EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 864-892
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

The Homotopy Analysis Method to Solve the Nonlinear System of Volterra Integral Equations and Applying the Genetic Algorithm to Enhance the Solutions

Rasha F. Ahmed[1], Waleed Al-Hayani[1,*], Abbas Y. Al-Bayati[2]

[1] *Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq*
[2] *Department of Mathematics, College of Basic Education, University of Telafer, Mosul, Iraq*

Abstract. This paper presents the application of the Homotopy Analysis Method (HAM) for solving nonlinear system of Volterra integral equations used to obtain a reasonably approximate solution. The HAM contains the auxiliary parameter h which provides a simple way to adjust and control the convergence region of the solution series. The results show that the HAM is a very effective method as well. The results were compared with the solutions obtained by developing a homotopy analysis method using the genetic algorithm (HAM-GA), considering the residual error function as a fitness function of the genetic algorithm.

2020 Mathematics Subject Classifications: 45D05, 45G15, 68W50

Key Words and Phrases: System of Volterra Integral Equations (SVIEs), Homotopy Analysis Method (HAM), h-curves, Genetic Algorithm (GA).

---

1. Introduction

The integral equations can be used to describe a wide variety of problems in science and engineering, such as the population dynamics, the spread of diseases, some Dirichlet problems in potential theory, electrostatics, mathematical modeling of radioactive equilibrium, particle transport problems of astrophysics and reactor theory, radiative energy and/or heat transfer problems, other general heat transfer problems, oscillation of strings and membranes, the problem of momentum representation in quantum mechanics, etc. However, various additional challenging problems in mechanics, astrophysics, chemistry, and mathematics can also be represented in terms of Volterra integral equations. In

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4693

Email addresses: rasha.20csp146@student.uomosul.edu.iq (Rasha F. Ahmed), waleedalhayani@uomosul.edu.iq (Waleed Al-Hayani), profabbasalbayati@yahoo.com (Abbas Y. Al-Bayati)

addition, there are several real-world problems where urges naturally develop. A differential equation, an integral equation, an integro-differential equation, or a system of these equations together can be used to describe problems that either arise naturally (as in population dynamics or many biological applications) or are brought about by a control system [9–11]. A numerical method is frequently required because systems of integral and/or integro-differential equations, particularly nonlinear systems of Volterra integral equations or with variable coefficients, are typically challenging to solve analytically. Therefore, there are many research papers to solve such systems using several analytical methods (see [4, 5, 7, 15, 29]), etc. Many researchers [3, 36, 37] have tried to solve the Volterra integral equation using different analytical and numerical methods. Recently, Younis and Al-Hayani [50] have been used Adomian decomposition method (ADM) for solving fuzzy system of Volterra integro-differential equations. The researchers [39, 40, 46–49] have been applied different analytical methods such as optimal homotopy asymptotic method (OHAM), homotopy perturbation method (HPM) and Padé approximation technique to solve the mathematical models as (porous rotating disk electrodes, ECE reactions at rotating disk electrodes, magnetohydrodynamic, etc.). So, in this paper, we consider the nonlinear system of Volterra Integral Equations (NLSVIEs) [2]:

$$\boldsymbol{U}(x) = \boldsymbol{F}(x) + \int_{f(x)}^{g(x)} \boldsymbol{K}(x,t)\, R\left[\boldsymbol{U}(t)\right] dt, \tag{1}$$

where

$$\begin{aligned} \boldsymbol{U}(x) &= \left[u_1(x), u_2(x), \cdots, u_n(x)\right]^T, \\ \boldsymbol{F}(x) &= \left[f_1(x), f_2(x), \cdots, f_n(x)\right]^T, \\ \boldsymbol{K}(x,t) &= \left[k_{ij}(x,t)\right], \quad i,j = 1,2,\cdots,n \end{aligned} \tag{2}$$

Consider the $i-th$ equation of Eq 1

$$u_i(x) = f_i(x) + \int_{f(x)}^{g(x)} \sum_{j=1}^{n} \left[K_{i,j}(x,t)\, R\left(u_i(t)\right)\right] dt, \qquad i = 1,2,\cdots,n \tag{3}$$

where the unknown functions $u_i(x) \in C[a,b], x \in [a,b]$, the functions $f_i(x) \in C[a,b]$, $f, g \in C[a,b],\ a \le f(x) \le g(x) \le b$. The non-negative kernel functions $K_{i,j}(x,t) \in C[a,b] \times C[a,b],\quad i = 1,2,\cdots,n$ and $R : C[a,b] \to C[a,b]$, are nonlinear operator. Therefore, R, the operator, is assumed to satisfy the Lipschitz condition, $\| R(v_1) - R(v_2) \| \le s \| v_1 - v_2 \|$ for every $v_1, v_2 \in C[a,b]$ and some $s > 0$. As the norm of the function, we take the supremum norm $\| v \| = \sup_{x\in\Omega} |v(x)|$, in particular $\| K \| = \sup_{(x,t)\in[a,b]\times[a,b]} |K(x,t)|$ and $\| F \| = \sup_{x\in[a,b]} |F(x)|$

Shijun Liao invented the HAM [24–27, 41] by which many types of equations can be resolved [24]. The method by Shijun Liao is mainly derived from topology, and can approach to the exact solution. Some examples of using this method for solving integral equations well discussed [17–19]. However, the researcher was able to prove the convergence of HAM [31]. This method has been successfully applied to solve many types of

linear and nonlinear problems [1, 13, 14, 16, 30, 32, 33, 38, 45], In 2020, researchers were able to solve Two-Dimensional nonlinear Volterra-Fredholm fuzzy integral equations by HAM [15]. Also, others were able to solve Mixed Volterra-Fredholm integro-differential equations using same method [16].
In the last years, the researchers [35, 43] have been applied the HAM to solve the mathematical models as (porous rotating disk electrodes and ECE reactions at rotating disk electrodes). Finally, Liao [28] published an explanation of how to avoid "small denominator problems" by the HAM.
The main objective of the present study is to improve the results obtained from HAM application to the system of Volterra integral equations using the genetic algorithm by considering the residual error function as a fitness function and finding the best values for the constants. In particular, this study demonstrates that for reasonable assumptions, the analyzed equations have both existent and unique solutions. Also, a comparison of the solutions with that obtained by developing a HAM using the genetic algorithm was performed considering the residual error function as a fitness function of the genetic algorithm, as shown in section 4 and listed in several examples.

2. Fundamental of the HAM

The HAM is applied for solving the operator equations

$$\mathcal{N}\left[u\left(x\right)\right]=0, \qquad x\in\Omega \tag{4}$$

where $\mathcal{N}$ refers to general operator (liner or non-linear), while $u\left(x\right)$ is an unknown function. First, we define the homotopy operator $\mathbb{H}$ as follows:

$$\mathbb{H}\left(\phi,q\right)=\left(1-q\right)\boldsymbol{\mathcal{L}}\left[\phi\left(x;q\right)-u_0\left(x\right)\right]-qh\boldsymbol{\mathcal{N}}\left[\phi\left(x;q\right)\right], \tag{5}$$

where $q\in[0,1]$is an embedding parameter, h is a non-zero auxiliary function (denotes the convergence control parameter [23, 26, 27, 44]), $\mathcal{L}$ is an auxiliary linear operator which has the property $\boldsymbol{\mathcal{L}}\left(0\right)=0, u_0\left(x\right)$ is an initial guess of $u\left(x\right)$ and $\phi\left(x;q\right)$ is an unknown function. The zero-order deformation equation of the operator $\mathbb{H}\left(\phi,q\right)=0$ is considered as follow:

$$\left(1-q\right)\boldsymbol{\mathcal{L}}\left[\phi\left(x;q\right)-u_0\left(x\right)\right]=qh\boldsymbol{\mathcal{N}}\left[\phi\left(x;q\right)\right], \tag{6}$$

Obviously, when q=0 we have $\boldsymbol{\mathcal{L}}\left[\phi\left(x;q\right)-u_0\left(x\right)\right]=0$, that implies $\phi\left(x;0\right)=u_0\left(x\right)$. While, when $q=1$, we have $\boldsymbol{\mathcal{N}}\left[\phi\left(x;q\right)\right]=0$, which leads to $\phi\left(x;1\right)=u\left(x\right)$, where $u\left(x\right)$ is the solution of Equation 4. in this case, the variation of parameter q from zero to one coincides with the variation of problem from the initial guess $u_0\left(x\right)$ to the non-trivial solution $u\left(x\right)$. Using the Maclaurin series for expanding the function $\phi\left(x;q\right)$ with respect the parameter q, we get:

$$\phi\left(x;q\right)=\phi\left(x;q\right)+\sum_{m=1}^{\infty}\frac{1}{m!}\frac{\partial^m\phi\left(x;q\right)}{\partial q^m}\bigg|_{q=0}, \tag{7}$$

By considering

$$u_m = \frac{1}{m!}\left.\frac{\partial^m \phi\left(x;q\right)}{\partial q^m}\right|_{q=0}, \qquad m = 1,2,3,\cdots \tag{8}$$

Equation 7 can be formulated as follow:

$$\phi\left(x;q\right) = u_0\left(x\right) + \sum_{m=1}^{\infty} u_m\left(x\right) q^m, \tag{9}$$

If the series in Equation 9 converge at $q = 1$ then we obtain the solution

$$u\left(x\right) = \sum_{m=0}^{\infty} u_m\left(x\right). \tag{10}$$

we differentiate both sides of Equation 10 m times with respect to the embedding parameter q in order to determine functions u_m. Then, we divide the received finding by $m!$and then setting $q = 0$. In this way, we deduce the so-called m_{th} order deformation equation $(m > 0)$:

$$\boldsymbol{\mathcal{L}}\left[u_m\left(x\right) - \mathcal{X}_m u_{m-1}\left(x\right)\right] = h\boldsymbol{R}_m\left(u_{m-1}\right) \tag{11}$$

where

$$\vec{u}_{m-1} = \left\{u_0\left(x\right), u_1\left(x\right), \cdots, u_{m-1}\left(x\right)\right\}.$$

and

$$\mathcal{X}_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}, \tag{12}$$

and

$$R_m\left(\vec{u}_{m-1}\left(x\right)\right) = \frac{1}{(m-1)!}\left.\left(\frac{\partial^{m-1}}{\partial q^{m-1}}\boldsymbol{\mathcal{N}}\left(\sum_{i=0}^{\infty} u_i\left(x\right) q^i\right)\right)\right|_{q=0}, \tag{13}$$

In case of inability to find the sum of the series in Equation 10, the partial sum was determined of this series.

$$\hat{u}_n\left(x\right) = \sum_{m=0}^{n} u_m\left(x\right), \tag{14}$$

is the approximate solution of the considered equation.
Choosing convenient amount of the parameter h has a great influence on the region of convergence of the series in Equation 10 and the convergence rate as well [27, 34, 42]. One of the methods for selecting the value of convergence control parameter is Genetic Algorithm (GA).

3. Description of the Method

Consider the operators $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{N}}$ are defined as follows

$$\boldsymbol{\mathcal{L}}(u_i) = u_i, \quad \boldsymbol{\mathcal{N}}(u_i) = u_i(x) - f_i(x) - \int_{f(x)}^{g(x)} \left[\sum_{j=1}^{n} K_{ij}(x,t)\, R(u_i(t))\right] dt, \quad i = 1, 2, \cdots, n, \tag{15}$$

Let $u_{i,0} \in C[a,b]$, $i = 1, 2, \cdots, n$. In this case, by considering the HAM, we obtain the following formula for function $u_{i,m}$, $i = 1, 2, \cdots, n$, $\quad m \geq 1$:

$$u_{i,m} = \chi_m u_{i,m-1}(x) + h\boldsymbol{R}_m(\vec{u}_{i,m-1}(x)) \tag{16}$$

where χ_m and $\boldsymbol{R}_m$ are defined as in Equations 12 and 13 respectively.
Using the definitions of the respective operators, we obtain

$$\boldsymbol{R}_{i,m}(\vec{u}_{i,m-1}, x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \boldsymbol{\mathcal{N}}\left(\sum_{k=0}^{\infty} u_{i,k}(x) q^k\right)\Bigg|_{q=0}, \tag{17}$$

$$= \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\sum_{k=1}^{\infty} u_{i,k}(x)q^k - f_i(x) - \int_{f(x)}^{g(x)} \left(\sum_{j=1}^{n} K_{ij}(x,t)\, R\left(\sum_{k=1}^{\infty} u_{j,k}(t) q^k\right)\right)\right]_{q=0},$$

$$= \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\sum_{k=1}^{\infty} u_{i,k}(x)q^k - f_i(x) - \sum_{k=1}^{\infty}\left[\int_{f(x)}^{g(x)} \sum_{j=1}^{n} K_{ij}(x,t)\, R(u_{i,k}(t))\, q^k dt\right]\right]_{q=0},$$

$$= \frac{1}{(m-1)!}\Bigg((m-1)! u_{i,m-1}(x) - (1-\chi_m) f_i(x) - \int_{f(x)}^{g(x)}$$
$$\sum_{j=1}^{n}\Big[K_{ij}(x,t)(m-1)! R(u_{i,m-1}(t))\Big]\Bigg),$$

$$= u_{i,m-1}(x) - \frac{1-\chi_m}{(m-1)!} f_i(x) - \int_{f(x)}^{g(x)} \left[\sum_{j=1}^{n} K_{ij}(x,t)\, R(u_{i,m-1}(t))\right] dt.$$

Utilizing the above relation and Equation 3 we get the following formula:

$$u_{1,i} = h\left(u_{0,i} - f_i(x) - \int_{f(x)}^{g(x)} \left[\sum_{j=1}^{n} K_{i,j}(x,t)\, R(u_{0,i}(t))\right] dt\right).$$

And form $m \geq 2$:

$$u_{i,m}(x) = (1+h)\, u_{i,m-1}(x) - \frac{h}{(m-1)!} \int_{f(x)}^{g(x)} \left[\sum_{j=1}^{n} K_{i,j}(x,t) \left(\frac{\partial^{m-1}}{\partial q^{m-1}} R \left(\sum_{i=0}^{\infty} u_{i,m-1}(t)\, q^{i} \right) \right) \right] dt. \quad (18)$$

In literature, one can find the expression $\frac{\partial^{m-1}}{\partial q^{m-1}} R \left(\sum_{i=0}^{\infty} u_{i,m-1}(t)\, q^{i} \right)$, calculated for various nonlinear operators R. Most of these results were collected in monograph [27]. Now, we turn to prove under proper assumptions Equation 6 which has a unique solution.

Theorem 1. *Consider the system of VIEs as in Equation 3, such that* $\| K_{ij}(x,t) \| = M_i$, *and* $\| \boldsymbol{M} \| = max\, |M_i|$, $i,j = 1,2,\cdots,n$. *If the following condition is satisfied*

$$sn \| \boldsymbol{M} \| (b-a) < 1. \quad (19)$$

Then Equation 3 possess at most one solution.

Proof. Suppose that there exit two solutions $u_{i,1}$ and $u_{i,2}$, we have

$$\| u_{i,1} - u_{i,2} \| = \left\| \int_{f(x)}^{g(x)} \left[\sum_{j=1}^{n} K_{ij}(x,t) (R\Big(u_{j,1}(t)\Big) - R\Big(u_{j,2}(t)\Big)) \right] dt \right\|$$

$$\leq \| \boldsymbol{M} \| \int_{f(x)}^{g(x)} \sum_{j=1}^{n} \| R(u_{i,1}) - R(u_{i,2}) \| \; dt,$$

$$\leq \| \boldsymbol{M} \| \int_{f(x)}^{g(x)} \sum_{j=1}^{n} \| R(u_{i,1}) - R(u_{i,2}) \| \; dt \leq sn \| \boldsymbol{M} \| (b-a) < 1 \| u_{i,1} - u_{i,2} \| .$$

Hence, we obtain

$$\Big(1 - sn \| \boldsymbol{M} \| (b-a)\Big) \leq 1. \quad (20)$$

So, if the condition 19 is satisfied, then the quality $u_{i,1} = u_{i,2}$, $i = 1,2,\cdots,n$ must be true. we now proceed to prove the theorem to ensure that the sum of the given series is the solution to the equation discussed.

Theorem 2. *Let the functions*$u_{i,m}$, $i = 1,2,\cdots,nm \geq 1$, *be defined in Equations 17 and 18. then, if* $(s<1)$ *and the series*

$$u_i(x) = \sum_{m=0}^{\infty} u_{i,m}(x), \quad i = 1,2,\cdots,n, \quad (21)$$

is convergent, then the sum of this series is the solution of Equation 3.

Proof. Let the series in Equation 10 is convergent. Then, from the necessary condition for the convergent series, we conclude that for any $x \in [a, b]$:

$$\lim_{m\to\infty} u_{i,m}(x) = 0, \quad i = 1, 2, \cdots, n.$$

Now, let $T_{i,m}$ has the form:

$$T_{i,m} = \frac{1}{m!}\left(\frac{\partial^m}{\partial q^m} R\left(\sum_{k=0}^{\infty} u_{i,k}(x) q^k\right)\right)\Bigg|_{q=0}, i = 1, 2, \cdots, n.$$

If R is the contraction mapping and $(s < 1)$, the series in Equation 21 converges to $u_i(x)$, then the series $\sum_{m=0}^{\infty} T_{i,m}$ converges to $R(u_i(x))$ (see [12]). Utilizing the definition of the operator $\boldsymbol{\mathcal{L}}$ we can write

$$\sum_{m=1}^{n} \boldsymbol{\mathcal{L}}\left(u_{i,m}(x) - \chi_m u_{i,m-1}(x)\right) = \sum_{m=1}^{n}\left(u_{i,m}(x) - \chi_m u_{i,m-1}(x)\right),$$

$= u_{i,1}(x) + (u_{i,2}(x) - u_{i,1}(x)) + (u_{i,3}(x) - u_{i,2}(x)) + \cdots + (u_{i,n}(x) - u_{i,n-1}(x))$
$= u_{i,n}(x)$.

Hence

$$\sum_{m=1}^{\infty} \boldsymbol{\mathcal{L}}\left(u_{i,m}(x) - \chi_m u_{i,m-1}(x)\right) = \lim_{n\to\infty} u_{i,n}(x) = 0$$

From Equation 11 we get

$$h\sum_{m=1}^{\infty} R_{i,m}\left(\vec{u}_{i,m-1}, x\right) = \sum_{m=1}^{\infty} \boldsymbol{\mathcal{L}}\left(u_{i,m}(x) - \chi_m u_{i,m-1}(x)\right), \;\; i = 1, 2, \cdots, n.$$

and since $h \neq 0$ thus we have

$$\sum_{m=1}^{\infty} R_{i,m}\left(\vec{u}_{i,m-1}, x\right) = 0, \quad i = 1, 2, \cdots, n.$$

As a result of some transformations, we successively get

$$0 = \sum_{m=1}^{\infty} R_{i,m}\left(\vec{u}_{i,m-1}, x\right),$$

$$= \sum_{m=1}^{\infty}\left(\frac{1}{(m-1)!} \quad \frac{\partial^{m-1}}{\partial q^{m-1}} \quad \sum_{k=1}^{\infty} u_{i,k}(x) q^k - f_i(x) - \int_{f(x)}^{g(x)}\left[\sum_{j=1}^{n} K_{i,j}(x,t)\, R\left(\sum_{k=1}^{\infty} u_{i,k}(t) q^k\right)\right] dt\Bigg|_{q=0}\right)$$

$$= \sum_{m=1}^{\infty}\left(u_{i,m-1}\quad(x)\quad-\frac{1-\chi_m}{(m-1)!}f_i\left(x\right)-\int_{f(x)}^{g(x)}\left[\sum_{j=1}^{n}K_{ij}(x,t)\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}R\left(\sum_{k=1}^{\infty}u_{i,k}(t)q^k\right)\right]\right]\bigg|_{q=0}dt\right)$$

$$= \sum_{m=1}^{\infty}\left(u_{i,m-1}\left(x\right)-\frac{1-\chi_m}{(m-1)!}f_i\left(x\right)-\int_{f(x)}^{g(x)}\left[\sum_{j=1}^{n}K_{ij}(x,t)T_{i,m-1}(t)\right]\bigg|_{q=0}dt\right),$$

$$= \sum_{m=1}^{\infty}u_{i,m-1}\left(x\right)-f_i\left(x\right)-\int_{f(x)}^{g(x)}\left[\sum_{j=1}^{n}K_{ij}(x,t)\sum_{m=1}^{\infty}T_{i,m-1}(t)\right]dt,$$

$$= u_i\left(x\right)-f_i\left(x\right)-\int_{f(x)}^{g(x)}\left[\sum_{j=1}^{n}K_{ij}\left(x,t\right)R\left(u_i\left(t\right)\right)\right]dt.$$

4. Genetic Algorithm (GA)

The genetic algorithm is a random search technique used to optimize difficult problems and solve nonlinear systems of equations. Instead of employing deterministic principles, GA makes the use of probabilistic transition rules and manages a population of alternative solutions known as individuals or iteratively evolving chromosomes [22]. The iterations of algorithm are known as "generations". A fitness function and genetic operators like reproduction, crossover, and mutation are used to mimic how solutions evolve [21]. The initial population of a genetic algorithm, as shown in Figure 1, is often random. A chromosome, which is a binary string or a real-valued number, is typically used to represent this population (mating pool). The objective function, which provides each person a corresponding number called its fitness, measures and evaluates the individual performance. The objective function, which provides each person a corresponding number called its fitness, measures and evaluates the individual performance. Each chromosome's fitness is evaluated, as well as a survival of the fittest approach is used. The fitness of each chromosome is evaluated in this work using the residual error value. A genetic algorithm primarily performs three operations: reproduction, crossover, and mutation. Figure 1 describes the GA operation sequences in detail.

4.1. Basic Steps of Genetic Algorithm [20]

Step 1: Use a population of random solutions to initialize the parameter, including crossover rate, mutation rate, number of clusters, and generations. Discover the coding mode.

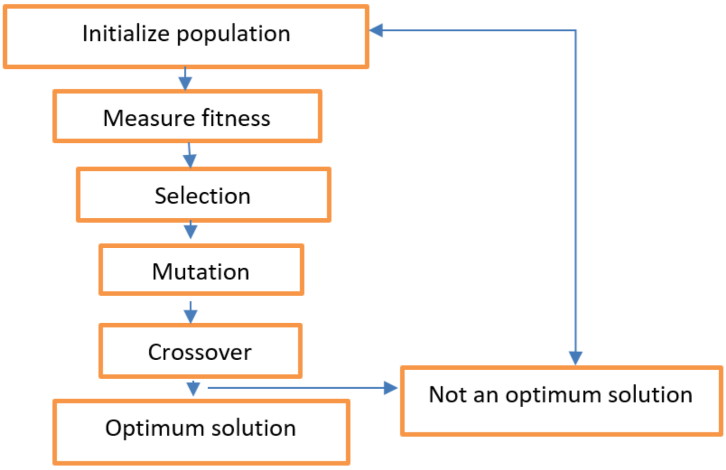

Figure 1: Flowchart of genetic algorithm.

Step 2: Calculate and assess the fitness function's value.
Step 3: Continue with the crossover and mutation process to create the new cluster.
Step 4: Repetition of Step 2 is necessary to get the best result.
Here, we use the genetic algorithm to improve the results of using the HAM in the following ways:

Algorithm 1 Genetic Algorithm for the best parameters $\boldsymbol{h}$ and λ_1, λ_2 in system of volterra integral equations

Input:
Set number of variables (var) ,
Set upper and lower limit for each variable (ub, lb),
Set size of population (a),
Set rate of crossover (rc),
Set rate of mutation (rm),
Set number of iterations ($Max_iteration$),
Set fitness function name
Output:
solution $\boldsymbol{\lambda_1}$, $\boldsymbol{\lambda_2}$ and $\boldsymbol{h}$
Initialization
1: Generate individual feasible solutions randomly with limit boundary
2: Save them in the population p;
3: Find fitness value for each population F
Loop until the terminal condition
4: **for** $i =$ 1 to $Max\_iteration$ **do** , Elitism based selection using Rank selection
5: $fitness$ = Sort individual descending to find ***minimum*** fitness value.
6: Select the best rc solutions in pop$_1$ and save them in pop$_t$ according to $fitness$;
7: **end for**

Crossover

number of crossover $nc = (\alpha - ne)/2$
for $j = 1$ to nc **do**
  randomly select two solutions X_A and X_B from $\mathrm{Pop_i}$;
  generate X_C and X_D by **Arithmetic crossover** to X_A and X_B;
  save X_C and X_D to $\mathrm{Pop_t}$;
end for

Mutation

for $j = 1$ to ne **do**
  select a solution X_j from $\mathrm{Pop_t}$;
  mutate **Random Resetting** of X_J under the rate rm and generate a new solution X_j';
  if thenX_j' is unfeasible
    update X_j' with a feasible solution by repairing X_j';
  end if
  update X_j with X_j' in $\mathrm{Pop_t}$;
end for

Updating

update $\mathrm{pop}_i + 1 = \mathrm{Pop}_i + \mathrm{pop}_t$;

Returning the best solution

return the best solution X in Pop

5. Applications and numerical results

In this section, the HAM applied to obtain an approximate-exact solution for NLSVIEs is displayed in the following three problems. To show the high accuracy of the solution results compared with the exact solution, the maximum absolute errors (MAE) are defined as:

$$\| \cdot \|_\infty = \| \, y_{Exact}\,(x_i) - \phi_n\,(x_i) \, \|_\infty$$

Moreover, giving the maximum residual error (MRE), the computations associated with the problems were performed using the Maple 18 package with a precision of 20 digits.

Problem 1. *Let us consider the following NLSVIEs [6]*

$$\begin{cases} u_1\,(x) = f_1\,(x) + \lambda_1 \int_0^x \left(u_1^2\,(t) + u_2^3(t)\right) dt, \\ u_2\,(x) = f_2\,(x) + \lambda_2 \int_0^x \left(u_1^3\,(t) - u_2^2(t)\right) dt, \end{cases} \tag{22}$$

where

$$\begin{cases} f_1\,(x) = \left(-\frac{1}{10}x^{10} - \frac{1}{5}x^5\right) \ \lambda_1 \ + x^2 \\ f_2\,(x) = x^3, \end{cases} \tag{23}$$

with the exact solutions

$$u_{Exact1}(x) = x^2,\ u_{Exact2}(x) = x^3. \tag{24}$$

To solve 22 and 23 by means of the standard HAM, we choose the initial approximation

$$u_{1,0}(x) = f_1(x),\ u_{2,0}(x) = f_2(x), \tag{25}$$

and the linear operator

$$L[\phi_1(x,q)] = \phi_1(x,q),\ L[\phi_2(x,q)] = \phi_2(x,q). \tag{26}$$

Furthermore, the system 22 suggests that we define the non-linear operator as

$$\begin{cases} N_1[\phi_1(x,q), \phi_2(x,q)] = \phi_1(x,q) - f_1(x) - \lambda_1 \int_0^x \left(\phi_1^2(t,q) + \phi_2^3(t,q)\right) dt, \\ N_2[\phi_1(x,q), \phi_2(x,q)] = \phi_2(x,q) - f_2(x) - \lambda_2 \int_0^x \left(\phi_1^3(t,q) - \phi_2^2(t,q)\right) dt, \end{cases} \tag{27}$$

Using the above definition, we construct the zeroth-order deformation equation as in 6 and 7 and the mth-order deformation equation for $m \geq 1$ which is

$$\begin{cases} L[u_{1,m}(x) - \chi_m u_{1,m-1}(x)] = h[R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \\ L[u_{2,m}(x) - \chi_m u_{2,m-1}(x)] = h[R_{1,m}(\vec{u}_{1,m-1}),\ R_{2,m}(\vec{u}_{2,m-1})], \end{cases} \tag{28}$$

where

$$\begin{cases} R_{1,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}) = u_{1,m-1}(x) - f_1(x) - \lambda_1 \int_0^x \left(u_{1,m-1}^2(t) + u_{2,m-1}^3(t)\right) dt, \\ R_{2,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}) = u_{2,m-1}(x) - f_2(x) - \lambda_2 \int_0^x \left(u_{1,m-1}^3(t) - u_{2,m-1}^2(t)\right) dt, \end{cases} \tag{29}$$

Now, for $m \geq 1$, the solutions of the mth-order deformation Equation 29 are

$$\begin{cases} u_{1,m}(x) = \chi_m u_{1,m-1}(x) + h[R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \\ u_{2,m}(x) = \chi_m u_{2,m-1}(x) + h[R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \end{cases} \tag{30}$$

Now, we successively obtain

$$\begin{aligned} u_{1,1}(x) &= -\frac{1}{2100}h\lambda_1^3 x^{21} - \frac{1}{400}h\lambda_1^3 x^{16} + \frac{1}{65}h\lambda_1^2 x^{13} \\ &\quad -\frac{1}{275}h\lambda_1^3 x^{11} + \frac{1}{20}h\lambda_1^2 x^8 - \frac{1}{5}h\lambda_1 x^5 - \frac{1}{10}h\lambda_1 x^{10} \end{aligned}$$

$$\begin{aligned} u_{2,1}(x) &= \frac{1}{31000}h\lambda_1^3\lambda_2 x^{31} + \frac{3}{13000}h\lambda_1^3\lambda_2 x^{26} - \frac{3}{2300}h\lambda_1^2\lambda_2 x^{23} \\ &\quad +\frac{1}{1750}h\lambda_1^3\lambda_2 x^{21} - \frac{1}{150}h\lambda_1^2\lambda_2 x^{18} + \frac{1}{50}h\lambda_1\lambda_2 x^{15} \\ &\quad +\frac{1}{2000}h\lambda_1^3\lambda_2 x^{16} - \frac{3}{325}h\lambda_1^2\lambda_2 x^{13} + \frac{3}{50}h\lambda_1\lambda_2 x^{10} \end{aligned}$$

$$\vdots$$

Thus, the approximate solution in a series form is given by

$$u_1(x) = u_{1,0}(x) + \sum_{i=1}^{6} u_{1,i}(x),\ u_2(x) = u_{2,0}(x) + \sum_{i=1}^{6} u_{2,i}(x),$$

This series has the closed form as $m \to \infty$

$$u_1(x) = x^2,\ u_2(x) = x^3.$$

which are the exact solutions of the Problem 1.

Tables 1 and 2 show a comparison of the numerical results with the errors applying the standard HAM ($m = 6$) and the numerical results applying the HAM developed by genetic algorithm HAM-GA with the exact solutions 24 within the interval $0 \leq x \leq 1$ for various values of λ_1, $\lambda_2$and h. Tables 3 and 4, we list the MAE and the MRE by the HAM and HAM-GA on the interval $[0, 1]$ for various values of λ_1, λ_2 and h. Table 5 gives the errors on the interval h-curves $[-1.2, -0.8]$ when $\lambda_1 = \lambda_2 = 1$.

Table 1: Numerical results when $\lambda_1 = \lambda_2 = 1$, for (Problem 1)

x	i	$u_{Exacti}(x)$	HAM h=-1	AE	HAM-GA h=-0.98213	AE
0.1	1	0.010	0.0100000000	2.113E-24	0.0099999999	3.899E-15
	2	0.001	0.0010000000	5.828E-26	0.0009999999	3.113E-18
0.3	1	0.090	0.0899999999	7.314E-15	0.0899999999	4.325E-12
	2	0.027	0.0269999999	1.689E-15	0.0269999999	4.316E-13
0.5	1	0.250	0.2499999997	2.022E-10	0.2499999982	1.763E-09
	2	0.125	0.1249999998	1.166E-10	0.1249999992	7.596E-10
0.7	1	0.490	0.4899997951	2.048E-07	0.4899995366	4.633E-07
	2	0.343	0.3429998134	1.865E-07	0.3429996197	3.802E-07
0.9	1	0.810	0.8099446465	5.535E-05	0.8099235084	7.649E-05
	2	0.729	0.7289423171	5.768E-05	0.7289244930	7.550E-05

Table 2: Numerical results when $\lambda_1 = 0.5, \lambda_2 = 1.33838729$ for (Problem 1)

x	i	$u_{Exacti}(x)$	HAM h=-1	AE	HAM-GA h=-0.98213	AE
0.1	1	0.010	0.010000000	3.302E-26	0.009999999	1.884E-15
	2	0.001	0.001000000	2.305E-27	0.000999999	2.043E-18
0.3	1	0.090	0.089999999	1.152E-16	0.089999999	1.040E-12
	2	0.027	0.026999999	5.883E-17	0.026999999	1.718E-13
0.5	1	0.250	0.249999999	3.432E-12	0.249999999	1.340E-10
	2	0.125	0.124999999	3.983E-12	0.124999999	9.760E-11
0.7	1	0.490	0.489999995	4.232E-09	0.489999979	2.034E-08
	2	0.343	0.342999990	9.081E-09	0.342999975	2.470E-08
0.9	1	0.810	0.809999004	9.954E-07	0.809996725	3.274E-06
	2	0.729	0.728994536	5.463E-06	0.728993970	6.029E-06

Table 3: MAE and MRE of HAM and HAM-GA for (Problem 1)

		HAM $\lambda_1 = \lambda_2 = 1h = -1$		HAM-GA $\lambda_1 = \lambda_2 = 1, h = -0.98213$	
m	i	MAE	MRE	MAE	MRE
2	1	5.877E-02	1.308E-02	6.308E-02	1.434E-02
	2	6.413E-02	1.935E-02	6.298E-02	1.858E-02
3	1	2.574E-02	6.445E-03	2.698E-02	6.684E-03
	2	1.335E-02	3.409E-03	1.513E-02	3.957E-03
4	1	5.897E-03	1.379E-03	6.974E-03	1.651E-03
	2	6.426E-03	1.897E-03	6.839E-03	1.990E-03
5	1	2.390E-03	6.002E-04	2.672E-03	6.635E-04
	2	1.947E-03	5.564E-04	2.273E-03	6.527E-04
6	1	6.738E-04	1.679E-04	8.337E-04	2.078E-04
	2	7.095E-04	2.088E-04	8.304E-04	2.429E-04

Table 4: MAE and MRE of HAM-GA for (Problem 1)

		HAM-GA $\lambda_1 = 0.5$, $\lambda_2 = 1.33838$, $h = -1$		HAM-GA $\lambda_1 = 0.5$, $\lambda_2 = 1.33838, h = -0.98213$	
m	i	MAE	MRE	MAE	MRE
2	1	1.551E-02	3.520E-03	1.792E-02	4.297E-03
	2	4.800E-02	1.655E-02	4.714E-02	1.596E-02
3	1	6.088E-03	1.764E-03	6.465E-03	1.833E-03
	2	1.283E-04	3.189E-04	1.772E-03	3.028E-04
4	1	3.162E-04	5.617E-05	6.297E-04	1.475E-04
	2	2.101E-03	6.938E-04	2.040E-03	6.561E-04
5	1	3.177E-04	9.251E-05	3.286E-04	9.307E-05
	2	1.741E-06	1.322E-05	1.426E-04	3.460E-05
6		4.388E-06	2.567E-07	3.172E-05	7.968E-06
		9.404E-05	3.052E-05	9.224E-05	2.888E-05

Table 5: Errors on the interval h-curves for (Problem 1) when $\lambda_1 = \lambda_2 = 1$

x	i	h=-1.2	h=-0.98213	h=-1	h=-0.8
-0.9	1	2.260E-03	1.646E-05	3.642E-05	5.252E-06
	2	3.258E-03	5.517E-05	9.355E-05	2.752E-06
-0.7	1	9.179E-05	5.215E-08	2.007E-07	1.512E-06
	2	7.466E-05	9.922E-08	3.115E-07	2.430E-06
-0.5	1	4.730E-06	2.689E-12	2.133E-10	1.005E-06
	2	1.023E-06	3.624E-12	1.693E-10	2.430E-07
-0.3	1	1.895E-07	9.567E-14	7.451E-15	1.354E-07
	2	3.866E-09	4.992E-14	2.062E-15	2.089E-09
-0.1	1	6.448E-10	3.398E-15	2.115E-24	6.368E-10
	2	5.785E-14	2.886E-18	6.186E-26	3.829E-14
0.1	1	6.352E-10	3.899E-15	2.113E-24	6.432E-10
	2	5.736E-14	3.113E-18	5.828E-26	3.849E-14
0.3	1	1.267E-10	4.325E-12	7.314E-15	1.778E-07
	2	3.074E-09	4.316E-13	1.689E-15	2.412E-09
0.5	1	7.503E-07	1.763E-09	2.022E-10	3.644E-06
	2	3.825E-07	7.596E-10	1.166E-10	4.909E-07
0.7	1	2.699E-06	4.633E-07	2.048E-07	5.064E-05
	2	9.241E-06	3.802E-07	1.865E-07	2.237E-05
0.9	1	1.872E-04	7.649E-05	5.535E-05	8.640E-04
	2	2.275E-04	7.550E-05	5.768E-05	6.502E-04

In Figures 2 and 3, an illustration shows how well the exact solution matches with the approximate solution using the genetic algorithm h=-0.98213 with $\lambda_1 = \lambda_2 = 1$. In Figures 4 and 5 present the h-curves for (Problem 1) when $\lambda_1 = \lambda_2 = 1$. We present the error residual $ER(u_i(x))$, $i = 1, 2$ of HAM-GA when $\lambda_1 = 0.5$, $\lambda_2 = 1.33838$, $h = -0.98213$ in Figures 6 and 7. The CPU time = 1.188 Seconds.

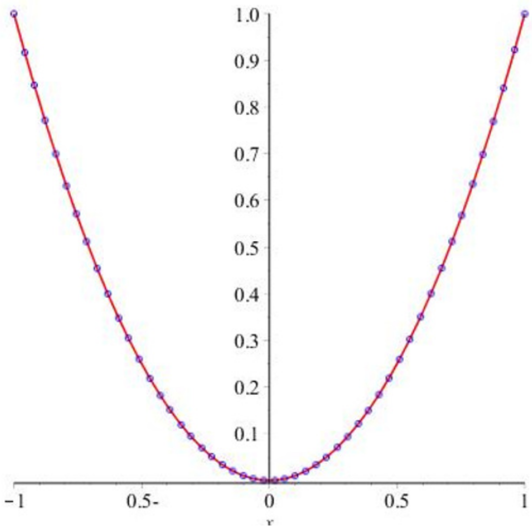

Figure 2: $Line: \ u_{Exact1}(x), \ o: HAM-GA_1(x)$

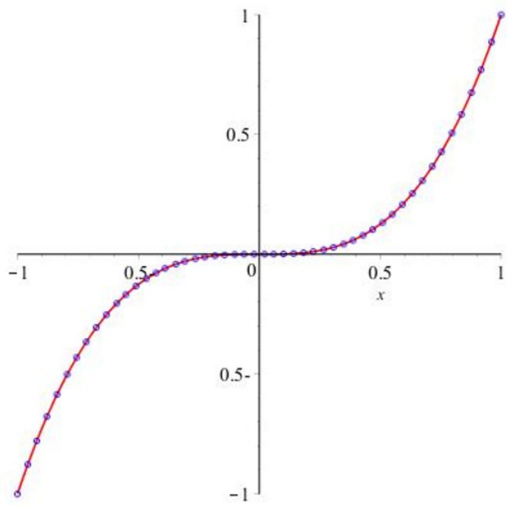

Figure 3: $Line: \ u_{Exact2}(x), \ o: HAM-GA_2(x)$

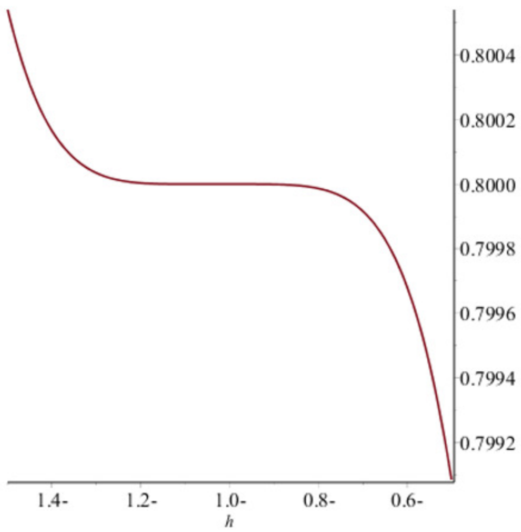

Figure 4: h-curve for $u_1'(0.4, h)$

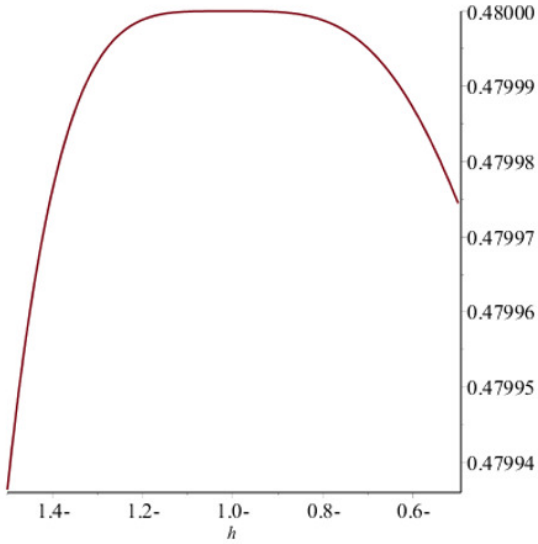

Figure 5: h-curve for $u_2'(0.4, h)$

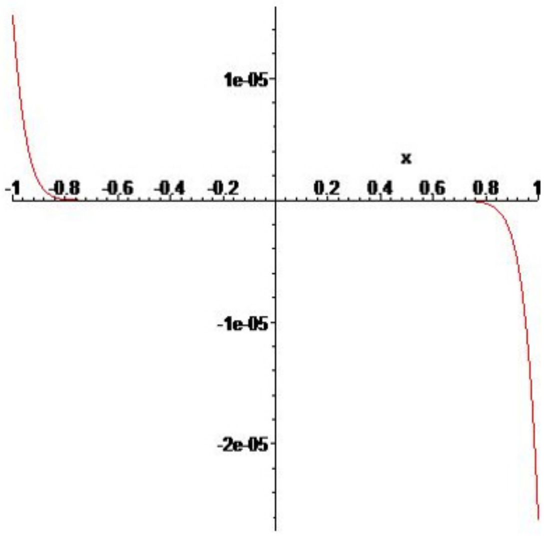

Figure 6: $ER(u_1(x))$ of HAM-GA

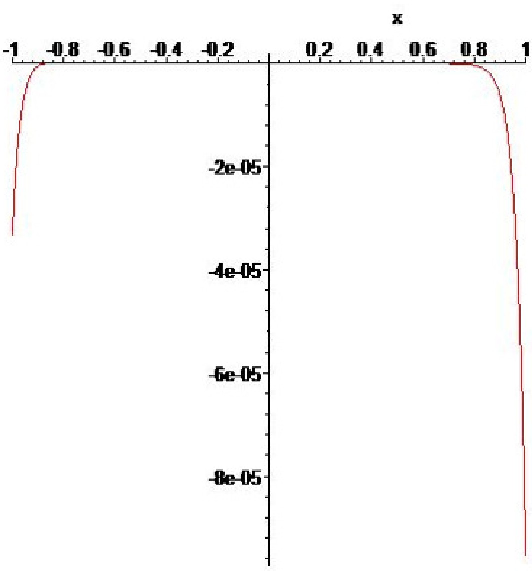

Figure 7: $ER(u_2(x))$ of HAM-GA

Problem 2. *Let us consider the following NLSVIEs [8]*

$$\begin{cases} u_1(x) = f_1(x) + \lambda_1 \int_0^x \left(u_1^2(t) + u_2^2(t)\right) dt, \\ u_2(x) = f_2(x) + \lambda_2 \int_0^x u_1(t) u_2(t) dt,, \end{cases} \tag{31}$$

where

$$\begin{cases} f_1(x) = \sin(x) - \lambda_1 x, \\ f_2(x) = \cos(x) - \frac{1}{2}\lambda_2 sin^2(x), \end{cases} \tag{32}$$

with the exact solutions

$$u_{Exact1}(x) = \sin(x),\ u_{Exact2}(x) = \cos(x), \tag{33}$$

To solve 31 and 32 by means of the standard HAM, we choose the initial approximation

$$u_{1,0}(x) = f_1(x),\ u_{2,0}(x) = f_2(x), \tag{34}$$

and the linear operator

$$L[\phi_1(x,q)] = \phi_1(x,q),\ L[\phi_2(x,q)] = \phi_2(x,q), \tag{35}$$

Furthermore, the system 31 suggests that we define the non-linear operator as

$$\begin{cases} N_1[\phi_1(x,q), \phi_2(x,q)] = \phi_1(x,q) - f_1(x) - \lambda_1 \int_0^x \left(\phi_1^2(t,q) + \phi_2^2(t,q)\right) dt, \\ N_2[\phi_1(x,q), \phi_2(x,q)] = \phi_2(x,q) - f_2(x) - \lambda_2 \int_0^x (\phi_1(t,q)\phi_2(t,q))\, dt, \end{cases} \tag{36}$$

Using the above definition, we construct the zeroth-order deformation equation as in 6 and 7 and the *mth*-order deformation equation for $m \geq 1$ is

$$\begin{cases} L[u_{1,m}(x) - \chi_m u_{1,m-1}(x)] = h[R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \\ L[u_{2,m}(x) - \chi_m u_{2,m-1}(x)] = h[R_{1,m}(\vec{u}_{1,m-1}),\ R_{2,m}(\vec{u}_{2,m-1})], \end{cases} \tag{37}$$

where

$$\begin{cases} R_{1,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}) = u_{1,m-1}(x) - f_1(x) - \lambda_1 \int_0^x \left(u_{1,m-1}^2(t) + u_{2,m-1}^2(t)\right) dt, \\ R_{2,m}(\vec{u}_{1,m-1}, \vec{u}_{2,m-1}) = u_{2,m-1}(x) - f_2(x) - \lambda_2 \int_0^x (u_{1,m-1}(t)\, u_{2,m-1}(t))\, dt,, \end{cases} \tag{38}$$

Now, for $m \geq 1$, the solutions of the *mth*-order deformation Equation 38 are

$$\begin{cases} u_{1,m}(x) = \chi_m u_{1,m-1}(x) + h[R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \\ u_{2,m}(x) = \chi_m u_{2,m-1}(x) + h[R_{1,m}(\vec{u}_{1,m-1}), R_{2,m}(\vec{u}_{2,m-1})], \end{cases} \tag{39}$$

we now successively obtain

$$\begin{aligned} u_{1,1}(x) &= -\frac{1}{3}h\lambda_1^3 x^3 - 2h\lambda_1^2 x\cos x + 2h\lambda_1^2 \sin x - h\lambda_1 x + \frac{1}{16}h\lambda_1\lambda_2^2 \sin^3 x\cos x \\ &\quad + \frac{3}{32}h\lambda_1\lambda_2^2 \sin x\cos x - \frac{3}{32}h\lambda_1\lambda_2^2 x + \frac{1}{3}h\lambda_1\lambda_2 \sin^3 x \end{aligned}$$

$$u_{2,1}(x) = -h\lambda_1\lambda_2 + \frac{1}{3}h\lambda_2^2 + \frac{1}{4}h\lambda_1\lambda_2^2 x\sin x\cos x - \frac{1}{8}h\lambda_1\lambda_2^2 x^2$$

$$-\frac{1}{8}h\lambda_1\lambda_2^2\sin^2 x + h\lambda_1\lambda_2 x\sin x + h\lambda_1\lambda_2\cos x$$
$$-\frac{1}{6}h\lambda_2^2\sin^2 x\cos x - \frac{1}{3}h\lambda_2^2\cos x - \frac{1}{2}h\lambda_2\sin^2 x$$

$$\vdots$$

Thus, the approximate solution in a series form is given by

$$u_1(x) = u_{1,0}(x) + \sum_{i=1}^{5} u_{1,i}(x),\ \ u_2(x) = u_{2,0}(x) + \sum_{i=1}^{5} u_{2,i}(x),$$

This series has the closed form as $m \to \infty$

$$u_1(x) = \sin x,\ \ u_2(x) = \cos x.$$

which are the exact solutions of the Problem 2.
Tables 6, 7 show a comparison of the numerical results with the errors applying the standard HAM ($m = 5$) and the numerical results applying the HAM developed by genetic algorithm HAM-GA with the exact solutions 33 within the interval $0 \leq x \leq 0.5$ for various values of λ_1, λ_2 and h. Tables 8, 9, we list the MAE and the MRE by the HAM and HAM-GA on the interval $[0, 0.5]$ for the various values of λ_1, λ_2 and h. Table 10 gives the errors on the interval h-curves $[-1.2, -0.8]$ when $\lambda_1 = \lambda_2 = 1$.

Table 6: Numerical results when $\lambda_1 = \lambda_2 = 1$, for (Problem 2)

x	i	$u_{Exacti}(x)$	HAM $h = -1$	AE	HAM-GA h=-0.98401	AE
0.1	1	0.099833416	0.099829110	4.306E-06	0.099828113	5.303E-06
	2	0.995004165	0.995003990	1.745E-07	0.995002111	2.054E-06
0.3	1	0.295520206	0.294524648	9.955E-04	0.294500710	1.019E-03
	2	0.955336489	0.955219396	1.170E-04	0.955085971	2.505E-04
0.5	1	0.479425538	0.467847859	1.157E-02	0.467776042	1.164E-02
	2	0.877582561	0.875456892	2.125E-03	0.874603732	2.978E-03

Table 7: Numerical results when $\lambda_1 = 0.5, \lambda_2 = 0.5$, for (Problem 2)

x	i	$u_{Exacti}(x)$	HAM $h = -1$	AE	HAM-GA h=-0.98401	AE
0.1	1	0.099833416	0.099833280	1.356E-07	0.099833132	2.841E-07
	2	0.995004165	0.995004155	9.445E-09	0.995004017	1.473E-07
0.3	1	0.295520206	0.295486811	3.339E-05	0.295479223	4.098E-05
	2	0.955336489	0.955329938	6.550E-06	0.955320666	1.582E-05
0.5	1	0.479425538	0.478989428	4.361E-04	0.478926206	4.993E-04
	2	0.877582561	0.877455542	1.270E-04	0.877391282	1.912E-04

Table 8: MAE and MRE of HAM and HAM-GA for (Problem 2)

		HAM $\lambda_1 = \lambda_2 = 1$ $h = -1$		HAM-GA $\lambda_1 = \lambda_2 = 1, h = -0.98401$	
m	i	MAE	MRE	MAE	MRE
2	1	7.494E-02	1.545E-02	8.174E-02	1.899E-02
	2	1.170E-01	5.382E-02	1.170E-01	5.295E-02
3	1	7.716E-02	3.207E-02	7.720E-02	3.158E-02
	2	1.534E-02	2.581E-03	1.857E-02	4.154E-03
4	1	1.114E-02	3.207E-03	1.425E-02	4.529E-03
	2	1.577E-02	6.432E-03	1.583E-02	6.287E-03
5	1	1.157E-02	4.556E-03	1.164E-02	4.514E-03
	2	2.125E-03	4.892E-04	2.978E-03	8.526E-04
6	1	1.642E-03	5.243E-04	2.413E-03	8.310E-04
	2	2.204E-03	8.691E-04	2.232E-03	8.547E-04

Table 9: MAE and MRE of HAM-GA for (Problem 2)

		HAM-GA $\lambda_1 = 0.5$, $\lambda_2 = 0.5, h = -1$		HAM-GA $\lambda_1 = 0.5$, $\lambda_2 = 0.5, h = -0.98401$	
m	i	MAE	MRE	MAE	MRE
2	1	2.411E-02	8.656E-03	2.772E-02	1.058E-02
	2	3.019E-02	1.452E-02	3.063E-02	1.451E-02
3	1	1.051E-02	4.590E-03	1.100E-02	4.749E-03
	2	2.735E-03	9.328E-04	3.613E-03	1.351E-03
4	1	1.221E-03	4.665E-04	1.671E-03	6.618E-04
	2	1.110E-03	4.649E-04	1.207E-03	4.968E-04
5	1	4.361E-04	1.779E-04	4.993E-04	2.020E-04
	2	1.270E-04	4.611E-05	1.912E-04	7.297E-05
6	1	5.861E-05	2.268E-05	9.015E-05	3.552E-06
	2	4.302E-05	1.721E-05	5.203E-05	2.049E-06

Table 10: Errors on the interval h-curves for (Problem 2) when $\lambda_1 = \lambda_2 = 1$

x	i	h=-1.2	h=-0.98213	h=-1	h=-0.8
-0.5	1	3.319E-02	1.164E-02	1.157E-02	2.387E-02
	2	1.640E-02	2.978E-03	2.125E-03	1.287E-02
-0.3	1	6.576E-03	1.019E-03	9.955E-04	4.423E-03
	2	3.711E-03	2.505E-04	1.170E-04	2.504E-03
-0.1	1	3.672E-04	5.303E-06	4.306E-06	2.826E-04
	2	2.132E-04	2.054E-06	1.745E-07	1.520E-04
0.1	1	3.672E-04	5.303E-06	4.306E-06	2.826E-04
	2	2.132E-04	2.054E-06	1.745E-07	1.520E-04
0.3	1	6.576E-03	1.019E-03	9.955E-04	4.423E-03
	2	3.711E-03	2.505E-04	1.170E-04	2.504E-03
0.5	1	3.319E-02	1.164E-02	1.157E-02	2.387E-02
	2	1.640E-02	2.978E-03	2.125E-03	1.287E-02

In Figures 8 and 9, an illustration shows how well the exact solution matches with the approximate solution using the genetic algorithm h=-0.98401 with $\lambda_1 = \lambda_2 = 1$. In Figures 10 and 11 the h-curves present for (Problem 2) when $\lambda_1 = \lambda_2 = 1$. We present the error residual $ER\left(u_i\left(x\right)\right),\ i = 1, 2$ of HAM-GA when $\lambda_1 \ = \ \lambda_2 \ = 0.5,\ h = -0.98401$ in Figures 12 and 13. The CPU time = 2.828 Seconds.

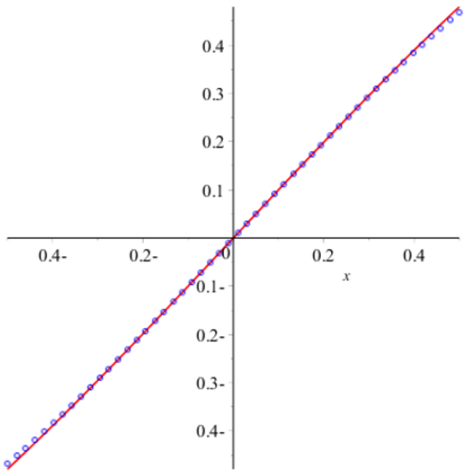

Figure 8: $Line:\ u_{Exact1}\left(x\right),\ o: HAM-GA_1(x)$

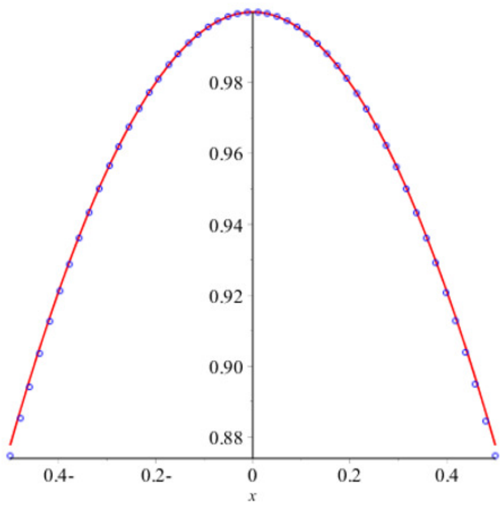

Figure 9: $Line:\ u_{Exact2}\left(x\right),\ o:\ HAM-GA_2(x)$

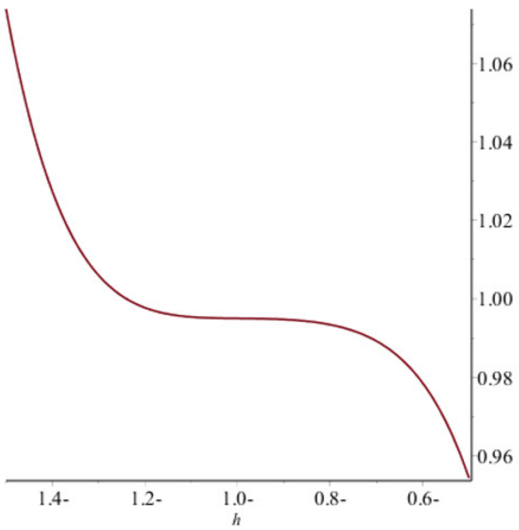

Figure 10: h-curve for $u'_1\left(0.1, h\right)$

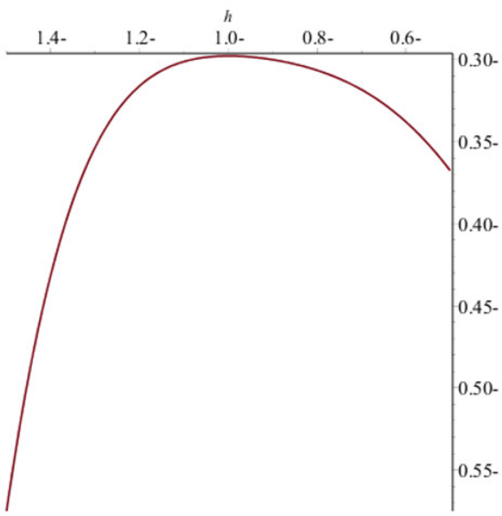

Figure 11: h-curve for $u'_2\left(0.4, h\right)$

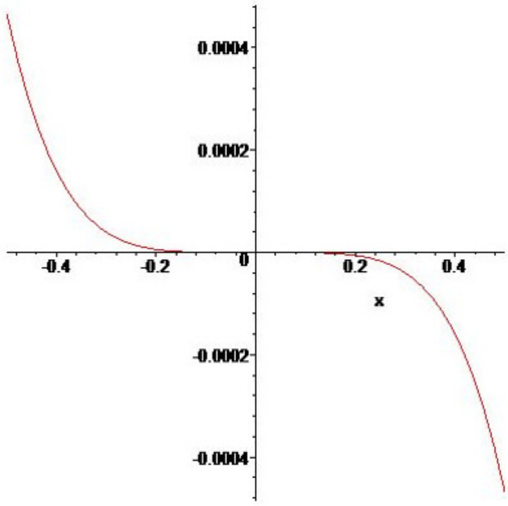

Figure 12: $ER\left(u_1\left(x\right)\right)$ of HAM-GA

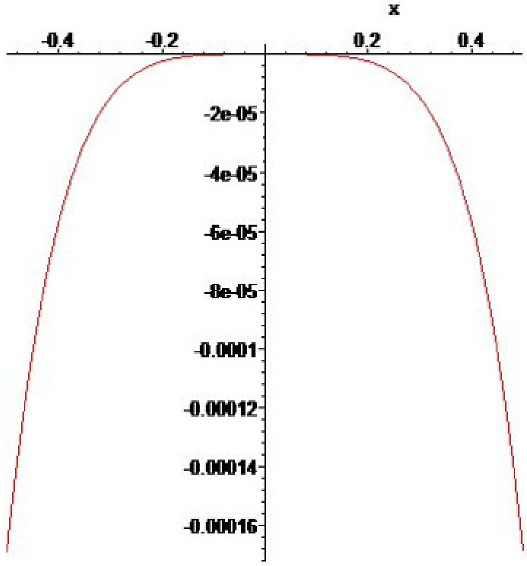

Figure 13: $ER\left(u_2\left(x\right)\right)$ of HAM-GA

Problem 3. *Finally, let us consider the following NLSVIEs [8]*

$$\begin{cases} u_1\left(x\right) = f_1\left(x\right) + 4\int_0^x u_1(t)u_2(t)dt, \\ u_2\left(x\right) = f_2\left(x\right) + \int_0^x tu_1(t)u_3^2(t)dt, \\ u_3\left(x\right) = f_3\left(x\right) - \frac{1}{3}\int_0^x tu_2(t)u_3(t)dt, \end{cases} \tag{40}$$

where

$$\begin{cases} f_1\left(x\right) = -2\ln\left(x\right)x^2 + x^2 + \ln\left(x\right), \\ f_2\left(x\right) = -\frac{1}{6}\ln\left(x\right)x^6 + \frac{1}{36}x^6 + x, \\ f_3\left(x\right) = \frac{1}{15}x^5 + x^2, \end{cases} \tag{41}$$

with the exact solutions

$$u_{Exact1}\left(x\right) = ln\left(x\right),\ u_{Exact2}\left(x\right) = x,\ u_{Exact3}\left(x\right) = x^2. \tag{42}$$

To solve 40 and 41 by means of the standard HAM, we choose the initial approximation

$$u_{1,0}\left(x\right) = f_1\left(x\right),\ u_{2,0}\left(x\right) = f_2\left(x\right),\ u_{3,0}\left(x\right) = f_3\left(x\right), \tag{43}$$

and the linear operator

$$L\left[\phi_1\left(x,q\right)\right] = \phi_1\left(x,q\right),\ L\left[\phi_2\left(x,q\right)\right] = \phi_2\left(x,q\right),\ L\left[\phi_3\left(x,q\right)\right] = \phi_3\left(x,q\right). \tag{44}$$

Furthermore, the system 40 suggests that we define the non-linear operator as

$$\begin{aligned} N_1\left[\phi_1\left(x,q\right),\phi_2\left(x,q\right),\phi_3\left(x,q\right)\right] &= \phi_1\left(x,q\right) - f_1\left(x\right) - 4\int_0^x \phi_1\left(t,q\right)\phi_2\left(t,q\right)dt, \\ N_2\left[\phi_1\left(x,q\right),\phi_2\left(x,q\right),\phi_3\left(x,q\right)\right] &= \phi_2\left(x,q\right) - f_2\left(x\right) - \int_0^x t\phi_1\left(t,q\right)\phi_3^2\left(t,q\right)dt, \\ N_3\left[\phi_1\left(x,q\right),\phi_2\left(x,q\right),\phi_3\left(x,q\right)\right] &= \phi_3\left(x,q\right) - f_3\left(x\right) + \frac{1}{3}\int_0^x t\phi_2\left(t,q\right)\phi_3\left(t,q\right)dt. \end{aligned} \tag{45}$$

Using the above definition, we construct the zeroth-order deformation equation as in 6 and 7 and the mth-order deformation equation for $m \geq 1$ is

$$\begin{array}{c} L\left[u_{1,m}\left(x\right)-\chi_m u_{1,m-1}\left(x\right)\right]=h\left[R_{1,m}\left(\vec{u}_{1,m-1}\right),R_{2,m}\left(\vec{u}_{2,m-1}\right),R_{3,m}\left(\vec{u}_{3,m-1}\right)\right], \\ L\left[u_{2,m}\left(x\right)-\chi_m u_{2,m-1}\left(x\right)\right]=h\left[R_{1,m}\left(\vec{u}_{1,m-1}\right),\ R_{2,m}\left(\vec{u}_{2,m-1}\right),R_{3,m}\left(\vec{u}_{3,m-1}\right)\right], \\ L\left[u_{3,m}\left(x\right)-\chi_m u_{3,m-1}\left(x\right)\right]=h\left[R_{1,m}\left(\vec{u}_{1,m-1}\right),\ R_{2,m}\left(\vec{u}_{2,m-1}\right),R_{3,m}\left(\vec{u}_{3,m-1}\right)\right]. \end{array} \tag{46}$$

where

$$\begin{array}{c} R_{1,m}\left(\vec{u}_{1,m-1},\vec{u}_{2,m-1}\ \ ,\vec{u}_{3,m-1}\ \ \right)=u_{1,m-1}\left(x\right)-f_1\left(x\right)-4\int_0^x u_{1,m-1}\left(t\right)u_{2,m-1}\left(t\right)dt, \\ R_{2,m}\left(\vec{u}_{1,m-1},\vec{u}_{2,m-1},\vec{u}_{3,m-1}\ \ \right)=u_{2,m-1}\left(x\right)-f_2\left(x\right)-\int_0^x tu_{1,m-1}\left(t\right)u_{3,m-1}^2\left(t\right)dt, \\ R_{3,m}\left(\vec{u}_{1,m-1},\vec{u}_{2,m-1},\vec{u}_{3,m-1}\ \ \right)=u_{3,m-1}\left(x\right)-f_3\left(x\right)+\frac{1}{3}\int_{30}^x tu_{2,m-1}\left(t\right)u_{3,m-1}\left(t\right)dt. \end{array} \tag{47}$$

Now, for $m \geq 1$, the solutions of the mth-order deformation Equation 47 are

$$\begin{array}{c} u_{1,m}\left(x\right)=\chi_m u_{1,m-1}\left(x\right)+h\left[R_{1,m}\left(\vec{u}_{1,m-1}\right),R_{2,m}\left(\vec{u}_{2,m-1}\right),R_{3,m}\left(\vec{u}_{2,m-1}\right)\right], \\ u_{2,m}\left(x\right)=\chi_m u_{2,m-1}\left(x\right)+h\left[R_{1,m}\left(\vec{u}_{1,m-1}\right),R_{2,m}\left(\vec{u}_{2,m-1}\right),R_{3,m}\left(\vec{u}_{2,m-1}\right)\right], \\ u_{3,m}\left(x\right)=\chi_m u_{3,m-1}\left(x\right)+h\left[R_{1,m}\left(\vec{u}_{1,m-1}\right),R_{2,m}\left(\vec{u}_{2,m-1}\right),R_{3,m}\left(\vec{u}_{2,m-1}\right)\right], \end{array} \tag{48}$$

we now successively obtain

$$\begin{aligned} u_{1,1}\left(x\right) &= h\Bigg(-2ln\left(x\right)x^2+x^2-\frac{4}{27}ln^2\left(x\right)x^9+\frac{32}{243}ln(x)x^9 \\ &\quad -\frac{59}{2187}x^9+2\ln\left(x\right)x^4-\frac{3}{2}x^4\Bigg). \end{aligned}$$

$$\begin{aligned} u_{2,1}\left(x\right)=h&\left(-\frac{1}{6}ln(x)x^6+\frac{1}{36}x^6+\frac{1}{1575}ln(x)\,x^{14}-\frac{4}{11025}x^{14}-\frac{1}{2700}ln(x)x^{12}\right. \\ &\left.+\frac{1}{32400}x^{12}+\frac{4}{165}ln(x)x^{11}-\frac{26}{1815}x^{11}-\frac{2}{135}ln(x)\,x^9+\frac{2}{1215}x^9+\frac{1}{4}ln(x)x^8-\frac{5}{32}x^8\right), \end{aligned}$$

$$u_{3,1}\left(x\right) = h\left(\frac{1}{15}x^5-\frac{1}{3510}ln\left(x\right)x^{13}+\frac{19}{273780}x^{13}-\frac{1}{180}ln\left(x\right)x^{10}+\frac{1}{675}x^{10}+\frac{1}{360}x^8\right),$$

$$\vdots$$

Thus, the approximate solution in a series form is given by

$$u_1\left(x\right)=u_{1,0}\left(x\right)+\sum_{i=1}^{6}u_{1,i}(x),$$

$$u_2\left(x\right)=u_{2,0}\left(x\right)+\sum_{i=1}^{6}u_{2,i}(x),$$

$$u_3(x) = u_{3,0}(x) + \sum_{i=1}^{6} u_{3,i}(x).$$

This series has the closed form $m \to \infty$

$$u_1(x) = ln(x),\ u_2(x) = x,\ u_3(x) = x^2.$$

which are the exact solutions of the Problem 3.
Table 11 shows a comparison of the numerical results with the errors applying the standard HAM (m=6) and the numerical results applying the HAM developed by genetic algorithm (HAM-GA) with the exact solutions 42 within the interval $0 \le x \le 0.6$. Table 12 the MAE is tabulated for different frequencies as well as the MRE. Table 13 gives the errors on the interval h-curves $[-1.2, -0.8]$.

Table 11: Numerical results for (Problem 3)

x	i	$u_{Exacti}(x)$	HAM $h=-1$	AE	HAM-GA h=-1.01064	AE
0.1	1	-2.30258509	-2.302585092	3.094E-13	-2.302585092	3.734E-13
	2	0.10000000	0.1000000000	5.644E-18	0.099999999	1.234E-17
	3	0.01000000	0.0099999999	2.932E-21	0.009999999	9.274E-17
0.3	1	-1.20397280	-1.203972712	9.189E-08	-1.203972808	4.117E-09
	2	0.30000000	0.300000000	1.477E-10	0.300000000	5.030E-12
	3	0.09000000	0.089999999	7.052E-13	0.089999999	9.033E-14
0.5	1	-0.69314718	-0.693128726	1.845E-05	-0.693144508	2.672E-06
	2	0.50000000	0.500000343	3.436E-07	0.500000159	1.596E-07
	3	0.25000000	0.249999995	4.934E-09	0.249999997	2.706E-09

Table 12: MAE and MRE of HAM and HAM-GA for (Problem 3)

		$HAMh=-1$		HAM-GA $h=-1.01064$	
m	i	MAE	MRE	MAE	MRE
2	1	3.266E-01	1.102E-01	3.223E-01	1.087E-01
	2	4.773E-03	1.241E-03	4.768E-03	1.251E-03
	3	7.305E-05	2.150E-05	1.289E-04	4.453E-05
3	1	8.783E-02	2.910E-02	8.276E-02	2.744E-02
	2	1.582E-03	4.708E-04	1.514E-03	4.544E-04
	3	1.783E-05	4.884E-06	1.606E-05	4.295E-06
4	1	1.626E-02	5.420E-03	1.404E-02	4.690E-03
	2	3.342E-04	1.057E-04	2.950E-04	9.430E-05
	3	4.897E-06	1.510E-06	4.505E-06	1.409E-06
5	1	2.020E-03	6.799E-04	1.452E-03	4.914E-04
	2	5.048E-05	1.679E-05	3.905E-05	1.319E-05
	3	8.725E-07	2.839E-07	7.070E-07	2.332E-07
6	1	8.969E-05	3.016E-05	2.672E-06	9.493E-07
	2	5.256E-06	1.880E-06	3.111E-06	1.169E-06
	3	1.138E-07	3.870E-08	7.714E-08	2.675E-08

Table 13: Errors on the interval h-curves for (Problem 3)

x	i	h=-1.2	h=-1.01064	h=-1	h=-0.8
-0.5	1	1.758E-04	1.337E-04	1.806E-04	9.697E-03
	2	2.235E-06	8.834E-07	1.363E-06	8.807E-05
	3	5.742E-07	9.568E-09	1.562E-08	1.219E-06
-0.3	1	4.996E-05	4.619E-08	2.375E-07	8.175E-04
	2	6.683E-08	4.114E-11	3.324E-10	8.066E-07
	3	5.017E-08	1.777E-13	1.546E-12	5.382E-08
-0.1	1	1.921E-05	8.053E-13	4.255E-13	3.286E-05
	2	1.200E-10	1.517E-17	8.126E-18	2.866E-10
	3	2.130E-10	8.919E-17	4.310E-21	2.135E-10
0.1	1	1.255E-05	3.734E-13	3.094E-13	2.206E-05
	2	6.870E-11	1.234E-17	5.644E-18	1.815E-10
	3	2.136E-10	9.274E-17	2.932E-21	2.131E-10
0.3	1	2.885E-05	4.117E-09	9.189E-08	4.264E-04
	2	3.622E-08	5.030E-12	1.477E-10	3.910E-07
	3	5.355E-08	9.033E-14	7.052E-13	4.991E-08
0.5	1	8.557E-05	2.672E-06	1.845E-05	4.074E-03
	2	1.003E-06	1.596E-07	3.436E-07	3.433E-05
	3	7.582E-07	2.706E-09	4.934E-09	3.348E-07

In Figures 14-16 an illustration of how well the exact solution matches with the approximate solution using the genetic algorithm h=-1.01064. In Figures 17-19 present the h-curves for (Problem 3). We present the error residual $ER\left(u_i\left(x\right)\right),\ i=1,2,3$ of HAM-GA when $h=-1.01064$ in Figures 20-22. The CPU time = 10.890 Seconds.

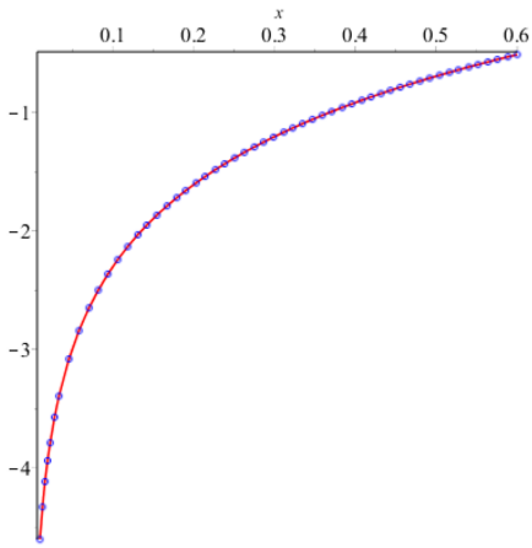

Figure 14: $Line:\ u_{Exact1}\left(x\right),\quad o:\ HAM-GA_1(x)$

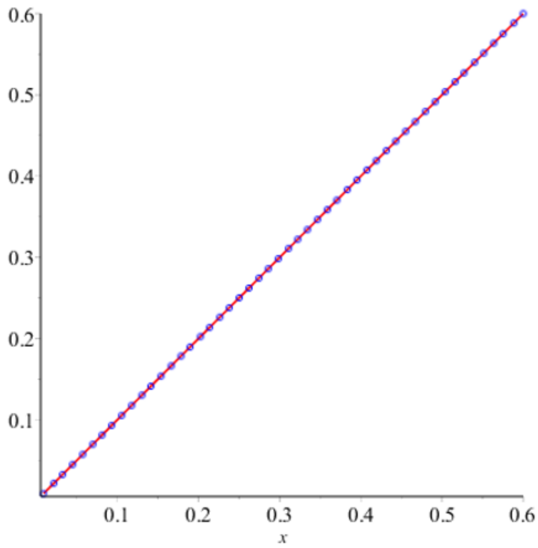

Figure 15: $Line:\ u_{Exact2}\left(x\right),\quad o:\ HAM-GA_2(x)$

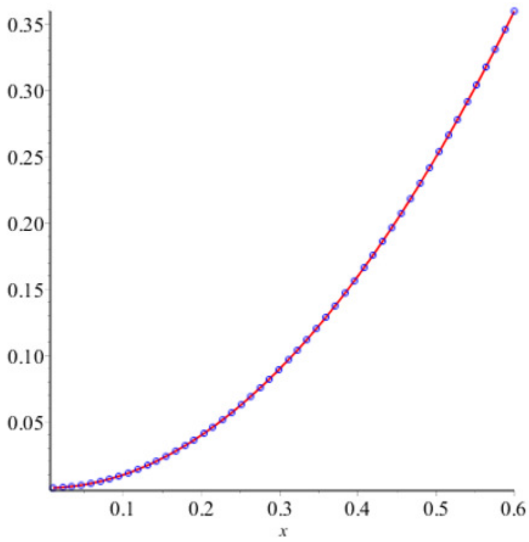

Figure 16: $Line: \ u_{Exact3}(x), \quad o: \ HAM - GA_3(x)$

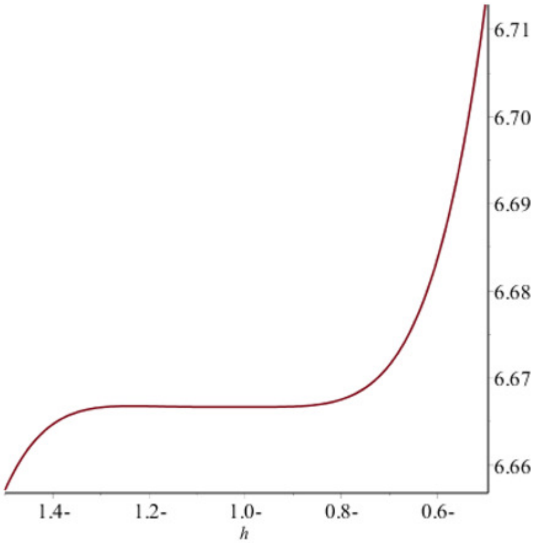

Figure 17: h-curve for $u_1'(0.15, h)$

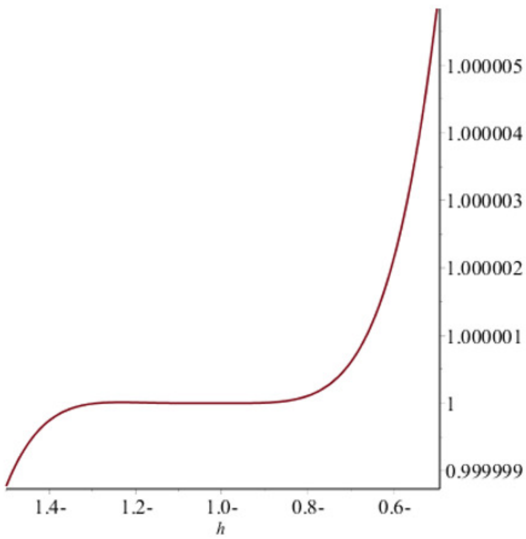

Figure 18: $h - curve for u_2'(0.15, h)$

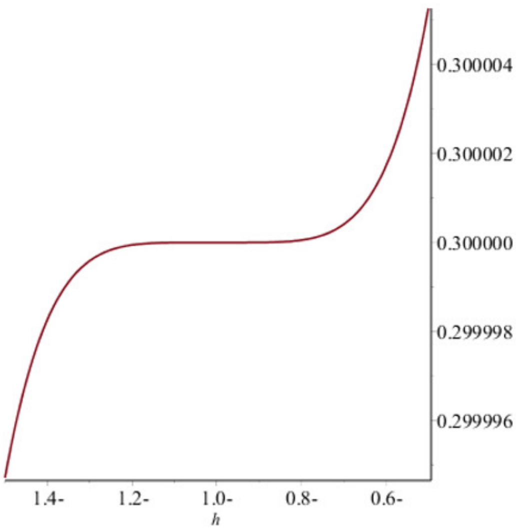

Figure 19: $h - curve for u_3'(0.15, h)$

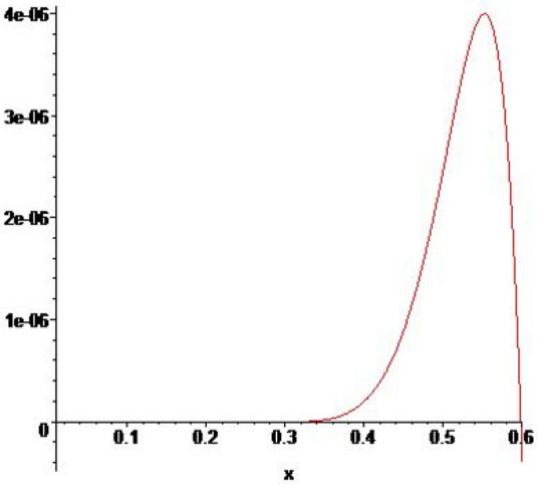

Figure 20: $ER(u_1(x))$ of HAM-GA

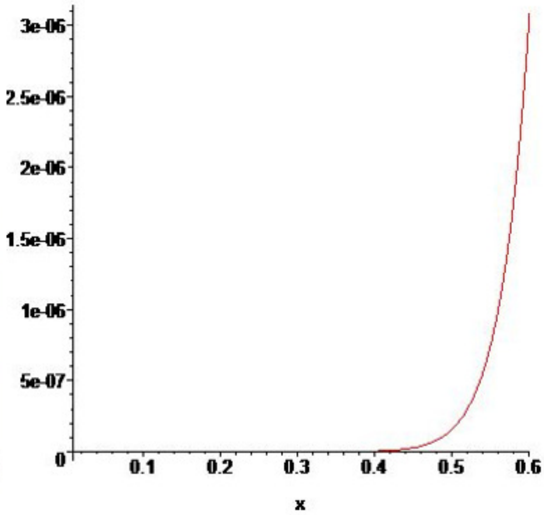

Figure 21: $ER(u_2(x))$ of HAM-GA

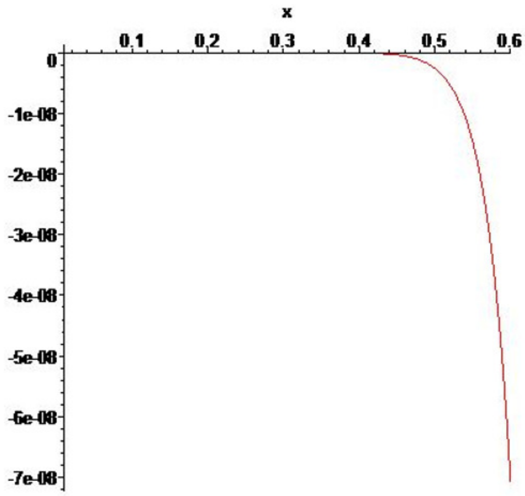

Figure 22: $ER\left(u_3\left(x\right)\right)$ of HAM-GA

6. Conclusion

The present study suggests a new algorithm (HAM-GA) merging both the HAM and the GA for solving the nonlinear system of Volterra integral equations. Based on four consecutive cases, the algorithm determines the solution. The algorithm deems the residual error function as a fitness function and based on which the optimum value of λ_1, λ_2 and h are selected. In the first case, the results were calculated according to the standard HAM, in the second case, the best value for h were chosen according to the genetic algorithm and based on which the results were calculated using HAM-GA. As for the third case, the best $\lambda_1$and λ_2 for the Volterra system of integral equations were chosen, then the results were calculated using HAM-GA. Finally, the results were calculated using HAM-GA depending on optimal λ_1, λ_2 and h. The results obtained using the best h were better than those obtained using the standard HAM. As for the results for the fourth case, they were optimal for the exact solution. The findings are in good agreement with the h-curves indicating that the proposed algorithm is successful in finding the solution.

References

[1] Hamdy R Abdl-Rahim, Mohra Zayed, and Gamal M Ismail. Analytical study of fractional epidemic model via natural transform homotopy analysis method. *Symmetry*, 14(8):1695, 2022.

[2] Samsul Ariffin Abdul Karim, Faheem Khan, and Muhammad Basit. Symmetric bernstein polynomial approach for the system of volterra integral equations on arbitrary interval and its convergence analysis. *Symmetry*, 14(7):1343, 2022.

[3] Muhammad Akbar, Rashid Nawaz, Sumbal Ahsan, Dumitru Baleanu, and Kottakkaran Sooppy Nisar. Analytical solution of system of volterra integral equations using oham. *Journal of Mathematics*, 2020, 2020.

[4] Yousef Al-Jarrah. On the approximation solutions of linear and nonlinear volterra integral equation of first and second kinds by using b-spline tight framelets generated by unitary extension principle and oblique extension principle. *International Journal of Difference Equations (IJDE)*, 15(2):165–189, 2020.

[5] Sadegh Amiri, Mojtaba Hajipour, and Dumitru Baleanu. A spectral collocation method with piecewise trigonometric basis functions for nonlinear volterra–fredholm integral equations. *Applied Mathematics and Computation*, 370:124915, 2020.

[6] Aytac Arikoglu and Ibrahim Ozkol. Solutions of integral and integro-differential equation systems by using differential transform method. *Computers & Mathematics with Applications*, 56(9):2411–2417, 2008.

[7] Pouria Assari and Mehdi Dehghan. The approximate solution of nonlinear volterra integral equations of the second kind using radial basis functions. *Appl. Numer. Math.*, 131(C):140–157, sep 2018.

[8] J Biazar and H Ghazvini. He's homotopy perturbation method for solving systems of volterra integral equations of the second kind. *Chaos, Solitons & Fractals*, 39(2):770–777, 2009.

[9] Fred Brauer. On a nonlinear integral equation for population growth problems. *SIAM Journal on Mathematical Analysis*, 6(2):312–317, 1975.

[10] Fred Brauer, Carlos Castillo-Chavez, and Carlos Castillo-Chavez. *Mathematical models in population biology and epidemiology*, volume 2. Springer, 2012.

[11] Ted A Burton. *Volterra integral and differential equations*, volume 202. Elsevier, 2005.

[12] Yves Cherruault. Convergence of adomian's method. *Kybernetes*, 1989.

[13] Eva de Botton, J Barry Greenberg, Alumah Arad, David Katoshevski, Visakh Vaikuntanathan, Matthias Ibach, and Bernhard Weigand. An investigation of grouping of two falling dissimilar droplets using the homotopy analysis method. *Applied Mathematical Modelling*, 104:486–498, 2022.

[14] Mohammad Ali Fariborzi Araghi and Samad Noeiaghdam. Finding optimal results in the homotopy analysis method to solve fuzzy integral equations. In *Advances in Fuzzy Integral and Differential Equations*, pages 173–195. Springer, 2022.

[15] Atanaska Georgieva and Snezhana Hristova. Homotopy analysis method to solve two-dimensional nonlinear volterra-fredholm fuzzy integral equations. *Fractal and Fractional*, 4(1):9, 2020.

[16] Ahmed HAMOUD, Nedal MOHAMMED, and Kirtiwant GHADLE. Solving mixed volterra-fredholm integro differential equations by using ham. *Turkish Journal of Mathematics and Computer Science*, 12(1):18–25, 2020.

[17] Edyta Hetmaniok, Damian Słota, Tomasz Trawiński, and Roman Wituła. Usage of the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind. *Numerical Algorithms*, 67(1):163–185, 2014.

[18] J Izadian, M MohammadzadeAttar, and M Jalili. Numerical solution of deformation equations in homotopy analysis method. *Applied Mathematical Sciences*, 6(8):357–367, 2012.

[19] Ahmad Jafarian, Pariya Ghaderi, Alireza K Golmankhaneh, and Dumitru Baleanu. Analytical treatment of system of abel integral equations by homotopy analysis method. *Rom. Rep. Phys*, 66(3):603–611, 2014.

[20] A Jayachitra and R Vinodha. Genetic algorithm based pid controller tuning approach for continuous stirred tank reactor. *Advances in Artificial Intelligence (16877470)*, 2014.

[21] Man KF, Tang KS, et al. Genetic algorithms: concepts and applications [in engineering design]. *IEEE transactions on industrial electronics*, 43(5):519–534, 1996.

[22] Oliver Kramer. Genetic algorithms. In *Genetic algorithm essentials*, pages 11–19. Springer, 2017.

[23] Jia-Xuan Li, Yan Yan, and Wen-Quan Wang. Time-delay feedback control of a cantilever beam with concentrated mass based on the homotopy analysis method. *Applied Mathematical Modelling*, 108:629–645, 2022.

[24] Shijun Liao. *Beyond perturbation: introduction to the homotopy analysis method.* Chapman and Hall/CRC, 2003.

[25] Shijun Liao. On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation*, 147(2):499–513, 2004.

[26] Shijun Liao. Notes on the homotopy analysis method: some definitions and theorems. *Communications in Nonlinear Science and Numerical Simulation*, 14(4):983–997, 2009.

[27] Shijun Liao. *Homotopy analysis method in nonlinear differential equations.* Springer, 2012.

[28] Shijun Liao. Avoiding small denominator problems by means of the homotopy analysis method. *Advances in Applied Mathematics and Mechanics*, 15(2):267–299, 2023.

[29] Bijan Hasani Lichae, Jafar Biazar, and Zainab Ayati. A class of runge–kutta methods for nonlinear volterra integral equations of the second kind with singular kernels. *Advances in Difference Equations*, 2018(1):1–19, 2018.

[30] Pedram Khaneh Masjedi and Paul M Weaver. Analytical solution for arbitrary large deflection of geometrically exact beams using the homotopy analysis method. *Applied Mathematical Modelling*, 103:516–542, 2022.

[31] M Matinfar, M Saeidy, and J Vahidi. Application of homotopy analysis method for solving systems of volterra integral equations. *Advances in Applied Mathematics and Mechanics*, 4(1):36–45, 2012.

[32] Saddam Mollah, Santosh Biswas, and Subhas Khajanchi. Impact of awareness program on diabetes mellitus described by fractional-order model solving by homotopy analysis method. *Ricerche di Matematica*, pages 1–26, 2022.

[33] Surya Kanta Mondal and Dulal Pal. Computational analysis of bioconvective flow of nanofluid containing gyrotactic microorganisms over a nonlinear stretching sheet with variable viscosity using ham. *Journal of Computational Design and Engineering*, 7(2):251–267, 2020.

[34] Zaid M Odibat. A study on the convergence of homotopy analysis method. *Applied Mathematics and Computation*, 217(2):782–789, 2010.

[35] MM Rashidi, SA Mohimanian Pour, T Hayat, and S Obaidat. Analytic approximate solutions for steady flow over a rotating disk in porous medium with heat transfer by homotopy analysis method. *Computers & Fluids*, 54:1–9, 2012.

[36] Rostam K Saeed and C Ahmed. Approximate solution for the system of non-linear volterra integral equations of the second kind by using block-by-block method. *Australian Journal of Basic and Applied Sciences*, 2(1):114–124, 2008.

[37] Niyazi Şahın, Şuayip Yüzbaşı, and Mustafa Gülsu. A collocation approach for solving systems of linear volterra integral equations with variable coefficients. *Computers & Mathematics with Applications*, 62(2):755–769, 2011.

[38] SR Saratha, M Bagyalakshmi, and G Sai Sundara Krishnan. Solving non linear differential equations by using ag-homotopy analysis method. In *Journal of Physics: Conference Series*, volume 1850, page 012065. IOP Publishing, 2021.

[39] S Sarwar and MM Rashidi. Approximate solution of two-term fractional-order diffusion, wave-diffusion, and telegraph models arising in mathematical physics using optimal homotopy asymptotic method. *Waves in random and complex media*, 26(3):365–382, 2016.

[40] R Shanthi, T Iswarya, J Visuvasam, L Rajendran, and Michael EG Lyons. Voltammetric and mathematical analysis of adsorption of enzymes at rotating disk electrode. *Int. J. Electrochem. Sci*, 17(220433):2, 2022.

[41] Liao Shijun. Homotopy analysis method: a new analytic method for nonlinear problems. *Applied Mathematics and Mechanics*, 19(10):957–962, 1998.

[42] Mustafa Turkyilmazoglu. Convergence of the homotopy analysis method. *arXiv preprint arXiv:1006.4460*, 2010.

[43] R Umadevi, J Visuvasam, K Venugopal, and L Rajendran. Mathematical models for ece reactions at rotating disk electrodes using homotopy analysis method. In *AIP Conference Proceedings*, volume 2277, page 130013. AIP Publishing LLC, 2020.

[44] Robert A Van Gorder and K Vajravelu. On the selection of auxiliary functions, operators, and convergence control parameters in the application of the homotopy analysis method to nonlinear differential equations: a general approach. *Communications in Nonlinear Science and Numerical Simulation*, 14(12):4078–4089, 2009.

[45] Lalchand Verma and Ramakanta Meher. Effect of heat transfer on jeffery–hamel cu/ag–water nanofluid flow with uncertain volume fraction using the double parametric fuzzy homotopy analysis method. *The European Physical Journal Plus*, 137(3):1–20, 2022.

[46] E Arul Vijayalakshmi, SS Santra, T Botmart, H Alotaibi, GB Loganathan, M Kannan, J Visuvasam, and V Govindan. Analysis of the magnetohydrodynamic flow in a porous medium. *AIMS Mathematics*, 7(8):15182–15194, 2022.

[47] E Arul Vijayalakshmi, James Visuvasam, and M Kannan. Theoretical analysis of mhd mixed convection from a vertical plate embedded in a porous medium with a convective boundary condition. *ECS Transactions*, 107(1):18507, 2022.

[48] J Visuvasam, A Meena, and L Rajendran. New analytical method for solving nonlinear equation in rotating disk electrodes for second-order ece reactions. *Journal of Electroanalytical Chemistry*, 869:114106, 2020.

[49] J Visuvasam, Angela Molina, E Laborda, and L Rajendran. Mathematical models of the infinite porous rotating disk electrode. *Int. J. Electrochem. Sci*, pages 9999–10022, 2018.

[50] Mahasin Thabet Younis and Waleed Mohammed Al-Hayani. Solving fuzzy system of volterra integro-differential equations by using adomian decomposition method. *European Journal of Pure and Applied Mathematics*, 15(1):290–313, 2022.