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Degenerate Apostol-Frobenius-Type Poly-Genocchi Polynomials of Higher Order with Parameters a and b

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Abstract. This paper introduces another variation of poly-Genocchi polynomials by mixing the concept of modified degenerate polyexponential function, Apostol-Genocchi polynomials and Frobenius polynomials. These polynomials are called the degenerate Apostol-Frobenius-type poly-Genocchi polynomials with parameters a and b. Several identities and formulas are derived including recurrence relations, explicit formulas and certain differential identity. Moreover, some relations are established connecting these polynomials to degenerate Stirling numbers of the first and second kind, higher order degenerate Bernoulli polynomials, and higher order degenerate Frobenius-Euler polynomials.

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1. Introduction

There are many ways of constructing a generalization of certain special function, polynomial or number. One of these is by mixing it with the concept of some other known functions and polynomials. For instance, multiplying the generating function of the Genocchi numbers G_n (see [12])

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}, \quad |t| < \pi,$$

and its variations with exponential polynomials yields the Genocchi polynomials, the Genocchi polynomials of higher order, the Apostol-Genocchi polynomials, and Apostol-Genocchi polynomials of higher order, which are respectively defined as follows:

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{xt}, \quad |t| < \pi, \tag{1}$$

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$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{xt}, \tag{2}$$

$$\sum_{n=0}^{\infty} G_n(x,\lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t+1} e^{xt}, \tag{3}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^k e^{xt}, \tag{4}$$

where $|t| < \pi$ when $\lambda = 1$ and $|t| < \log(-\lambda)$ when $\lambda \neq 1, \lambda \in \mathbb{C}$ (see [1–3, 5, 6, 40]). The first two polynomials are well-studied and two of the most recent studies are the works of Corcino-Corcino [14, 15] on asymptotic approximations, while the last two were given asymptotic approximation and Fourier series expansion in [13, 16, 20].

Also, incorporating the concept of Frobenius polynomials yields the so-called Frobenius-Genocchi polynomials, which are given by

$$\sum_{n=0}^{\infty} G_n^F(x;u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t-u} e^{xt}, \tag{5}$$

(see [4, 6, 7, 19, 21, 24, 32, 43, 44] for other interesting studies related to these polynomials). Moreover, mixing the Genocchi numbers with the concept of polylogarithm $\mathrm{Li}_k(z)$

$$\mathrm{Li}_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^k}, k \in \mathbb{Z}, \tag{6}$$

yields the poly-Genocchi polynomials, which are defined as follows

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{x^n}{n!} = \frac{2Li_k(1-e^t)}{e^t+1} e^{xt}. \tag{7}$$

Furthermore, with a slight modification of the generating function, another generalization, denoted by $G_{n,2}^{(k)}(x)$, was defined by Kim et al. [9, 25, 34] as follows

$$\sum_{n=0}^{\infty} G_{n,2}^{(k)}(x)\frac{x^n}{n!} = \frac{Li_k(1-e^{-2t})}{e^t+1} e^{xt}. \tag{8}$$

These polynomials are called modified poly-Genocchi polynomials. Note that, when $k = 1$, equations (7) and (8) give the Genocchi polynomials in (1). That is,

$$G_n^{(1)}(x) = G_{n,2}^{(1)}(x) = G_n(x).$$

Kim et. al [25] obtained several properties of these polynomials.

By introducing additional three parameters a, b, and c, Kurt [33] defined two forms of generalized poly-Genocchi polynomials as follows

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x;a,b,c)\frac{x^n}{n!} \tag{9}$$

$$\frac{2Li_k(1-(ab)^{-2t})}{a^{-t}+b^t}e^{xt}=\sum_{n=0}^{\infty}G_{n,2}^{(k)}(x;a,b,c)\frac{x^n}{n!}. \tag{10}$$

These were motivated by the generalizations introduced in (7) and (8), respectively. Note that, when $x=0$, (7) reduces to

$$\frac{2Li_k(1-e^t)}{e^t+1}=\sum_{n=0}^{\infty}G_n^{(k)}\frac{x^n}{n!}, \tag{11}$$

where $G_n^{(k)}$ are called the poly-Genocchi numbers. It is worth-mentioning that, using multi-polylogarithm, the generalized poly-Genocchi polynomials in (9) and (10) have been extended further in [39]. Recently, a new variation of poly-Genocchi polynomials with parameters a, b and c was defined in [17] by mixing the definitions of polylogarithm, Apostol-Genocchi polynomials and Frobenius polynomials, namely, the Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c. More precisely, the said polynomials, denoted by $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$, are defined as coefficients of the following generating function

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}=\left(\frac{Li_{k,\rho}(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^{\alpha}c^{xt}. \tag{12}$$

It is worth-mentioning the following interesting applications of Genocchi numbers, Genocchi polynomials and poly-Genocchi polynomials:

(i) The Genocchi numbers were used to bound the number of finite languages over a two-letter alphabet accepted by a deterministic finite automation (DFA) with n states (see [18]);

(ii) Certain wavelets method was constructed using Genocchi polynomials which has been used to obtain a numerical solution for the classical and time-fractional Rosenau-Hyman equation arising in the formation of patterns in liquid drops (see [37]);

(iii) The orthogonal version of poly-Genocchi polynomials coincides with the shifted Legendre polynomials. Also, the poly-Genocchi polynomials were used to solve the fractional differential equation, including the delay fractional differential equation via the operational matrix method with a collocation scheme (see [8]).

The degenerate exponential function, denoted by $e_\lambda^x(t)$, was defined in [10, 11])

$$e_\lambda^x(t)=(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\quad \lambda\in\mathbb{R}^+\cup\{0\}, \tag{13}$$

where $e_\lambda(t)=e_\lambda^1(t)=(1+\lambda t)^{1/\lambda}$ and $(x)_{0,\lambda}=1$,

$$(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\dots(x-(n-1)\lambda), n\geq 1.$$

It can easily be seen that

$$
\begin{aligned}
e_\lambda^{x+y}(t) &= (1+\lambda t)^{(x+y)/\lambda} = (1+\lambda t)^{(x/\lambda)+(y/\lambda)} \\
&= (1+\lambda t)^{(x/\lambda)}(1+\lambda t)^{(y/\lambda)} \\
&= e_\lambda^x(t) e_\lambda^y(t), \qquad (14)
\end{aligned}
$$

and

$$\frac{d}{dx} e_\lambda^x(t) = \log(1+\lambda t)^{1/\lambda} e_\lambda^x(t). \qquad (15)$$

The degenerate Bernoulli polynomials $\mathcal{B}_{n,\lambda}(x)$ and degenerate Euler polynomials $\mathcal{E}_{n,\lambda}(x)$ were defined by Carlitz [11] by means of the following generating functions

$$
\begin{aligned}
\frac{t}{e_\lambda(t)+1} e_\lambda^x(t) &= \frac{t}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}(x)\frac{t^n}{n!} \\
\frac{2}{e_\lambda(t)+1} e_\lambda^x(t) &= \frac{2}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}.
\end{aligned}
$$

Parallel to these, Lim [36] defined the degenerate Genocchi polynomials as follows

$$\frac{2t}{e_\lambda(t)+1} e_\lambda^x(t) = \frac{2t}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{G}_{n,\lambda}(x)\frac{t^n}{n!}. \qquad (16)$$

Kim and Kim [42] introduced the degenerate Frobenius-Euler polynomials as coefficients of the following generating function

$$\frac{1-u}{e_\lambda(t)-u} e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u)\frac{t^n}{n!} \qquad (17)$$

and Kim et al. [26] derived formulas that express any polynomial in terms of $h_{n,\lambda}(x|u)$. In their separate paper, Kim and Kim [27] defined the generalized degenerate Euler-Genocchi polynomials, denoted by $A_{n,\lambda}^{(r)}(x)$, as coefficients of the following generating function

$$\frac{2t^r}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} A_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},$$

which reduce to the degenerate Genocchi polynomials in (26) when $r = 1$. That is,

$$A_{n,\lambda}^{(1)}(x) = \mathcal{G}_{n,\lambda}(x).$$

The degenerate Stirling numbers of the first and second kind, denoted by $S_{1,\rho}(n,k)$ and $S_{2,\rho}(n,k)$, were defined in ([23, 29, 31])

$$\frac{(\log_\rho(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_{1,\rho}(n,k)\frac{t^n}{n!}, \quad \frac{(e_\rho(t)-1)^k}{k!} = \sum_{n=0}^{\infty} S_{2,\rho}(n,k)\frac{t^n}{n!}, \qquad (18)$$

where

$$\log_\rho(e_\rho(t)) = e_\rho(\log_\rho(t)) = t. \tag{19}$$

When $\rho \to 0$,

$$\lim_{\rho\to 0} S_{1,\rho}(n,k) = S_1(n,k), \quad \lim_{\rho\to 0} S_{2,\rho}(n,k) = S_2(n,k)$$

where $S_1(n,k)$ and $S_2(n,k)$ are the classical Stirling numbers of the first and second kind. Also, the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{(1+t)^x}{\log_\rho(1+t)}t = \sum_{n=0}^{\infty} b_{n,\rho}(x)\frac{t^n}{n!}. \tag{20}$$

The degenerate Stirling numbers of the second kind appeared in the probability distribution of the random variable given as the sum of a finite number of random variables with degenerate zero-truncated Poisson distributions and a random variable with degenerate Poisson distribution, all having the same parameter (see [31]).

The polyexponential functions are defined by the following generating functions [22, 23, 28, 30]

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k(n-1)!}, \quad k \in \mathbb{Z}. \tag{21}$$

For $k = 1$, $\mathrm{Ei}_1(x) = e^x - 1$. The modified degenerate polyexponential function are given by ([22, 23, 28, 30])

$$\mathrm{Ei}_{k,\rho}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\rho}x^n}{n^k(n-1)!}, \quad \rho \in \mathbb{R}. \tag{22}$$

Note that

$$\mathrm{Ei}_{1,\rho}(x) = \sum_{n=1}^{\infty}(1)_{n,\rho}\frac{x^n}{n!} = e_\rho(x) - 1, \quad \rho \in \mathbb{R}. \tag{23}$$

Also, we have

$$\frac{d}{dx}\mathrm{Ei}_{k,\rho}(\log_\rho(1+x)) = \frac{(1+x)^{\rho-1}}{\log_\rho(1+x)}\mathrm{Ei}_{k-1,\rho}(\log_\rho(1+x)), \tag{24}$$

which implies

$$\begin{aligned}
\mathrm{Ei}_{k,\rho}(\log_\rho(1+x)) &= \int_0^x \frac{(1+t)^{\rho-1}}{\log_\rho(1+t)}\int_0^y \cdots \frac{(1+x)^{\rho-1}}{\log_\rho(1+x)}\int_0^y \frac{(1+x)^{\rho-1}}{\log_\rho(1+x)}x dx \dots dx \\
&= \sum_{m=0}^{\infty}\sum_{m_1+m_2+\ldots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}} \\
&\quad \times \frac{b_{m_1,\rho}(\rho-1)}{m_1+1}\frac{b_{m_2,\rho}(\rho-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\rho}(\rho-1)}{m_1+\ldots+m_{k-1}+1}\frac{x^{m+1}}{m!}. \qquad (25)
\end{aligned}$$

The degenerate poly-Euler polynomials were defined in [35] by means of the following generating function

$$2\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+t))}{te_\rho(t)+1}e_\rho^x(t)=\sum_{n=0}^{\infty}\mathcal{E}_{n,\rho}^{(k)}(x)\frac{t^n}{n!}, \tag{26}$$

where $k\in\mathbb{Z}$.

In this paper, a new variation of poly-Genocchi polynomials is constructed by mixing the concepts of modified degenerate polyexponential function, Apostol-Genocchi polynomials and Frobenius polynomials. These polynomials are called the degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b. Some special cases of these polynomials are enumerated and some identities that contain a number of relations of this new variation with some Genocchi-type polynomials are provided. Finally, some connections of these degenerate Apostol-Frobenius-type poly-Genocchi polynomials to degenerate Stirling numbers of the first and second kind, higher order degenerate Bernoulli polynomials, and higher order degenerate Frobenius-Euler polynomials are discussed.

2. Definition and Some Explicit Formulas

Analogous to the definition of degenerate poly-Euler polynomials in (26), the desired variation of Apostol-type poly-Genocchi polynomials can be constructed by introducing the parameter u to incorporate the concept of Frobenius polynomials as well as the parameters a and b. The following contains the formal definition of the desired polynomials.

Definition 2.1. The degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b, denoted by $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$, are defined as coefficients of the following generating function:

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}e_\rho^x(t), \tag{27}$$

where $|t|<\frac{\sqrt{(\ln(\frac{\lambda}{u}))^2+4\pi^2}}{|\ln a+\ln b|}$. When $\alpha=1$, (27) yields

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}=\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}e_\rho^x(t). \tag{28}$$

where $\widehat{\mathcal{G}}_n^{(k)}(x;\lambda,\rho,u,a,b)=\widehat{\mathcal{G}}_n^{(k,1)}(x;\lambda,\rho,u,a,b)$ denotes the degenerate Apostol-Frobenius-type poly-Genocchi polynomials with parameters a and b.

Now, if $x=(1-u)t\ln ab$, then (25) yields

$$\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))$$

$$= t\sum_{m=0}^{\infty}((1-u)\ln ab)^{m+1}\sum_{m_1+m_2+\ldots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}}$$
$$\times\frac{b_{m_1,\rho}(\rho-1)}{m_1+1}\frac{b_{m_2,\rho}(\rho-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\rho}(\rho-1)}{m_1+\ldots+m_{k-1}+1}\frac{t^m}{m!}.$$

Also,

$$\frac{e_\rho^x(t)}{\lambda b^t-ua^{-t}}=\sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{n=0}^{\infty}\binom{m}{j}(x)_{j,\rho}\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{m-j}\frac{t^m}{m!}.$$

With

$$\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(m,\rho-1)$$
$$=((1-u)\ln ab)^{m+1}\sum_{m_1+m_2+\ldots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}}$$
$$\times\frac{b_{m_1,\rho}(\rho-1)}{m_1+1}\frac{b_{m_2,\rho}(\rho-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\rho}(\rho-1)}{m_1+\ldots+m_{k-1}+1}, \tag{29}$$

we have

$$\sum_{m=0}^{\infty}\widehat{\mathcal{G}}_m^{(k)}(x;\lambda,\rho,u,a,b)\frac{t^m}{m!}=\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}e_\rho^x(t)$$
$$=t\sum_{m=0}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{n=0}^{\infty}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)\binom{m-i}{j}\binom{m}{i}(x)_{j,\rho}\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{m-i-j}\frac{t^m}{m!}.$$

Note that the right-hand side of the preceding equation has no constant term. Hence, when $m=0$, $\widehat{\mathcal{G}}_0^{(k)}(x;\lambda,\rho,u,a,b)=0$. Moreover,

$$\sum_{m=0}^{\infty}\frac{1}{m}\widehat{\mathcal{G}}_m^{(k)}(x;\lambda,\rho,u,a,b)\frac{t^{m-1}}{(m-1)!}$$
$$=\sum_{m=0}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{n=0}^{\infty}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)\binom{m-i}{j}\binom{m}{i}(x)_{j,\rho}\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{m-i-j}\frac{t^m}{m!}.$$

By comparing the coefficients of $\frac{t^m}{m!}$, we obtain the following explicit formula:

$$\frac{1}{m+1}\widehat{\mathcal{G}}_{m+1}^{(k)}(x;\lambda,\rho,u,a,b)$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{n=0}^{\infty}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)\binom{m-i}{j}\binom{m}{i}(x)_{j,\rho}\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{m-i-j}.$$

Furthermore, using the arithmetic-geometric series formula in [12, p.245], we can write

$$\sum_{n=0}^{\infty}\left(\frac{u}{\lambda}\right)^n n^{m-i-j}=\frac{A_{m-i-j}\left(\frac{u}{\lambda}\right)}{\left(1-\frac{u}{\lambda}\right)^{m-i-j+1}},$$

where $A_n(u)$ is the Eulerian polynomial

$$A_n(u)=\sum_{k=0}^{n}A(n,k)u^k \tag{30}$$

with $A(n,k)$, the Eulerian number, satisfying

$$A(n,k)=(n-k+1)A(n-1,k-1)+kA(n-1,k).$$

Thus,

$$\begin{aligned}&\frac{1}{m+1}\widehat{\mathcal{G}}^{(k)}_{m+1}(x;\lambda,\rho,u,a,b)\\&\quad=\sum_{i=0}^{m}\sum_{j=0}^{m-i}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)A_{m-i-j}\left(\frac{u}{\lambda}\right)\binom{m-i}{j}\binom{m}{i}\frac{(x)_{j,\rho}(-\log ab)^{m-i-j}}{\left(1-\frac{u}{\lambda}\right)^{m-i-j+1}}.\end{aligned}$$

To state formally this result, we have the following theorem.

Theorem 2.2. *The degenerate Apostol-Frobenius-type poly-Genocchi polynomials with parameters a and b are equal to*

$$\begin{aligned}&\frac{1}{m+1}\widehat{\mathcal{G}}^{(k)}_{m+1}(x;\lambda,\rho,u,a,b)\\&\quad=\sum_{i=0}^{m}\sum_{j=0}^{m-i}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)A_{m-i-j}\left(\frac{u}{\lambda}\right)\binom{m-i}{j}\binom{m}{i}\frac{(x)_{j,\rho}(-\log ab)^{m-i-j}}{\left(1-\frac{u}{\lambda}\right)^{m-i-j+1}},\end{aligned}$$

where $\widehat{\mathcal{G}}^{(k,\alpha)}_0(x;\lambda,\rho,u,a,b)=0$, $A_n(u)$ *is the Eulerian polynomial and* $\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)$ *satisfies* (29).

Now, let us extend the explicit formula to higher order degenerate Apostol-Frobenius-type poly-Genocchi polynomials with parameters a and b. First, we have to find the expansion of the following function:

$$\begin{aligned}&\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\\&\quad=t\sum_{m=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{n=0}^{\infty}\frac{1}{m!}\binom{m}{j}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{m-j}\right\}t^m.\end{aligned}$$

Raising this to power α gives

$$\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^\alpha$$
$$=t^\alpha\sum_{m=0}^{\infty}\sum_{k_1+k_2+\ldots+k_\alpha=m}\prod_{i=1}^{\alpha}\left\{\sum_{j=0}^{k_i}\sum_{n=0}^{\infty}\frac{1}{k_i!}\binom{k_i}{j}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{k_i-j}\right\}t^m.$$

Hence,

$$\sum_{m=0}^{\infty}\widehat{\mathcal{G}}_m^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^m}{m!}$$
$$=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^\alpha e_\rho^x(t)=t^\alpha\sum_{m=0}^{\infty}\sum_{q=0}^{m}\binom{m}{q}(x)_{q,\rho}(m-q)!$$
$$\times\sum_{k_1+k_2+\ldots+k_\alpha=m-q}\prod_{i=1}^{\alpha}\left\{\sum_{j=0}^{k_i}\sum_{n=0}^{\infty}\frac{1}{k_i!}\binom{k_i}{j}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{k_i-j}\right\}\frac{t^m}{m!}.$$

Note that, when $0\le m\le\alpha-1$, $\widehat{\mathcal{G}}_m^{(k,\alpha)}(x;\lambda,\rho,u,a,b)=0$. Now, we can further rewrite the preceding equation as follows:

$$\sum_{m=-\alpha}^{\infty}\widehat{\mathcal{G}}_{m+\alpha}^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^m}{(m+\alpha)!}$$
$$=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^\alpha e_\rho^x(t)=\sum_{m=0}^{\infty}\sum_{q=0}^{m}\binom{m}{q}(x)_{q,\rho}(m-q)!$$
$$\times\sum_{k_1+k_2+\ldots+k_\alpha=m-q}\prod_{i=1}^{\alpha}\left\{\sum_{j=0}^{k_i}\sum_{n=0}^{\infty}\frac{1}{k_i!}\binom{k_i}{j}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{k_i-j}\right\}\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ and using the arithmetic-geometric formula yield the following explicit formula.

Theorem 2.3. *The degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b are equal to*

$$\frac{1}{(m+\alpha)_\alpha}\widehat{\mathcal{G}}_{m+\alpha}^{(k,\alpha)}(x;\lambda,\rho,u,a,b)=\sum_{q=0}^{m}\binom{m}{q}(x)_{q,\rho}(m-q)!\sum_{k_1+k_2+\ldots+k_\alpha=m-q}\prod_{i=1}^{\alpha}\sum_{j=0}^{k_i}\frac{1}{k_i!}\binom{k_i}{j}$$
$$\times\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)A_{k_i-j}\left(\frac{u}{\lambda}\right)\frac{(-\log ab)^{k_i-j}}{\left(1-\frac{u}{\lambda}\right)^{k_i-j+1}}.$$

where $\widehat{\mathcal{G}}_m^{(k,\alpha)}(x;\lambda,\rho,u,a,b)=0$ for $m=0,1,\ldots,\alpha-1$, $A_n(u)$ is the Eulerian polynomial defined in (30) *and $\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)$ satisfies* (29).

Remark 2.4. It can easily be seen that, when $\alpha = 1$, the explicit formula in Theorem 2.3 reduces to that in Theorem 2.2.

3. Relation With Some Genocchi-type Polynomials

By giving special values to the parameters involved, $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ reduces to some interesting Genocchi-type polynomials.

(i) Using (23), when $k = 1$, (27) yields

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!} = \left(\frac{(1-u)t\ln ab}{\lambda b^t - ua^{-t}}\right)^{\alpha} e_{\rho}^{x}(t), \tag{31}$$

where the polynomials $\widehat{\mathcal{G}}_n^{(\alpha)}(x;\lambda,\rho,u,a,b) = \widehat{\mathcal{G}}_n^{(1,\alpha)}(x;\lambda,\rho,u,a,b)$ are called the degenerate Apostol-Frobenius-type Genocchi polynomials of higher order with parameters a, b and c. When $\alpha = 1$, (31) yields

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(1)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!} = \frac{(1-u)t\ln ab}{\lambda b^t - ua^{-t}} e_{\rho}^{x}(t), \tag{32}$$

where the polynomials $\widehat{\mathcal{G}}_n^{(1)}(x;\lambda,\rho,u,a,b)$ are called the degenerate Apostol-Frobenius-type Genocchi polynomials with parameters a and b.

(ii) When $x = 0$, equation (27) reduces to

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda,\rho,u,a,b)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,\rho}(\log_{\rho}(1+(1-u)t\ln ab))}{\lambda b^t - ua^{-t}}\right)^{\alpha}, \tag{33}$$

the degenerate Apostol-Frobenius-type poly-Genocchi numbers with parameters a and b.

(iii) When $a = 1, b = e$, (27) will reduce to

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,\rho}(\log_{\rho}(1+(1-u)t))}{\lambda e^t - u}\right)^{\alpha} e_{\rho}^{x}(t), \tag{34}$$

and call $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u)$, the degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order. When $x = 0$, we get

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda,\rho,u)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,\rho}(\log_{\rho}(1+(1-u)t)}{\lambda e^t - u}\right)^{\alpha}, \tag{35}$$

the degenerate Apostol-Frobenius-type Genocchi numbers of higher order.

(iv) When $\rho \to 0$, equation (27) reduces to

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,0,u,a,b)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,0}(\log_0(1+(1-u)t\ln ab))}{\lambda b^t - ua^{-t}}\right)^{\alpha} e_0^x(t)$$
$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,u,a,b)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k}(\log(1+(1-u)t\ln ab))}{\lambda b^t - ua^{-t}}\right)^{\alpha} e^{xt}, \tag{36}$$

where the polynomials $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b)$ are the type 2 Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a,b and c with $c=e$ in [17].

(v) When $\lambda = 1$, (34) gives

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x;u,1,e)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t))}{e^t - u}\right)^{\alpha} e^{xt}. \tag{37}$$

which is the higher order version of equation (8) and are called the higher order poly-Genocchi polynomials. We may use $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;u)$ to denote $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;u,1,e)$.

(vi) When $k=1$, $a=1$ and $b=e$, (36) gives

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(1,\alpha)}(x;\lambda,u)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^{\alpha} e^{xt}, \tag{38}$$

and when $\lambda = 1$, (38) gives

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(1,\alpha)}(x;1,u)\frac{t^n}{n!} = \left(\frac{(1-u)t}{e^t - u}\right)^{\alpha} e^{xt},$$

where $\widehat{\mathcal{G}}_n^{(1,\alpha)}(x;\lambda,u) = \widehat{\mathcal{G}}_n^{(\alpha)}(x;\lambda,u)$ and $\widehat{\mathcal{G}}_n^{(1,\alpha)}(x;1,u) = \widehat{\mathcal{G}}_n^{(\alpha)}(x;u)$ are called the degenerate Apostol-Frobenius-type Genocchi polynomials and Frobenius-Genocchi polynomials of higher order in (4) and (2), respectively. Furthermore, when $\alpha = 1$, we have

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n(x;\lambda,u)\frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t - u}e^{xt}, \tag{39}$$

and

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n(x;u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t - u}e^{xt},$$

where $\widehat{\mathcal{G}}_n(x;\lambda,u)$ and $\widehat{\mathcal{G}}_n(x;u)$ are called the degenerate Apostol-Frobenius-type Genocchi polynomials and Frobenius-Genocchi polynomials in (4) and (2), respectively.

Now, let us consider some some relations of $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ with other Genocchi-type polynomials. First is to establish a kind of addition formula for $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ expressing them as polynomials in x.

Theorem 3.1. *The degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b satisfy the relation*

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)=\sum_{m=0}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)(x)_{m,\rho}. \tag{40}$$

Moreover, the expression of $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ as polynomial in x is given by

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)=\sum_{j=0}^{n}\widehat{\mathcal{G}}_{n,j,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b)x^j, \tag{41}$$

where

$$\widehat{\mathcal{G}}_{n,j,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b)=\sum_{m=j}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\widetilde{w}_\rho(m,j),$$

Proof. Using (33), we can write (27) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}&=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^\alpha e_\rho^x(t)\\&=\left(\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda,\rho,u,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\rho}\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)(x)_{m,\rho}\right)\frac{t^n}{n!}\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (40). To prove (41), we first recall that the r-Whitney numbers of the first kind, denoted by $w_{m,r}(n,k)$ were defined by Mező [38] by means of the following horizontal generating function:

$$m^n(x)_n=\sum_{j=0}^{n}w_{m,r}(n,j)(mx+r)^j, \tag{42}$$

where $(x)_n=x(x-1)(x-2)\dots(x-n+1)$. Replacing x with x/m and letting $r=0$ yield

$$(x)_{n,m}=\sum_{j=0}^{n}\widetilde{w}_m(n,j)x^j,$$

where $\widetilde{w}_m(n,j) = w_{m,0}(n,j)$, a certain of generalization of Stirling numbers of the first kind, i.e. $S_1(n,j) = \widetilde{w}_0(n,j)$. Using (42), equation (40) can further be written as polynomial in x

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b) = \sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\widetilde{w}_\rho(m,j)\right\}x^j,$$

with coefficients

$$\widehat{\mathcal{G}}_{n,j,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b) = \sum_{m=j}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\widetilde{w}_\rho(m,j),$$

the convolution of $\widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda,\rho,u,a,b)$ and $\widetilde{w}_\rho(n,k)$.

The relation in (41) is useful in constructing the orthogonal version of $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ using Gram-Schmidt process. It can easily be verified from Theorem 2.3 that

$$\widehat{\mathcal{G}}_m^{(k,\alpha)}(x;\lambda,\rho,u,a,b) = 0, m = 0,1,\ldots,\alpha-1, \quad \widehat{\mathcal{G}}_\alpha^{(k,\alpha)}(x;\lambda,\rho,u,a,b) = 1.$$

Let $\phi_\alpha(x),\phi_{\alpha+1}(x),\ldots,\phi_s(x)$ be the orthogonal version of of poly-Genocchi polynomials obtained from Gram-Schmidt process in which the polynomial is orthogonal with respect to the inner product

$$< f, g >= \int_0^1 w(x)f(x)g(x)dx.$$

Then

$$\phi_{s+1} = \widehat{\mathcal{G}}_{s+1}^{(k,\alpha)}(x;\lambda,\rho,u,a,b) - \sum_{i=1}^{s}\lambda_i\phi_i(x) \tag{43}$$

satisfies

$$< \phi_{s+1}, \phi_j >= \int_0^1 w(x)\phi_{s+1}(x)\phi_j dx = 0, \quad j = 0,1,\ldots,s$$

with

$$\lambda_j = \frac{< \widehat{\mathcal{G}}_{s+1}^{(k,\alpha)}(x;\lambda,\rho,u,a,b), \phi_j >}{< \phi_j, \phi_j >}.$$

Clearly, $\phi_i(x) = 0$ when $0 \le i \le \alpha-1$ and $\phi_\alpha(x) = 1$. Then,

$$\begin{aligned}
\phi_{\alpha+1}(x) &= \widehat{\mathcal{G}}_{\alpha+1}^{(k,\alpha)}(x;\lambda,\rho,u,a,b) - \sum_{i=1}^{\alpha}\lambda_i\phi_i(x)\\
&= \sum_{j=0}^{\alpha+1}\widehat{\mathcal{G}}_{\alpha+1,j,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b)x^j - \frac{\int_0^1\sum_{j=0}^{\alpha+1}\widehat{\mathcal{G}}_{\alpha+1,j,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b)x^j\phi_\alpha(x)dx}{\int_0^1(\phi_\alpha(x))^2dx}\phi_\alpha(x)\\
&= \sum_{j=0}^{\alpha+1}\widehat{\mathcal{G}}_{\alpha+1,j,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b)\left(x^j - \frac{1}{j+1}\right)
\end{aligned}$$

$$= \widehat{\mathcal{G}}^{(k,\alpha)}_{\alpha+1,0,\widetilde{w}}(\lambda,\rho,u,a,b)\left(x-\frac{1}{2}\right)+\widehat{\mathcal{G}}^{(k,\alpha)}_{\alpha+1,1,\widetilde{w}}(\lambda,\rho,u,a,b)\left(x^2-\frac{1}{3}\right)$$
$$= 2\left(x-\frac{1}{2}\right).$$

$$\begin{aligned}
\phi_3(x) &= \widehat{\mathcal{G}}^{(k,\alpha)}_3(x;\lambda,\rho,u,a,b)-\sum_{i=1}^{2}\lambda_i\phi_i(x)\\
&= \sum_{j=0}^{3}\widehat{\mathcal{G}}^{(k,\alpha)}_{3,j,\widetilde{w}}(\lambda,\rho,u,a,b)x^j-\frac{\int_0^1\sum_{j=0}^{3}\widehat{\mathcal{G}}^{(k,\alpha)}_{3,j,\widetilde{w}}(\lambda,\rho,u,a,b)x^j\phi_2(x)dx}{\int_0^1(\phi_2(x))^2dx}\phi_2(x)\\
&\quad -\frac{\int_0^1\sum_{j=0}^{3}\widehat{\mathcal{G}}^{(k,\alpha)}_{3,j,\widetilde{w}}(\lambda,\rho,u,a,b)x^j\phi_1(x)dx}{\int_0^1(\phi_1(x))^2dx}\phi_1(x)\\
&= \sum_{j=0}^{3}\widehat{\mathcal{G}}^{(k,\alpha)}_{3,j,\widetilde{w}}(\lambda,\rho,u,a,b)\left(x^j-\frac{3j(2x-1)}{(j+2)(j+1)}-\frac{1}{j+1}\right)\\
&= \widehat{\mathcal{G}}^{(k,\alpha)}_{3,1,\widetilde{w}}(\lambda,\rho,u,a,b)\left(x-\frac{2x-1}{2}-\frac{1}{2}\right)+\widehat{\mathcal{G}}^{(k,\alpha)}_{3,2,\widetilde{w}}(\lambda,\rho,u,a,b)\left(x^2-\frac{2x-1}{2}-\frac{1}{3}\right)\\
&= \widehat{\mathcal{G}}^{(k,\alpha)}_{3,2,\widetilde{w}}(\lambda,\rho,u,a,b)\left(x^2-x+\frac{1}{6}\right)=3x^2-3x+\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
&\phi_4(x) = \widehat{\mathcal{G}}^{(k,\alpha)}_4(x;\lambda,\rho,u,a,b)-\sum_{i=1}^{3}\lambda_i\phi_i(x)\\
&= \sum_{j=0}^{4}\widehat{\mathcal{G}}^{(k,\alpha)}_{4,j,\widetilde{w}}(\lambda,\rho,u,a,b)x^j-\frac{\int_0^1\sum_{j=0}^{4}\widehat{\mathcal{G}}^{(k,\alpha)}_{4,j,\widetilde{w}}(\lambda,\rho,u,a,b)x^j\phi_3(x)dx}{\int_0^1(\phi_3(x))^2dx}\phi_3(x)\\
&-\frac{\int_0^2\sum_{j=0}^{4}\widehat{\mathcal{G}}^{(k,\alpha)}_{4,j,\widetilde{w}}(\lambda,\rho,u,a,b)x^j\phi_2(x)dx}{\int_0^1(\phi_2(x))^2dx}\phi_2(x)-\frac{\int_0^2\sum_{j=0}^{4}\widehat{\mathcal{G}}^{(k,\alpha)}_{4,j,\widetilde{w}}(\lambda,\rho,u,a,b)x^j\phi_1(x)dx}{\int_0^1(\phi_1(x))^2dx}\phi_1(x)\\
&= \sum_{j=0}^{4}\widehat{\mathcal{G}}^{(k,\alpha)}_{4,j,\widetilde{w}}(\lambda,\rho,u,a,b)\left(x^j-\frac{9(20)j(j-1)\left(x^2-x+\frac{1}{6}\right)}{6(j+3)(j+2)(j+1)}-\frac{3j(2x-1)}{(j+2)(j+1)}-\frac{1}{j+1}\right)\\
&= \widehat{\mathcal{G}}^{(k,\alpha)}_{4,3,\widetilde{w}}(\lambda,\rho,u,a,b)\left(x^3-\frac{3}{2}x^2+\frac{3}{5}x-\frac{1}{20}\right)=4x^3-6x^2+\frac{12}{5}x-\frac{1}{5}
\end{aligned}$$

The next identity gives the relation between

$$\widehat{\mathcal{G}}^{(k,\alpha)}_n(x;\lambda,\rho,u,a,b)\ \text{ and }\ \widehat{\mathcal{G}}^{(k,\alpha)}_n(x;\lambda,\rho,u).$$

Theorem 3.2. *The degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b satisfy the relation,*

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)=\sum_{m=0}^{n}\binom{n}{m}(\ln a)^m(\ln ab)^{n-m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}\left(\frac{x}{\ln ab};\lambda,\rho,u\right). \tag{44}$$

Proof. Using (34), we can rewrite (27) as follows

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}&=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{a^{-t}(\lambda(ab)^t-u)}\right)^{\alpha}e_{\rho\ln ab}^{x\ln ab}(t\ln ab)\\&=a^{\alpha t}\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda e^{t\ln ab}-u}\right)^{\alpha}e_{\rho\ln ab}^{x\ln ab}(t\ln ab)\\&=\left(\sum_{n=0}^{\infty}(\ln ab)^n\widehat{\mathcal{G}}_n^{(k,\alpha)}\left(\frac{x}{\ln ab};\lambda,\rho,u\right)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(t\ln a)^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(\ln a)^m(\ln ab)^{n-m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}\left(\frac{x}{\ln ab};\lambda,\rho,u\right)\frac{t^n}{n!}\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result.

The next result is another form of addition formula for $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$.

Theorem 3.3. *The degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b satisfy the relation*

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x+y;\lambda,\rho,u,a,b)=\sum_{m=0}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(x;\lambda,\rho,u,a,b)(y)_{m,\rho}.$$

Proof. We can write (27) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x+y;\lambda,\rho,u,a,b)\frac{t^n}{n!}&=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}e_\rho^{x+y}(t)\\&=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}e_\rho^{x}(t)e_\rho^{y}(t)\\&=\left(\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y)_{n,\rho}\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(x;\lambda,\rho,u,a,b)(y)_{m,\rho}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem.

4. Differential and Integral Formulas

In the following theorem, certain differential equation will be established containing the degenerate Apostol-type poly-Genocchi polynomials of higher order with parameters a and b. Here, we consider $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ as polynomial in x.

Theorem 4.1. *The degenerate Apostol-Frobenius-type poly-Genocchi polynomials with parameters a and b satisfy the relation,*

$$\frac{d}{dx}\widehat{\mathcal{G}}_{n+1}^{(k,\alpha)}(x;\lambda,\rho,u,a,b)=\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{n-j}}{n-j+1}\rho^{n-j}\widehat{\mathcal{G}}_{j}^{(k,\alpha)}(x;\lambda,\rho,u,a,b). \tag{45}$$

Proof. Applying the first derivative to equation (27) with respect to x and using (15) yield

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{d}{dx}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}\\&=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{(\lambda b^t-ua^{-t})}\right)^{\alpha}e_\rho^x(t)\log(1+\rho t)^{1/\rho}\\&=t\left(\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\frac{(\rho t)^n}{n!}\right),\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{d}{dx}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}\\&=t\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\widehat{\mathcal{G}}_j^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{(-1)^{n-j}}{n-j+1}\rho^{n-j}\frac{t^n}{n!}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{1}{n+1}\frac{d}{dx}\widehat{\mathcal{G}}_{n+1}^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\widehat{\mathcal{G}}_j^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{(-1)^{n-j}}{n-j+1}\rho^{n-j}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired differential identity.

Remark 4.2. When $\rho\to 0$, equation (45) reduces to the following differential identity

$$\frac{d}{dx}\widehat{\mathcal{G}}_{n+1}^{(k,\alpha)}(x;\lambda,u,a,b)=(n+1)\widehat{\mathcal{G}}_{n}^{(k,\alpha)}(x;\lambda,u,a,b), \tag{46}$$

where $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,u,a,b)$ is the type 2 Apostol-Frobenius-type poly-Genocchi polynomials in (36). Equation (46) was used to classify $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,u,a,b)$ as an Appell polynomial.

To derive the integral formula

$$GI_n = \int \widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,u,a,b)dx,$$

we need to consider the following lemma.

Lemma 4.3. *If*

$$a_n = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{n-j+1}b_j,$$

then

$$b_n - na_{n-1} = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}b_j. \tag{47}$$

Proof. Using the fact that

$$\binom{n+1}{j}\frac{1}{n+1} = \binom{n}{j}\frac{1}{n-j+1},$$

we have

$$-(n+1)a_n = \sum_{j=0}^{n}(-1)^{n+1-j}\binom{n+1}{j}\frac{1}{n-j+1}b_j,$$

which is equivalent to (47).

To compute the value of the integral GI_n, we can use the following recurrence relation.

Theorem 4.4. *The integral GI_n satisfies the following recurrence relation*

$$GI_n = -\frac{\rho^n}{n+1}\left\{\sum_{j=0}^{n-1}\binom{n+1}{j}\frac{1}{\rho^j}GI_j - \sum_{j=0}^{n+1}\binom{n+1}{j}\frac{j}{\rho^{j-1}}\widehat{\mathcal{G}}_j^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\right\}.$$

Proof. Integrating both sides of (45) gives

$$\frac{1}{\rho^n}\widehat{\mathcal{G}}_{n+1}^{(k,\alpha)}(x;\lambda,\rho,u,a,b) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{n-j+1}\frac{1}{\rho^j}\int \widehat{\mathcal{G}}_j^{(k,\alpha)}(x;\lambda,\rho,u,a,b)dx.$$

Applying Lemma 4.3 yields

$$\frac{1}{\rho^n}GI_n - n\frac{1}{\rho^{n-1}}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{\rho^j}GI_j.$$

Using the inversion formula

$$a_n = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}b_j \Longleftrightarrow b_n = \sum_{j=0}^{n}\binom{n}{j}a_j,$$

we have

$$\frac{1}{\rho^n}GI_n = \sum_{j=0}^{n}\binom{n}{j}\left\{\frac{1}{\rho^j}GI_j - j\frac{1}{\rho^{j-1}}\widehat{\mathcal{G}}_j^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\right\}.$$

Thus,

$$\sum_{j=0}^{n-1}\binom{n}{j}\frac{1}{\rho^j}GI_j = \sum_{j=0}^{n}\binom{n}{j}\frac{j}{\rho^{j-1}}\widehat{\mathcal{G}}_j^{(k,\alpha)}(x;\lambda,\rho,u,a,b).$$

$$n\frac{1}{\rho^{n-1}}GI_{n-1} = -\sum_{j=0}^{n-2}\binom{n}{j}\frac{1}{\rho^j}GI_j + \sum_{j=0}^{n}\binom{n}{j}\frac{j}{\rho^{j-1}}\widehat{\mathcal{G}}_j^{(k,\alpha)}(x;\lambda,\rho,u,a,b).$$

This completes the proof of the theorem.

Remark 4.5. With the aid of the explicit formula in Theorem 2.3, one can easily compute the value of GI_n recursively.

5. Connections with Some Special Polynomials

Here, some connections of the higher order degenerate Apostol-type poly-Genocchi polynomials $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ with other well-known special numbers and polynomials will be established.

Carlitz [10, 11] defined the degenerate Bernoulli polynomials, denoted by $\beta_{n,\rho}(x)$, as follows:

$$\frac{t}{e_\rho(t)-1}e_\rho^x(t) = \sum_{n=0}^{\infty}\beta_{n,\rho}(x)\frac{t^n}{n!}. \tag{48}$$

One may extend this to higher order degenerate Bernoulli polynomials, which can be defined as follows

$$\left(\frac{t}{e_\rho(t)-1}\right)^s e_\rho^x(t) = \sum_{n=0}^{\infty}\beta_{n,\rho}^{(s)}(x)\frac{t^n}{n!}. \tag{49}$$

The degenerate Frobenius-Euler polynomials of higher order, denoted by $H_{n,\rho}^{(s)}(x;\mu)$, are defined in [42] as follows

$$\left(\frac{1-\mu}{e_\rho(t)-\mu}\right)^s e_\rho^x(t) = \sum_{n=0}^{\infty}H_{n,\rho}^{(s)}(x;\mu)\frac{t^n}{n!}. \tag{50}$$

When $s=1, w=0$, (50) gives $E_n^{(s)}(x;\mu,\lambda)$, the Apostol-type Frobenius-Euler polynomials in [41]. Now, if $\lambda=0$, we can define the Frobenius-Euler polynomials, denoted by $E_{n,H}^{(s)}(x;\mu)$, as follows:

$$\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{xt} = \sum_{n=0}^{\infty}E_n^{(s)}(x;\mu)\frac{t^n}{n!}. \tag{51}$$

The following theorem contains an identity that relates the degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c to the degenerate Stirling numbers of the first kind $S_{1,\rho}(n,k)$ in (18). Here, it is important to note that if $(c_0, c_1, \ldots, c_j, \ldots)$ is any sequence of numbers and l is a positive integer, then

$$\begin{aligned}\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!}\right)^l &= \prod_{i=1}^{l}\left(\sum_{n_i=0}^{\infty} \frac{c_{n_i}}{n_i!} t^{n_i}\right)\\ &= \sum_{n=0}^{\infty}\left\{\sum_{n_1+n_2+\ldots+n_\alpha=n}\prod_{i=1}^{l} c_{n_i}\binom{n}{n_1, n_2, \ldots, n_\alpha}\right\}\frac{t^n}{n!}. \end{aligned} \tag{52}$$

(see [12]). Now, we are ready to introduce the following theorem.

Theorem 5.1. *The degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c satisfies the relation,*

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x; \lambda, \rho, u, a, b) = \sum_{j=0}^{n}\binom{n}{j}(\ln ab)^{n-j}\widehat{\mathcal{G}}_{n-j}^{(\alpha)}\left(\frac{x}{\ln ab}; \lambda, \rho, u\right) d_j \tag{53}$$

where

$$\begin{aligned} d_j &= \sum_{n_1+n_2+\ldots+n_\alpha=j}\prod_{i=1}^{\alpha} c_{n_i}\binom{j}{n_1, n_2, \ldots, n_\alpha}\\ c_j &= \sum_{m=0}^{j}(-1)^{m+j+1}\frac{((1-u)\ln ab)^j (1)_{m+1,\rho} S_{1,\rho}(j+1, m+1)}{(j+1)(m+1)^{k-1}}. \end{aligned}$$

Proof. Now, (27) can be written as

$$\begin{aligned} &\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x; \lambda, \rho, u, a, b)\frac{t^n}{n!}\\ &\quad= \frac{e_\rho^x(t)}{(\lambda b^t - ua^{-t})^\alpha}\left(\sum_{m=1}^{\infty}\frac{(1)_{m,\rho}}{m^{k-1}}\frac{(\log_\rho(1+(1-u)t\ln ab))^m}{m!}\right)^\alpha\\ &\quad= \frac{e_\rho^x(t)}{(\lambda b^t - ua^{-t})^\alpha}\left(\sum_{m=0}^{\infty}\frac{(1)_{m+1,\rho}}{(m+1)^{k-1}}\frac{(\log_\rho(1+(1-u)t\ln ab))^{m+1}}{(m+1)!}\right)^\alpha\\ &\quad= \frac{e_\rho^x(t)}{(\lambda b^t - ua^{-t})^\alpha}\left(\sum_{m=0}^{\infty}\frac{(1)_{m+1,\rho}}{(m+1)^{k-1}}\sum_{j=m+1}^{\infty}S_{1,\rho}(j, m+1)\frac{((1-u)t\ln ab)^j}{j!}\right)^\alpha\\ &\quad= \frac{e_\rho^x(t)}{(\lambda b^t - ua^{-t})^\alpha}\left(\sum_{m=0}^{\infty}\frac{(1)_{m+1,\rho}}{(m+1)^{k-1}}\sum_{j=m}^{\infty}S_{1,\rho}(j+1, m+1)\frac{((1-u)t\ln ab)^{j+1}}{(j+1)!}\right)^\alpha \end{aligned}$$

$$= \frac{e_\rho^x(t)}{(\lambda b^t - ua^{-t})^\alpha}\left(\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(1)_{m+1,\rho}}{(m+1)^{k-1}}S_{1,\rho}(j+1,m+1)\frac{((1-u)t\ln ab)^{j+1}}{(j+1)!}\right)^\alpha$$

$$= e_\rho^x(t)\left(\frac{(1-u)t\ln ab}{\lambda b^t - ua^{-t}}\right)^\alpha\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha,$$

where

$$c_j = \sum_{m=0}^{j}(-1)^{m+j+1}\frac{((1-u)\ln ab)^j(1)_{m+1,\rho}S_{1,\rho}(j+1,m+1)}{(j+1)(m+1)^{k-1}}.$$

Using (31), we get

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha.$$

Note that, using (52), $\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha$ can be expressed as

$$\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right)^\alpha = \sum_{n=0}^{\infty}d_n\frac{t^n}{n!},$$

where

$$d_n = \sum_{n_1+n_2+\ldots+n_\alpha=n}\prod_{i=1}^{\alpha}c_{n_i}\binom{n}{n_1,n_2,\ldots,n_\alpha}.$$

It follows that

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}\widehat{\mathcal{G}}_{n-j}^{(\alpha)}(x;\lambda,\rho,u,a,b)d_j\right\}\frac{t^n}{n!}.$$

Comparing the coefficients and using equation (44) complete the proof of the theorem.

Remark 5.2. When $\alpha = 1$, $d_j = c_j$.

The identities in the following theorem are derived using the fact that the polynomials $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b,)$ with parameters a and b satisfy the relation in (27).

Theorem 5.3. *The degenerate Apostol-type poly-Genocchi polynomials of higher order with parameters a, b satisfy the following explicit formulas:*

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$$
$$= \sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}S_{2,\rho}(l+s,s)\frac{\binom{n-l}{m}}{\binom{l+s}{s}}\beta_{m,\rho}^{(s)}(x)\widehat{\mathcal{G}}_{n-l-m}^{(k,\alpha)}(\lambda,\rho,u,a,b), \quad (54)$$

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$$
$$=\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,j,a,b)E_n^{(s)}(x\ln c;\mu). \tag{55}$$

Proof. Using (49), (27) may be expressed as

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}$$
$$=\left(\frac{(e_\rho(t)-1)^s}{s!}\right)\left(\frac{t^s e_\rho^x(t)}{(e_\rho(t)-1)^s}\right)\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^\alpha\frac{s!}{t^s}$$
$$=\left(\sum_{n=0}^{\infty}S_{2,\rho}(n+s,s)\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}\beta_{m,\rho}^{(s)}(x)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda,\rho,u,a,b)\frac{t^m}{m!}\right)\frac{s!}{t^s}$$
$$=\left(\sum_{n=0}^{\infty}S_{2,\rho}(n+s,s)\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\rho}^{(s)}(x\ln c)\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}$$
$$=\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}S_{2,\rho}(l+s,s)\frac{t^{l+s}}{(l+s)!}\sum_{m=0}^{n-l}\binom{n-l}{m}\beta_{m,\rho}^{(s)}(x)\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\frac{t^{n-l}}{(n-l)!}\right)\frac{s!}{t^s}.$$

This can further be written as

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}$$
$$=\left(\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{m=0}^{n-l}\left\{\begin{matrix}l+s\\ s\end{matrix}\right\}\frac{l!s!}{(l+s)!}\binom{n-l}{m}B_m^{(s)}(x\ln c)\widehat{\mathcal{G}}_{n-l-m}^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{n!}{(n-l)!l!}\frac{t^n}{n!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}S_{2,\rho}(l+s,s)\frac{\binom{n-l}{m}}{\binom{l+s}{s}}\beta_{m,\rho}^{(s)}(x)\widehat{\mathcal{G}}_{n-l-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (54).

Now, to prove relation (55), (27) may be expressed as

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}$$
$$=\left(\frac{(1-\mu)^s}{(e_\rho(t)-\mu)^s}e_\rho^x(t)\right)\left(\frac{(e_\rho(t)-\mu)^s}{(1-\mu)^s}\right)\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^\alpha$$
$$=\frac{1}{(1-\mu)^s}\left(\sum_{n=0}^{\infty}H_{n,\rho}^{(s)}(x;\mu)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\times\right.$$
$$\left.\times\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^\alpha(e_\rho(t))^j\right)$$

$$= \frac{1}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\left(\sum_{n=0}^{\infty}H_{n,\rho}^{(s)}(x\ln c;\mu)\frac{t^n}{n!}\right)\times$$
$$\times\left(\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda,\rho,u,j,a,b)\frac{t^n}{n!}\right)$$

$$= \frac{1}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,j,a,b)\times\right.$$
$$\left.\times H_{n,\rho}^{(s)}(x\ln c;\mu)\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,j,a,b)\times\right.$$
$$\left.\times H_{n,\rho}^{(s)}(x\ln c;\mu)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (55).

6. Conclusion and Recommendations

In this paper, a certain variation of poly-Genocchi polynomials, called the degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameter a, b and c was constructed using the concept of modified degenerate polylexponential function, Apostol-Genocchi polynomials and Frobenius polynomials. Some interesting properties and identities of these polynomials were explored parallel to those of the poly-Genocchi, poly-Euler and poly-Bernoulli polynomials. The paper was concluded by expressing these degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order in terms of degenerate Stirling numbers of the first and second kind, higher order degenerate Bernoulli polynomials, and higher order degenerate Frobenius-Euler polynomials.

For future research work, one may try investigate more identities and properties for $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ to describe further the structure of these polynomials that may eventually be used to find some applications of these polynomials to other areas in mathematics. For instance, it would be interesting to establish the orthogonal version of $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ as well as to derive new operational matrix based on these polynomials in order to provide possible application to solve some fractional differential equation (see [8]). Lastly, it would also be interesting to construct other variations of poly-Genocchi polynomials by mixing $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$ with 2-variable generalization of Hermite polynomials.

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