$

This proves the assertion. □

Corollary 1. *Let n be a positive integer. Then each of the following statements holds.*

(i) *There exists a connected graph G such that $\gamma_{gr}^h(G) - \gamma_{gr}(G) = n$.*

(ii) *There exists a connected graph H such that $\gamma_{gr}(H) - \gamma_{gr}^h(H) = n$.*

In other words, the absolute difference $|\gamma_{gr}^h(G) - \gamma_{gr}(G)|$ can be made arbitrarily large.

Proof. Let n be a positive integer and let $a = n + 2$ and $b = 2n + 2$. By Theorem 1(i), there exists a connected graph G with $\gamma_{gr}(G) = a$ and $\gamma_{gr}^h(G) = b$. Hence, $\gamma_{gr}^h(G) - \gamma_{gr}(G) = n$. By Theorem 1$(ii)$, there exists a connected graph H such that $\gamma_{gr}^h(H) = a$ and $\gamma_{gr}(H) = b$. Hence, $\gamma_{gr}(H) - \gamma_{gr}^h(H) = n$. □

The following result shows that difference of the Grundy domination number (Grundy hop domination number) and domination number (resp. hop domination number) of a graph G can be made arbitrarily large.

Proposition 1. *Let n be a positive integer. Then each of the following statements holds.*

(i) *There exists a connected graph G such that $\gamma_{gr}(G) - \gamma(G) = n$.*

(ii) *There exists a connected graph G' such that $\gamma^h_{gr}(G') - \gamma_h(G') = n$.*

Proof. (i) Consider the graph G in Figure 4. Let $D = \{u\}$ and $S = (v_1, v_2, \dots, v_{n+1})$. Then D is a γ-set and S is a γ_{gr}-sequence of G. Hence, $\gamma(G) = 1$ and $\gamma_{gr}(G) = n + 1$, that is, $\gamma_{gr}(G) - \gamma(G) = n$.

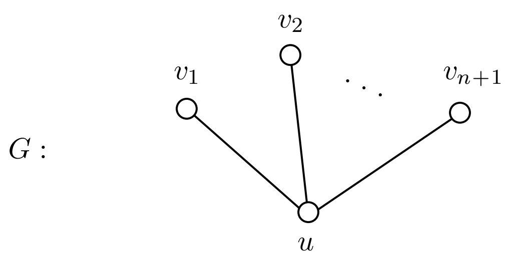

Figure 4: A graph G with $\gamma_{gr}(G) - \gamma(G) = n$

(ii) Consider the graph G' given in Figure 5. Let $S_1 = \{u, w\}$ and $S_2 = (x_1, x_2, \dots, x_{n+1}, w)$. Then S_1 is a γ_h-set and S_2 is a γ^h_{gr}-sequence of G'. Therefore, $\gamma_h(G') = 2$ and $\gamma^h_{gr}(G') = n + 2$, that is, $\gamma^h_{gr}(G') - \gamma_h(G') = n$.

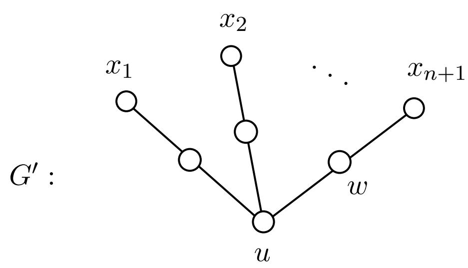

Figure 5: A graph G' with $\gamma^h_{gr}(G') - \gamma_h(G') = n$

This proves the assertion. □

Proposition 2. *If $D = \{v_1, \ldots, v_k\}$ is a γ-set of G, then $(v_1, \ldots, v_k)$ is a Grundy dominating sequence. In particular, $\gamma(G) \leq \gamma_{gr}(G)$.*

Proof. Let $D = \{v_1, \ldots, v_k\}$ be a γ-set of G. Suppose $S = (v_1, v_2, \ldots, v_k)$ is not a legal closed neighborhood sequence of G. Then there exists $i \in \{2, 3, \ldots, k\}$ such that $N_G[v_i] \setminus \cup_{j=1}^{i-1} N_G[v_j] = \varnothing$. This implies that $N_G[v_i] \subseteq \cup_{j=1}^{i-1} N_G[v_j]$. Thus, $\hat{S} \setminus \{v_i\} = D \setminus \{v_i\}$ is a dominating set of G, contradicting the the minimality of D. Therefore, $N_G[v_i] \setminus \cup_{j=1}^{i-1} N_G[v_j] \neq \varnothing$ for each $i \in \{2, 3, \ldots, k\}$, showing that S is a Grundy dominating sequence of G. □

Proposition 3. *Let G be a graph on n vertices. If $S = (u_1, v_2, \cdots, u_k)$ is a Grundy dominating sequence of smallest length k, then $\gamma(G) = |\hat{S}| = k$.*

Proof. Since $\hat{S}$ is a dominating set of G, it follows that $\gamma(G) \leq |\hat{S}|$. On the other hand, by Proposition 2 and the assumption that S a Grundy dominating sequence of smallest length k, $|\hat{S}| \leq \gamma(G)$. Consequently, $\gamma(G) = |\hat{S}| = k$. □

Theorem 2. *Let G be a graph. Then $S = (s_1, s_2, \cdots, s_k)$ is a maximum legal closed neighborhood sequence of G if and only if S is a Grundy dominating sequence of G with $\gamma_{gr}(G) = k$.*

Proof. Let $S = (s_1, s_2, \cdots, s_k)$ be a maximum legal closed neighborhood sequence of G. Suppose $\hat{S}$ is not a dominating set of G. Then there exists $v \in V(G) \setminus N_G[\hat{S}]$. This implies that $v \notin N_G[u]$ for every $u \in \hat{S}$. Let $S^* = (s_1, s_2, \cdots, s_k, v)$. Then $N_G[v] \setminus \cup_{j=1}^{k} N_G[s_i] \neq \varnothing$. It follows that S^* is a legal closed neighborhood sequence of G, a contradiction to the maximality of S. Thus, $\hat{S}$ is a dominating set of G. Therefore, by assumption, S is a Grundy dominating sequence of G and $\gamma_{gr}(G) = k$.

The converse is clear. □

The next result follows from Theorem 2

Corollary 2. *Let G be a graph and let $S = (s_1, s_2, \ldots, s_m)$ be a legal closed neighborhood sequence of G. Then $|\hat{S}| = m \leq \gamma_{gr}(G)$.*

Theorem 3. *Let G be a graph on n vertices. Then $\gamma_{gr}(G) = \alpha(G)$ if and only if every α-set is induced by a maximum legal closed neighborhood sequence of G.*

Proof. Suppose $\gamma_{gr}(G) = \alpha(G)$. Let $D = \{v_1, \ldots, v_k\}$ be a maximum independent set of G and let $S = (v_1, \ldots, v_k)$. Then $\hat{S} = D$ is a dominating set of G. Moreover, since $\hat{S}$ is an independent set, $v_i \in N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j]$ for each $i \in \{2, \ldots, k\}$. Hence, S is a legal closed neighborhood sequence of G. Since $\gamma_{gr}(G) = \alpha(G)$, S is a maximum legal closed neighborhood sequence of G by Theorem 2. Therefore, α-set is induced by a maximum legal closed neighborhood sequence of G.

For the converse, suppose that every α-set is induced by a maximum legal closed neighborhood sequence of G. By Theorem 2, $\gamma_{gr}(G) = \alpha(G)$. □

Corollary 3. *$\gamma_{gr}(K_{1,n}) = \alpha(K_{1,n}) = n$ for every positive integer n.*

Proposition 4. *Let G be a graph with components $G_1, G_2, \ldots, G_k, k \geq 2$. Then*

$$\gamma_{gr}(G) = \sum_{i=1}^{k} \gamma_{gr}(G_i).$$

Proof. For each $i \in \{1, 2, \ldots, k\}$, let S_i be a γ_{gr}-sequence of G_i. Then $S = S_1 \oplus S_2 \oplus \cdots \oplus S_k$ is a Grundy dominating sequence of G. Thus,

$$\gamma_{gr}(G) \geq |\hat{S}| = \sum_{i=1}^{k} |\hat{S}_i| = \sum_{i=1}^{k} \gamma_{gr}(G_i).$$

Next, suppose that $S' = (w_1, w_2, \ldots, w_m)$ is a γ_{gr}-sequence of G. For each $i \in \{1, 2, \ldots, k\}$, let $S'_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,m_i})$ be a subsequence of S' such that $\hat{S}'_i = \hat{S}' \cap V(G_i)$. Since S' is a Grundy dominating sequence of G, S'_i is a Grundy dominating sequence of G_i for each $i \in \{1, 2, \ldots, k\}$. Hence, $S^* = S'_1 \oplus S'_2 \oplus \cdots \oplus S'_k$ is a Grundy dominating sequence of G. Therefore,

$$\gamma_{gr}(G) = |\hat{S}'| = |\hat{S}^*| = \sum_{i=1}^{k} |\hat{S}'_i| \leq \sum_{i=1}^{k} \gamma_{gr}(G_i).$$

Consequently,

$$\gamma_{gr}(G) = \sum_{i=1}^{k} \gamma_{gr}(G_i). \qquad \square$$

Lemma 1. *Let G be a graph on n vertices and k be any positive integer. If $|N_G[a]| \geq k$ for every $a \in V(G)$, then $\gamma_{gr}(G) \leq n - k + 1$.*

Proof. Suppose $S = (s_1, s_2, \ldots, s_t)$ is γ_{gr}-sequence of G. Then $|N_G[s_1]| \geq k$ by assumption. It follows that there are only at most $n - k$ remaining vertices that can be footprinted by the remaining terms of S. Thus, $\gamma_{gr}(G) = t \leq n - k + 1$. $\square$

Theorem 4. *Let G be any non-trivial graph on $n \geq 1$ vertices. Then $1 \leq \gamma_{gr}(G) \leq n$. Moreover, each of the following statements holds.*

(i) *$\gamma_{gr}(G) = 1$ if and only if G is a complete graph.*

(ii) *$\gamma_{gr}(G) = 2$ if and only if G is non-complete and $\{a, b\}$ is a dominating set of G for each pair of distinct vertices $a, b \in V(G)$ with $N_G[a] \neq N_G[b]$.*

(iii) *$\gamma_{gr}(G) = n$ if and only if $G = \overline{K}_n$.*

Proof. Clearly, $1 \leq \gamma_{gr}(G) \leq n$.

(i) Assume that $\gamma_{gr}(G) = 1$. Suppose G is not a complete graph. Then there exists $a, b \in V(G)$ such that $d_G(a, b) = 2$. This implies that $b \in N_G[b] \setminus N_G[a]$. Hence, (a, b) is a legal closed neighborhood sequence of G. Therefore, $\gamma_{gr}(G) \geq 2$, a contradiction.

Conversely, suppose G is a complete graph. Then $\gamma_{gr}(G) = 1$.

(ii) Suppose that $\gamma_{gr}(G) = 2$. Then G is non-complete by (i). Let a, b be two distinct vertices of G such that $N_G[a] \neq N_G[b]$. Then $N_G[a] \setminus N_G[b] \neq \varnothing$ or $N_G[b] \setminus N_G[a] \neq \varnothing$. We may ssume that $N_G[b] \setminus N_G[a] \neq \varnothing$. This implies that (a, b) is a legal closed neighborhood sequence of G. Suppose there exists $c \in V(G) \setminus (N_G[a] \cup N_G[b])$. Since $c \in N_G[c]$, it follows that $N_G[c] \setminus (N_G[a] \cup N_G[b]) \neq \varnothing$. Hence, (a, b, c) is a legal closed neighborhood sequence of G. Thus, $\gamma_{gr}(G) \geq 3$, a contradiction to the assumption that $\gamma_{gr}(G) = 2$. Therefore, $\{a, b\}$ is a dominating set of G.

Conversely, suppose that G is non-complete and $\{a, b\}$ is a dominating set of G for each pair of distinct vertices $a, b \in V(G)$ with $N_G[a] \neq N_G[b]$. Then $\gamma_{gr}(G) \geq 2$. Let $(s_1, s_2, \cdots, s_t)$ be a γ_{gr}-sequence of G. Then $N_G[s_2] \setminus N_G[s_1] \neq \varnothing$, that is, $N_G[s_2] \neq N_G[s_1]$. By assumption, $\{s_1, s_2\}$ is a dominating set of G, that is, $V(G) = N_G^2[s_1] \cup N_G^2[s_2]$. Therefore, $\gamma_{gr}(G) = t = 2$.

(iii) Suppose $\gamma_{gr}(G) = n$. Suppose there exists component C of G which is non-trivial. Then $|N_C[v]| \geq 2$ for every $v \in V(C)$. By Lemma 1, $\gamma_{gr}(C) \leq |V(C)| - 1$. By Proposition 4, it follows that $\gamma_{gr}(G) \leq n - 1$, a contradiction. Therefore, every component C of G is trivial, i.e., $G = \overline{K}_n$.

Conversely, suppose that every component C of G is trivial. Then by (i) and Proposition 4, $\gamma_{gr}(G) = n$. □

The next result follows from Theorem 4 and Proposition 4.

Corollary 4. *Let G be a graph on n vertices. Then each of the following statements holds.*

(i) $\gamma_{gr}(G) \geq 2$ *if and only if G is non-complete graph.*

(ii) *If G has k components and every component is complete, then $\gamma_{gr}(G) = k$.*

(iii) *If G is complete, then $\gamma_{gr}(G) + \gamma_{gr}(\overline{G}) = n + 1$.*

(iv) *If G is non-complete, then*

(a) $3 \leq \gamma_{gr}(G) + \gamma_{gr}(\overline{G}) \leq 2n - 1$, *and*

(b) $2 \leq \gamma_{gr}(G) \cdot \gamma_{gr}(\overline{G}) \leq n^2 - n$.

In particular, equality in (a) and (b) holds if and only if $G = \overline{K}_2$.

Proposition 5. *For any positive integer n, each of the following holds.*

(i) $\gamma_{gr}(C_n) = \begin{cases} 1 \text{ if } n = 3 \\ n - 2 \text{ if } n \geq 4. \end{cases}$

(ii) $\gamma_{gr}(P_n) = \begin{cases} 1 \ if \, n = 1, 2 \\ n - 1 \ if \, n \geq 3. \end{cases}$

(iii) $\gamma_{gr}(\overline{P}_n) = \begin{cases} 2, & n = 2, 3 \\ 3, & n \geq 4. \end{cases}$

(iv) $\gamma_{gr}(\overline{C}_n) = \begin{cases} 2, & if n = 4 \\ 3, & if \ n = 3 \ or \ n \geq 5. \end{cases}$

Proof. (i) Clearly, $\gamma_{gr}(C_n) = 1$ for $n = 3$. Suppose $n \geq 4$. Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ and $S = (v_1, v_2, \cdots, v_{n-2})$. Clearly, $\hat{S}$ is a dominating set of C_n. Observe that $v_{i+1} \in N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j]$ for each $i \in \{2, \ldots, n-2\}$. Thus, S is a Grundy dominating sequence of C_n and $\gamma_{gr}(C_n) \geq n - 2$. On the other hand, let $S_0 = (a_1, a_2, \ldots, a_k)$ be a Grundy dominating sequence of C_n. Since $|N_G[v_i]| = 3$ for each $i \in \{1, \ldots, n\}$, it follows that $\gamma_{gr}(C_n) = k \leq n - 2$ by Lemma 1. Consequently, $\gamma_{gr}(G) = n - 2$ for all $n \geq 4$.

(ii) Clearly, $\gamma_{gr}(P_n) = 1$ for $n = 1, 2$. Suppose $n \geq 3$. Let $P_n = [v_1, v_2, \ldots, v_n]$ and let $D = (v_1, v_2, \cdots, v_{n-1})$. Clearly, $\hat{D}$ is a dominating set of P_n. Observe that $v_{i+1} \in N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j]$ for each $i \in \{2, \ldots, n-2\}$. Thus, D is a Grundy dominating sequence of P_n showing that $\gamma_{gr}(P_n) \geq n - 1$. Since $|N_{P_n}[a]| \geq 2$ for every $a \in V(P_n)$, $\gamma_{gr}(P_n) \leq n - 1$ by Lemma 1. Therefore, $\gamma_{gr}(P_n) = n - 1$ for all $n \geq 3$.

(iii) Clearly, $\gamma_{gr}(\overline{P}_n) = 2$ for $n = 2, 3$. Suppose $n \geq 4$. Let $\{v_1, v_2, \cdots, v_n\}$ be a vertex set of $\overline{P}_n$ and consider $S = (v_2, v_4, v_3)$. Then S is a Grundy dominating sequence of $\overline{P}_n$. Hence, $\gamma_{gr}(\overline{P}_n) \geq 3$. On the other hand, let $S' = (w_1, \cdots, w_k)$ be a Grundy dominating sequence of $\overline{P}_n$. Notice that $|N_{\overline{P}_n}[v_i]| \geq n - 2$ for every $i \in \{1, \ldots, n\}$. Thus, $\gamma_{gr}(\overline{P}_n) = k \leq 3$ by Lemma 1. Consequently, $\gamma_{gr}(\overline{P}_n) = 3$ for all $n \geq 4$.

(iv) Clearly, $\gamma_{gr}(\overline{C}_3) = 3$ and $\gamma_{gr}(\overline{C_4}) = 2$. Suppose $n \geq 5$. Let $\{v_1, v_2, \cdots, v_n\}$ be a vertex set of $\overline{C}_n$ and consider $S' = \{v_1, v_3, v_2\}$. Then S' is a Grundy dominating sequence of $\overline{C}_n$. Hence, $\gamma_{gr}(\overline{C}_n) \geq 3$. On the other hand, let $S'' = (u_1, \cdots, u_l)$ be a Grundy dominating sequence of $\overline{C}_n$. Observe that $|N_{\overline{C}_n}[v_i]| = n - 2$ for every $i \in \{1, \cdots, n\}$. Thus, $\gamma_{gr}(\overline{C}_n) = l \leq 3$ by Lemma 1. Therefore, $\gamma_{gr}(\overline{C}_n) = 3$ for all $n \geq 5$. □

Theorem 5. *Let G and H be two non-complete graphs. A sequence D of distinct vertices of $G + H$ is a Grundy dominating sequence in $G + H$ if and only if one of the following condition holds:*

(i) *D is a Grundy dominating sequence of G.*

(ii) *D is a Grundy dominating sequence of H.*

(iii) *$D = D_G \oplus (w)$ for some non-dominating legal closed neighborhood sequence D_G of G and $w \in V(H)$.*

(*iv*) $D = D_H \oplus (v)$ *for some non-dominating legal closed neighborhood sequence* D_H *of* H *and* $v \in V(G)$.

Proof. Let D_G and D_H be subsequences D such that $\hat{D_G} = \hat{D} \cap V(G)$ and $\hat{D_H} = \hat{D} \cap V(H)$. If $\hat{D_H} = \varnothing$, then $D = D_G$ is a Grundy dominating sequence of G. If $\hat{D}_G = \varnothing$, then $D = D_H$ is a Grundy dominating sequence of H. Hence, (i) or (ii) holds. Suppose now that $\hat{D_G}$ and $\hat{D_H}$ are both non-empty. Since D is a legal closed neighborhood sequence of $G + H$, D_G and D_H are legal closed neighborhood sequences of G and H, respectively. If $|\hat{D_G}| = |\hat{D_H}| = 1$, then both ($iii$) and ($iv$) hold. Next, suppose that $|\hat{D_G}| \geq 2$. Suppose further that $\hat{D_G}$ is dominating. Let $D = (x_1, x_2, \ldots, x_m)$. Since $\hat{D_G}$ is dominating, none of the terms in D_H comes after (succeeds) all the terms of D_G in D by the legality property of D. Hence, if $x_j \in \hat{D_H}$ and $x_k \in \hat{D_G}$, then $k > j$. However, if $x_j \in \hat{D_H}$ for some j, then $V(G) \subseteq N_{G+H}[x_j]$. This would imply that $|\hat{D_G}| = 1$, a contradiction. Thus, $\hat{D_H} = \varnothing$, a contradiction. Thus, D_G is a non-dominating legal closed neighborhood sequence of G. Let $D_G = (v_1, v_2, \ldots, v_r)$ and $D_H = (w_1, w_2, \ldots, w_t)$. If in D the term w_1 does not precede v_r, where $r \geq 2$, then v_r does not satisfy the property in the legality condition, a contradiction. Hence, in D, w_1 comes after the term v_r in D_G. Since $V(H) \subseteq \bigcup_{j=1}^{r} N_{G+H}[v_j]$ and $V(G) \setminus (\bigcup_{j=1}^{r} N_G[v_j]) \subseteq N_{G+H}[w_1]$, D_H cannot have other terms, i.e., $t = 1$. Thus, $D_H = (w)$, where $w = w_1$, and $D = D_G \oplus (w)$. This shows that (iii) holds. Similarly, (iv) holds if $|\hat{D_H}| \geq 2$.

For the converse, suppose first that (i) or (ii) holds. Then clearly, D is a Grundy dominating sequence of $G + H$. Next, suppose that (iii) holds. Then $\hat{D}$ is a dominating set of $G + H$. Let $D_G = (v_1, v_2, \ldots, v_r)$. Since D_G is a legal closed neighborhood sequence of G, it is legal closed neighborhood sequence of $G + H$. Moreover, since $\hat{D_G}$ is non-dominating in G, $N_{G+H}[w] \setminus (\bigcup_{j=1}^{r} N_{G+H}[v_j]) \neq \varnothing$. Thus, $D = D_G \oplus (w)$ is a legal closed neighborhood sequence of $G + H$. Similarly, D is a Grundy dominating sequence of $G + H$ if (iv) holds. □

The next result follows from Proposition 5 and Theorem 5.

Corollary 5. *Let G and H be two non-complete graphs. Then*

$$\gamma_{gr}(G + H) = max\{\gamma_{gr}(G), \gamma_{gr}(H)\}$$

In particular, each of the following holds:

(*i*) $\gamma_{gr}(P_n + P_m) = \begin{cases} n - 1 \ \textit{if}\ n \geq m \geq 3 \\ m - 1 \ \textit{if}\ m \geq n \geq 3. \end{cases}$

(*ii*) $\gamma_{gr}(P_n + C_m) = \begin{cases} n - 1 \ \textit{if}\ n \geq m = 4 \\ m - 2 \ \textit{if}\ m \geq n + 1 = 4. \end{cases}$

(*iii*) $\gamma_{gr}(C_n + C_m) = \begin{cases} n - 2 \ \textit{if}\ n \geq m \geq 4 \\ m - 2 \ \textit{if}\ m \geq n \geq 4. \end{cases}$

(iv) $\gamma_{gr}(K_{m,n}) = max\ \{m, n\}$ *for all* $m, n \geq 2$.

Then next result can be proved easily.

Theorem 6. *Let G be a complete graph and let H be a non-complete graph. A sequence D of distinct vertices of $G + H$ is a Grundy dominating sequence in $G + H$ if and only if one of the following condition holds:*

(i) $D = (v)$ *for some* $v \in V(G)$.

(ii) D *is a Grundy dominating sequence of* H.

(iii) $D = D_H \oplus (v)$ *for some non-dominating legal closed neighborhood sequence* D_H *of* H *and* $v \in V(G)$.

Since $\gamma_{gr}(H) \geq 2$ for any non-complete graph H, the following result follows from Theorem 6.

Corollary 6. *Let G be a complete graph and let H be any non-complete graph. Then*

$$\gamma_{gr}(G + H) = \gamma_{gr}(H).$$

In particular, each of the following holds:

(i) $\gamma_{gr}(K_{1,n}) = n$ *for all* $n \geq 1$.

(ii) $\gamma_{gr}(W_n) = \gamma_{gr}(K_1 + C_n) = \gamma_{gr}(C_n) = n - 2$ *for all* $n \geq 4$.

(iii) $\gamma_{gr}(F_n) = \gamma_{gr}(K_1 + P_n) = \gamma_{gr}(P_n) = n - 1$ *for all* $n \geq 3$.

4. Conclusion

The study revisited the concepts of Grundy domination and Grundy hop domination in graphs which have been considered previously by various authors. In general, the Grundy domination and Grundy hop domination numbers do not satisfy a consistent relationship as any one of them can be larger than the other. In fact, it was shown that the absolute difference of these two parameters can be made arbitrarily large. The Grundy domination numbers of some graphs were determined. For the join of two graphs, the Grundy domination number was obtained by first characterizing all the Grundy dominating sequences in the graph. These two parameters can still be studied for other graphs; in particular, for graphs under some binary operations not yet considered in previous studies. Moreover, it still remains a conjecture that the Grundy hop dominating set problem is NP-complete.

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