EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 430-439
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Some Forms of Open Multifunctions in Ideal Topological Spaces

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Abstract. By using m-open multifunctions from an m-space into an m-space, we establish the unified theory for several weak forms of open multifunctions between topological spaces.

2020 Mathematics Subject Classifications: 54A05, 54C10, 54C60

Key Words and Phrases: m-structure, m-space, m-open multifunction, ideal

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1. Introduction

The notion of ideal topological spaces is introduced in [15] and [27]. In [14], the authors introduced the notion of I-open sets in an ideal topological space. As generalizations of open sets and I-open sets, semi-I-open sets, pre-I-open sets, α-I-open sets, β-I-open sets and b-I-open sets are introduced and used to obtain decompositions of continuity.

Recently, in [24] and [25] the present authors introduced the notions of minimal structures and m-spaces as a generalization of topological spaces. The notion of m-open multifunctions is introduced in [21]. The notion of m-I-open functions is introduced in [22]. In this paper, the authors introduce a minimal structure $mIO(X)$ determined by operations Int, Cl, Cl* in an ideal topological space (X, τ, I). By using $mIO(X)$, the authors introduce and study the notion of mI-open multifunctions. As special case of mI-open multifunctions, we obtain semi-I-open functions [12], pre-I-open functions [2], α-I-open functions [2], b-I-open functions [3], weakly semi-I-open functions [9] and weakly $b-I$-open functions [19].

In Section 3, we introduce the notion of an m-open multifunction from an m-space into an m-space. We obtain the characterizations of m-open multifunctions and characterize the set of all points at which a multifunction is not m-open. In the last part, a new modification of m-open multifunctions, called mI-open multifunctions, is introduced and investigated.

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DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4691

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2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 1. Let (X,τ) be a topological space. A subset A of X is said to be *α-open* [20] (resp. *semi-open* [16], *preopen* [18], *β-open* [1], *b-open* [4]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, b-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$).

Throughout the present paper, (X,τ) and (Y,σ) always denote topological spaces and $F : X \to Y$ presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2. A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be *open* [5] (resp. *semi-open* [23], *preopen* [7], *α-open* [6], *β-open* [21]) if $F(U)$ is open (resp. semi-open, preopen, α-open, β-open) for each open set U of X.

Definition 3. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (or briefly *m-structure*) [24], [25] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Definition 4. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* and the *m_X-interior* of A are defined in [17] as follows:
(1) $m_X\mathrm{Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) $m_X\mathrm{Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Lemma 1. (Maki et al. [17]) *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*
(1) $m_X\mathrm{Cl}(X - A) = X - m_X\mathrm{Int}(A)$ and $m_X\mathrm{Int}(X - A) = X - m_X\mathrm{Cl}(A)$,
(2) If $(X - A) \in m_X$, then $m_X\mathrm{Cl}(A) = A$ and if $A \in m_X$, then $m_X\mathrm{Int}(A) = A$,
(3) $m_X\mathrm{Cl}(\emptyset) = \emptyset, m_X\mathrm{Cl}(X) = X$, $m_X\mathrm{Int}(\emptyset) = \emptyset$ and $m_X\mathrm{Int}(X) = X$,
(4) If $A \subset B$, then $m_X\mathrm{Cl}(A) \subset m_X\mathrm{Cl}(B)$ and $m_X\mathrm{Int}(A) \subset m_X\mathrm{Int}(B)$,
(5) $A \subset m_X\mathrm{Cl}(A)$ and $m_X\mathrm{Int}(A) \subset A$,
(6) $m_X\mathrm{Cl}(m_X\mathrm{Cl}(A)) = m_X\mathrm{Cl}(A)$ and $m_X\mathrm{Int}(m_X\mathrm{Int}(A)) = m_X\mathrm{Int}(A)$.

Definition 5. An m-structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [17] if the union of any family of subsets belonging to m_X belongs to m_X.

Remark 1. Let (X, τ) be a topological space and $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$), then m_X is an m-structure having property $\mathcal{B}$.

Lemma 2. (Popa and Noiri [26]) *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. Then for a subset A of X, the following properties hold:*
(1) $A \in m_X$ if and only if $m_X\mathrm{Int}(A) = A$,
(2) A is m-closed if and only if $m_X\mathrm{Cl}(A) = A$,
(3) $m_X\mathrm{Int}(A) \in m_X$ and $m_X\mathrm{Cl}(A)$ is m_X-closed.

3. m-open multifunctions

Definition 6. Let (X, m_X) and (Y, m_Y) be two m-spaces. A multifunction $F : (X, m_X) \to (Y, m_Y)$ is said to be *m-open* at $x \in X$ if for each m_X-open set U containing x, there exists $V \in m_Y$ containing $F(x)$ such that $V \subset F(U)$. If F is m-open at each point $x \in X$, then F is said to be *m-open*.

Theorem 1. *A multifunction $F : (X, m_X) \to (Y, m_Y)$ is m-open at $x \in X$, where m_Y has property $\mathcal{B}$, if and only if for each m_X-open set U containing x, $x \in F^+(m_Y\mathrm{Int}(F(U)))$.*

Proof. *Necessity.* Let U be any m_X-open set containing x. Then, there exists $V \in m_Y$ such that $F(x) \subset V \subset F(U)$ and hence $F(x) \subset m_Y\mathrm{Int}(F(U))$. Therefore, we obtain that $x \in F^+(m_Y\mathrm{Int}(F(U)))$.

Sufficiency. Suppose that $x \in F^+(m_Y\mathrm{Int}(F(U)))$ for each m_X-open set U containing x. Then $F(x) \subset m_Y\mathrm{Int}(F(U))$. Set $V = m_Y\mathrm{Int}(F(U))$, then by Lemma 2 $V \in m_Y$ and $F(x) \subset V \subset F(U)$. Therefore, F is m-open at x.

Theorem 2. *A multifunction $F : (X, m_X) \to (Y, m_Y)$ is m-open, where (Y, m_Y) has property $\mathcal{B}$, if and only if $F(U)$ is m_Y-open for each m_X-open set U of X.*

Proof. *Necessity.* Let U be any m_X-open set of X and $x \in U$. Since F is m-open at $x \in X$, by Theorem 1 we have $F(x) \subset m_Y\mathrm{Int}(F(U))$ and $F(U) = m_Y\mathrm{Int}(F(U))$. By Lemma 2, $F(U)$ is m_Y-open.

Sufficiency. Let x be an arbitrary point of X and U any m_X-open set of X containing x. Then, we have $F(x) \subset F(U) = m_Y\mathrm{Int}(F(U))$. Therefore, $x \in F^+(m_Y\mathrm{Int}(F(U))$. By Theorem 1, F is m-open at an arbitrary point $x \in X$.

Remark 2. For a multifunction $F : (X, m_X) \to (Y, m_Y)$, let $m_X = \tau$ and $m_Y = \sigma$ (resp. $\mathrm{SO}(Y)$, $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$), then we obtain Definition 2, that is, the definition of an open (resp. semi-open, preopen, α-open, β-open) multifunction.

Theorem 3. *For a multifunction $F : (X, m_X) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*
(1) F is m-open at x;
(2) If $x \in m_X\mathrm{Int}(A)$ for any $A \in \mathcal{P}(X)$, then $x \in F^+(m_Y\mathrm{Int}(F(A)))$;
(3) If $x \in m_X\mathrm{Int}(F^+(B))$ for any $B \in \mathcal{P}(Y)$, then $x \in F^+(m_Y\mathrm{Int}(B))$;
(4) If $x \in F^-(m_Y\mathrm{Cl}(B))$ for any $B \in \mathcal{P}(Y)$, then $x \in m_X\mathrm{Cl}(F^-(B))$.

Proof. (1) $\Rightarrow$ (2): Let $A \in \mathcal{P}(X)$ and $x \in m_X\text{Int}(A)$. Then, there exists an m_X-open set U such that $x \in U \subset A$ and hence $F(x) \subset F(U) \subset F(A)$. Since F is m-open at x, by Theorem 1 and Lemma 1, we obtain $x \in F^+(m_Y\text{Int}(F(U))) \subset F^+(m_Y\text{Int}(F(A)))$.

(2) $\Rightarrow$ (3): Let $B \in \mathcal{P}(Y)$ and $x \in m_X\text{Int}(F^+(B))$. Then, $x \in F^+(m_Y\text{Int}(F(F^+(B)))) \subset F^+(m_Y\text{Int}(B))$.

(3) $\Rightarrow$ (4): Let $B \in \mathcal{P}(Y)$ and $x \notin m_X\text{Cl}(F^-(B))$. Then $x \in X - m_X\text{Cl}(F^-(B)) = m_X\text{Int}(X - F^-(B)) = m_X\text{Int}(F^+(Y - B))$. By (3) we have $x \in F^+(m_Y\text{Int}(Y - B)) = X - F^-(m_Y\text{Cl}(B))$. Hence, $x \notin F^-(m_Y\text{Cl}(B))$. Therefore, if $x \in F^-(m_Y\text{Cl}(B))$, then $x \in m_X\text{Cl}(F^-(B))$.

(4) $\Rightarrow$ (1): Let U be any m_X-open set of X containing x and $B = Y - F(U)$. Since $m_X\text{Cl}(F^-(B)) = m_X\text{Cl}(F^-(Y - F(U))) = m_X\text{Cl}(X - F^+(F(U))) \subset X - m_X\text{Int}(U) = X - U$ and $x \in U$, we obtain that $x \notin m_X\text{Cl}(F^-(B))$. By (4), we have $x \notin F^-(m_Y\text{Cl}(B)) = F^-(m_Y\text{Cl}(Y - F(U))) = X - F^+(m_Y\text{Int}(F(U)))$. Therefore, $x \in F^+(m_Y\text{Int}(F(U)))$. By Theorem 1, F is m-open at x.

Theorem 4. *For a multifunction $F : (X, m_X) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) F is m-open;

(2) $F(m_X\text{Int}(A)) \subset m_Y\text{Int}(F(A))$ for any subset A of X;

(3) $m_X\text{Int}(F^+(B)) \subset F^+(m_Y\text{Int}(B))$ for any subset B of Y;

(4) $F^-(m_Y\text{Cl}(B)) \subset m_X\text{Cl}(F^-(B))$ for any subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X and $x \in m_X\text{Int}(A)$. Since F is m-open at each $x \in A$, by Theorem 3 $F(x) \subset m_Y\text{Int}(F(A))$. Hence $F(m_X\text{Int}(A)) \subset m_Y\text{Int}(F(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. By (2), we have $F(m_X\text{Int}(F^+(B))) \subset m_Y\text{Int}(F(F^+(B))) \subset m_Y\text{Int}(B)$. Hence, we have $m_X\text{Int}(F^+(B)) \subset F^+(m_Y\text{Int}(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. By (3), we have $X - m_X\text{Cl}(F^-(B)) = m_X\text{Int}(X - F^-(B)) = m_X\text{Int}(F^+(Y - B)) \subset F^+(m_Y\text{Int}(Y - B)) = X - F^-(m_Y\text{Cl}(B))$. Hence, $F^-(m_Y\text{Cl}(B)) \subset m_X\text{Cl}(F^-(B))$.

(4) $\Rightarrow$ (1): Let U be any m_X-open set of X and $B = Y - F(U)$. By (4), we have $F^-(m_Y\text{Cl}(Y - F(U))) \subset m_X\text{Cl}(F^-(Y - F(U)))$. Now, $F^-(m_Y\text{Cl}(Y - F(U))) = F^-(Y - m_Y\text{Int}(F(U))) = X - F^+(m_Y\text{Int}(F(U)))$. And also we have $m_X\text{Cl}(F^-(Y - F(U))) = m_X\text{Cl}(X - F^+(F(U))) \subset X - m_X\text{Int}(U) = X - U$. Therefore, we obtain $U \subset F^+(m_Y\text{Int}(F(U)))$ and hence $F(U) \subset m_Y\text{Int}(F(U))$. Consequently, we obtain $F(U) = m_Y\text{Int}(F(U))$ and $F(U)$ is m_Y-open. Therefore, by Theorem 2 F is m-open.

For a multifunction $F : (X, m_X) \to (Y, m_Y)$, we denote

$$D^0(F) = \{x \in X\colon \textit{ F is not m-open at x }\}.$$

Theorem 5. *For a multifunction $F : (X, m_X) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties hold:*

$D^0(F) = \cup_{U \in m_X}\{U - F^+(m_Y\text{Int}(F(U)))\}$

$= \cup_{A \in P(X)}\{m_X\text{Int}(A) - F^+(m_Y\text{Int}(F(A)))\}$

$= \cup_{B\in P(Y)}\{m_X\mathrm{Int}(F^+(B)) - F^+(m_Y\mathrm{Int}(B))\}$
$= \cup_{B\in P(Y)}\{F^-(m_Y\mathrm{Cl}(B)) - m_X\mathrm{Cl}(F^-(B))\}$.

Proof. Let $x \in D^0(F)$. Then, by Theorem 1, there exists an m_X-open set U_0 containing x such that $x \notin F^+(m_Y\mathrm{Int}(F(U_0)))$. Hence, $x \in U_0 \cap (X - F^+(m_Y\mathrm{Int}(F(U_0)))) = U_0 - F^+(m_Y\mathrm{Int}(F(U_0))) \subset \cup_{U\in m_X}\{U - F^+(m_Y\mathrm{Int}(F(U)))\}$.

Conversely, let $x \in \cup_{U\in m_X}\{U - F^+(m_Y\mathrm{Int}(F(U)))\}$. Then, there exists $U_0 \in m_X$ such that $x \in U_0 - F^+(m_Y\mathrm{Int}(F(U_0)))$. Therefore, by Theorem 1 $x \in D^0(F)$.

For the second equation, let $x \in D^0(F)$. Then, by Theorem 3, there exists $A_1 \in \mathcal{P}(X)$ such that $x \in m_X\mathrm{Int}(A_1)$ and $x \notin F^+(m_Y\mathrm{Int}(F(A_1)))$. Therefore, $x \in m_X\mathrm{Int}(A_1) - F^+(m_Y\mathrm{Int}(F(A_1))) \subset \cup_{A\in P(X)}\{m_X\mathrm{Int}(A) - F^+(m_Y\mathrm{Int}(F(A)))\}$.

Conversely, $x \in \cup_{A\in P(X)}\{m_X\mathrm{Int}(A) - F^+(m_Y\mathrm{Int}(F(A)))\}$. Then, there exists $A_1 \in \mathcal{P}(X)$ such that $x \in m_X\mathrm{Int}(A_1) - F^+(m_Y\mathrm{Int}(F(A_1)))$. By Theorem 3, $x \in D^0(F)$.

The other equations are similarly proved.

4. Ideal topological spaces

Let (X,τ) be a topological space. The notion of ideals has been introduced in [15] and [27] and further investigated in [13]

Definition 7. A nonempty collection I of subsets of a set X is called an *ideal on* X if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X,τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X,τ,I). Let (X,τ,I) be an ideal topological space. For any subset A of X, $A^\star(I,\tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [13]. Hereafter, $A^\star(I,\tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Lemma 3. (Janković and Hamlett [13]) *Let (X,τ,I) be an ideal topological space and A, B be two subsets of X. Then, the following properties hold:*
(1) $A \subset B$ *implies* $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(2) $\mathrm{Cl}^\star(X) = X$ *and* $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

A subset A is said to be *I-open* [14] if $A \subset \mathrm{Int}(A^\star)$. As generalizations of open sets and I-open sets, the following subsets are introduced and investigated.

Definition 8. Let (X,τ,I) be an ideal topological space. A subset A of X is said to be
(1) *α-I-open* [11] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$,
(2) *semi-I-open* [12] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(3) *pre-I-open* [8] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$,

(4) *b-I-open* [3] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(5) *β-I-open* [11] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(6) *weakly semi-I-open* [9] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(7) *weakly b-I-open* [19] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A))) \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(8) *strongly β-I-open* [10] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$.

Between the sets in Definition 8, we have the following relations:

DIAGRAM 1

$$\begin{array}{ccccccc}
\text{open} & \Rightarrow & \alpha\text{-}I\text{-open} & \Rightarrow & \text{semi-}I\text{-open} & \Rightarrow & \text{weakly semi-}I\text{-open} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
I\text{-open} & \Rightarrow & \text{pre-}I\text{-open} & \Rightarrow & b\text{-}I\text{-open} & \Rightarrow & \text{weakly } b\text{-}I\text{-open} \\
 & & & & \Downarrow & & \Uparrow \\
 & & & & \text{strongly } \beta\text{-}I\text{-open} & \Rightarrow & \beta\text{-}I\text{-open}
\end{array}$$

The family of all α-I-open (resp. semi-I-open, pre-I-open, b-I-open, β-I-open, weakly semi-I-open, weakly b-I-open, strongly β-I-open) sets in an ideal topological space (X, τ, I) is denoted by $\alpha\mathrm{IO}(X)$ (resp. $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$).

Definition 9. By $\mathrm{mIO}(X)$, we denote each one of the families $\tau^\star$, $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$.

Lemma 4. *Let (X, τ, I) be an ideal topological space. Then, $\mathrm{mIO}(X)$ is an m-structure on X and has property $\mathcal{B}$.*

Proof. We shall show that $\mathrm{SIO}(X)$ is an m-structure with property $\mathcal{B}$.

(1) It is obvious that by Lemma 3 $\mathrm{Cl}^\star(\mathrm{Int}(\emptyset)) = \mathrm{Cl}^\star(\emptyset) = \emptyset$ and $\mathrm{Cl}^\star(\mathrm{Int}(X)) = \mathrm{Cl}^\star(X) = X$. Hence, $\mathrm{SIO}(X)$ is an m-structure.

(2) Let $\{A_\alpha : \alpha \in \Delta\}$ be any family of semi-I-open sets. Then, for each $\alpha \in \Delta$, by Lemma 3 we have $A_\alpha \subset \mathrm{Cl}^\star(\mathrm{Int}(A_\alpha)) \subset \mathrm{Cl}^\star(\mathrm{Int}(\cup\{A_\alpha : \alpha \in \Delta\}))$. Therefore, $\cup\{A_\alpha : \alpha \in \Delta\} \subset \mathrm{Cl}^\star(\mathrm{Int}(\cup\{A_\alpha : \alpha \in \Delta\}))$ and hence $\cup\{A_\alpha : \alpha \in \Delta\} \in \mathrm{SIO}(X)$. Hence $\mathrm{SIO}(X)$ has property $\mathcal{B}$.
For other families, the proofs are similar.

Definition 10. Let (X, τ, I) be an ideal topological space. For a subset A of X, $m\mathrm{Cl}_I(A)$ and $m\mathrm{Int}_I(A)$ are defined as follows:
(1) $m\mathrm{Cl}_I(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{mIO}(X)\}$,
(2) $m\mathrm{Int}_I(A) = \cup\{U : U \subset A, U \in \mathrm{mIO}(X)\}$.

Let (X, τ, I) be an ideal topological space and $\mathrm{mIO}(X)$ the m-structure on X. If $\mathrm{mIO}(X) = \tau^\star$ (resp. $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$), then we have the following:

(1) $m\mathrm{Cl}_I(A) = \mathrm{Cl}^\star(A)$ (resp. $\alpha\mathrm{Cl}_I(A)$, $s\mathrm{Cl}_I(A)$, $p\mathrm{Cl}_I(A)$, $b\mathrm{Cl}_I(A)$, $\beta\mathrm{Cl}_I(A)$, $ws\mathrm{Cl}_I(A)$, $wb\mathrm{Cl}_I(A)$, $s\beta\mathrm{Cl}_I(A)$).

(2) $\mathrm{mInt}_I(A) = \mathrm{Int}^\star(A)$ (resp. $\alpha\mathrm{Int}_I(A)$, $s\mathrm{Int}_I(A)$, $p\mathrm{Int}_I(A)$, $b\mathrm{Int}_I(A)$, $\beta\mathrm{Int}_I(A)$, $ws\mathrm{Int}_I(A)$, $wb\mathrm{Int}_I(A)$, $s\beta\mathrm{Int}_I(A)$).

5. mI-open multifunctions

Definition 11. Let (X, m_X) be an m-space and (Y, τ, I) be an ideal topological space. A multifunction $F : (X, m_X) \to (Y, \tau, I)$ is said to be *mI-open* at $x \in X$ if for each m_X-open set U containing x, there exists $V \in mIO(Y)$ containing $F(x)$ such that $V \subset F(U)$. If F is mI-open at each point $x \in X$, then F is said to be *mI-open.*

Then $F : (X, m_X) \to (Y, \tau, I)$ is mI-open at $x \in X$ (resp. on X) if and only if $F : (X, m_X) \to (Y, mIO(X))$ is m-open at $x \in X$ (resp. on X). Therefore, by the results of Section 3, we obtain the following properties of mI-open multifunctions.

Theorem 6. *A multifunction $F : (X, m_X) \to (Y, \tau, I)$ is mI-open at $x \in X$ if and only if for each m_X-open set U containing x, $x \in F^+(m\mathrm{Int}_I(F(U)))$.*

Proof. The proof follows from Theorem 1 and Lemma 4.

Theorem 7. *A multifunction $F : (X, m_X) \to (Y, \tau, I)$ is mI-open if and only if $F(U)$ is mI-open for each m_X-open set U of X.*

Proof. The proof follows from Theorem 2 and Lemma 4.

Theorem 8. *For a multifunction $F : (X, m_X) \to (Y, \tau, I)$, the following properties are equivalent:*
(1) F is mI-open at x;
(2) If $x \in m_X\mathrm{Int}(A)$ for $A \in \mathcal{P}(X)$, then $x \in F^+(m\mathrm{Int}_I(F(A)))$;
(3) $x \in m_X\mathrm{Int}(F^+(B))$ for $B \in \mathcal{P}(Y)$, then $x \in F^+(m\mathrm{Int}_I(B))$;
(4) If $x \in F^-(m\mathrm{Cl}_I(B))$ for $B \in \mathcal{P}(Y)$, then $x \in m_X\mathrm{Cl}(F^-(B))$.

Proof. The proof follows from Theorem 3 and Lemma 4.

Theorem 9. *For a multifunction $F : (X, m_X) \to (Y, \tau, I)$, the following properties are equivalent:*
(1) F is mI-open;
(2) $F(m_X\mathrm{Int}(A)) \subset m\mathrm{Int}_I(F(A))$ for any subset A of X;
(3) $m_X\mathrm{Int}(F^+(B)) \subset F^+(m\mathrm{Int}_I(B))$ for any subset B of Y;
(4) $F^-(m\mathrm{Cl}_I(B)) \subset m_X\mathrm{Cl}(F^-(B))$ for any subset B of Y.

Proof. The proof follows from Theorem 4 and Lemma 4.

For a multifunction $F : (X, m_X) \to (Y, \tau, I)$, we denote

$$D_I^0(F) = \{x \in X\colon\ F \text{ is not mI-open at } x\}.$$

Theorem 10. *For a multifunction $F : (X, m_X) \to (Y, \tau, I)$, the following properties hold:*

$$D_I^0(F) = \cup_{U \in m_X}\{U - F^-(m\mathrm{Int}_I(F(U)))\}$$
$$= \cup_{A \in P(X)}\{m_X\mathrm{Int}(A) - F^+(m\mathrm{Int}_I(F(A)))\}$$
$$= \cup_{B \in P(Y)}\{m_X\mathrm{Int}(F^+(B)) - F^+(m\mathrm{Int}_I(B))\}$$
$$= \cup_{B \in P(Y)}\{F^-(m\mathrm{Cl}_I(B)) - m_X\mathrm{Cl}(F^-(B))\}.$$

Proof. The proof follows from Theorem 5 and Lemma 4.

Remark 3. 1) Let $F : (X, \tau) \to (Y, \sigma, J)$ be a multifunction, where (X, τ) is a topological space. Since $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$) is an m-structure having property $\mathcal{B}$, an mJ-open multifunction $F : (X, m_X) \to (Y, \sigma, J)$ is defined and it is equivalent to an m-open multifunction $F : (X, m_X) \to (Y, \mathrm{mJO}(Y))$. For example, let $m_X = \mathrm{SO}(X)$ and $\mathrm{mJO}(Y) = \mathrm{SJO}(Y)$, then an m-open multifunction $F : (X, \mathrm{SO}(X)) \to (Y, \mathrm{SJO}(Y))$ is defined and we obtain the properties from the results of Sections 3 and 5.

2) An mIJ-open multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is defined by (i) an mJ-open multifunction $F : (X, \mathrm{mIO}(X)) \to (Y, \sigma, J)$ or (ii) an m-open multifunction $F : (X, \mathrm{mIO}(X)) \to (Y, \mathrm{mJO}(Y))$.

For example, let $\mathrm{mIO}(X) = \mathrm{SIO}(X)$ and $\mathrm{mJO}(Y) = \mathrm{SJO}(Y)$, then an m-open multifunction $F : (X, \mathrm{SIO}(X)) \to (Y, \mathrm{SJO}(Y))$ is defined and we obtain the properties from the results of Sections 3 and 5.

Corollary 1. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:*

(1) $F : (X, \tau, I) \to (Y, \sigma, J)$ is mIJ-open;
(2) $F : (X, \mathrm{SIO}(X)) \to (Y, \sigma, J)$ is mJ-open;
(3) $F : (X, \mathrm{SIO}(X)) \to (Y, \mathrm{SJO}(Y))$ is m-open;
(4) $F(s\mathrm{Int}_I(A)) \subset s\mathrm{Int}_J(F(A))$ for any subset A of X;
(5) $s\mathrm{Int}_I(F^+(B)) \subset F^+(s\mathrm{Int}_J(B))$ for any subset B of Y;
(6) $F^-(s\mathrm{Cl}_J(B)) \subset s\mathrm{Cl}_I(F^-(B))$ for any subset B of Y.

Proof. The proof easily follows from Theorem 9.

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