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Relations Between Vertex–Edge Degree Based Topological Indices and M_{ve}-Polynomial of $r-$Regular Simple Graph

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Abstract. One of the more exciting polynomials among the newly presented graph algebraic polynomials is the $M-$Polynomial, which is a standard method for calculating degree$-$based topological indices. In this paper, we define the $M_{ve}-$polynomials based on vertex–edge degree and derive various vertex–edge degree based topological indices from them. Thus, for any graph, we provide some relationships between vertex–edge degree topological indices. Also, we discuss the general $M_{ve}-$polynomial of $r-$regular simple graph. Finally, we computed the $M_{ve}-$polynomial of the 2$-$ary tree graph.

2020 Mathematics Subject Classifications: 05C05, 05C07, 05C10, 94C15
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1. Introduction

Let G be a connected simple graph and let w be a weight given to its vertices. That is, for each $v \in V(G)$ "$V = V(G)$ be the vertex set", $w(v)$ is a positive integer. A $W_{ve}-$polynomial of G is defined by:

$$W_{ve}(G; x, y) = \Sigma_{uv \in E(G)} x^{w(u)} y^{w(v)}, w(u) \leq w(v), \tag{1.1}$$

where "$E = E(G)$ be the edge set". Collecting all similar terms $x^i y^j$ we can rewrite this polynomial as:

$$W_{ve}(G; x, y) = \Sigma_{uv \in E(G)} n_{ij} x^i y^j, i \leq j, \tag{1.2}$$

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where n_{ij} is the number of all edges uv such that $w(u) = i$ and $w(v) = j$.

The degree of a vertex $u \in V$ is the number of edges incident on u, denoted as d_u. The neighborhood of a vertex $u \in V, N_G(u)$ is a set of all neighbors of u, i.e., $N_G(u) = \{v|uv \in E(G)\}$ and its called open neighborhood. The closed neighborhood of a vertex u, denoted by $N_G[u]$, is obtained by adding a vertex u to $N_G(u)$, that is, $N_G[u] = N_G(u) \cup \{u\}$. The δ_u denotes the degrees sum of neighbors of u in G.

It is clear that (1.1) is a general $W_{ve}-$polynomial. If $w(u) = d_u$ for all $u \in V(G)$, then (1.1) is $M-$polynomial of G, which may simplified as:

$$M(G; x, y) = \Sigma_{uv\in E(G)} m_{ij} x^i y^j, i \leq j, \tag{1.3}$$

where m_{ij} is the number of edges $uv \in E(G)$ such that $\{d_u, d_v\} = \{i, j\}$. This polynomial was first introduced by Deutsch and Klavžar [6].

And, if $w(u) = \delta_u$ for all $u \in V(G)$, then from (1.1), we get $NM-$polynomial of G:

$$NM(G; x, y) = \Sigma_{uv\in E(G)} m'_{ij} x^i y^j, i \leq j, \tag{1.4}$$

where m'_{ij} is the total number of edges $uv \in E(G)$ such that $\{\delta_u, \delta_v\} = \{i, j\}$. Mondal and others [13] developed a $M-$polynomial into a $NM-$polynomial of a graph G.

Now, for each $u \in V(G)$, τ_u is defined as the number of all edges in G incident to a vertex of $N_G[u]$. From (1.1), substituting $w(u) = \tau_u$, we get $M_{ve}-$polynomial:

$$M_{ve}(G) = \Sigma_{uv\in E(G)} x^{\tau_u} y^{\tau_v}, \tau_u \leq \tau_v, \tag{1.5}$$

Simplifying (1.5) by collecting similar terms, we get

$$M_{ve}(G) = \Sigma_{uv\in E(G)} c_{ij} x^{\tau_u} y^{\tau_v}, \tau_u \leq \tau_v,$$

in which c_{ij} is the number of all edges $uv \in E(G)$ such that $\{\tau_u, \tau_v\} = \{i, j\}$.

The terms vertex–edge degree of the graphs, τ_u were first introduced by Chellali and others [4]. The authors defined these novel degree concepts in relation to the vertex$-$edge domination parameters [4, 10]. The vertex–edge degree ($ve-$degree) concepts of the graphs were extensively used in chemical graph theory [3, 7, 8] and [19].

The $M-$polynomial is the most general polynomial that may generate a wide range of degree$-$based topological indices [5, 6, 9, 11–13, 15, 17] and [18].

From the above three definitions, we get some properties:

(i) $\Sigma_{i\le j} m_{ij} x^i y^j|_{x=y=1} = \Sigma_{i\le j} n_{ij} x^i y^j|_{x=y=1} = \Sigma_{i\le j} c_{ij} x^i y^j|_{x=y=1} = q$.

(ii) For any vertex u in a graph G, we have: $d_u \le \tau_u \le \delta_u$.

(iii) Let $f(h_u, h_v)$ be the index function, where $h_z \in \{d_z, \tau_z, \delta_z\}$. Then $f(d_u, d_v) \le f(\tau_u, \tau_v) \le f(\delta_u, \delta_v)$, where $f \in$ [first and second Zagreb, reduced first and second Zagreb, hyper Zagreb index, forgotten index, albertson index, sigma index].

Finally, there are many polynomials that have been found over the current century that have chemical applications; see [1, 2] and [14]

2. Some Relations Between Vertex – Edge Degree Topological Indices

In this section, we give some relations between vertex–edge degree topological indices for any graph G. Also, the lower and upper bounds for vertex–edge degree topological indices via $W_{ve}-$polynomial are determined; some of them are shown in Table 1. for any graph G.

Table 1: Some vertex–edge–degree based topological indices for $W_{ve}-$polynomial.

Topological index	Symbol Index	Formula $f(w(u), w(v))$	Derivation from $W_{ve}(G; x, y)$
First Zagreb	$W^1_{ve}(G)$	$\Sigma_{uv\in E(G)}(w(u) + w(v))$	$(D_x + D_y)W_{ve}(G; x, y)\|_{x=y=1}$
Second Zagreb	$W^2_{ve}(G)$	$\Sigma_{uv\in E(G)}(w(u)w(v))$	$(D_xD_y)W_{ve}(G; x, y)\|_{x=y=1}$
Reduced First Zagreb	$RW^1_{ve}(G)$	$\Sigma_{uv\in E(G)}(w(u) + w(v) - 2)$	$(D_x + D_y - 2)W_{ve}(G; x, y)\|_{x=y=1}$
Reduced Second Zagreb	$RW^2_{ve}(G)$	$\Sigma_{uv\in E(G)}(w(u) - 1)(w(v)) - 1)$	$(D_x - 1)(D_y - 1)W_{ve}(G; x, y)\|_{x=y=1}$
Hyper Zagreb index	$HypW_{ve}(G)$	$\Sigma_{uv\in E(G)}(w(u) + w(v))^2$	$D^2_x JW_{ve}(G; x, y)\|_{x=y=1}$
Forgotten index	$FW_{ve}(G)$	$\Sigma_{uv\in E(G)}((w(u))^2 + (w(v))^2)$	$(D^2_x + D^2_y)W_{ve}(G; x, y)\|_{x=y=1}$
Albertson index	$AlbW_{ve}(G)$	$\Sigma_{uv\in E(G)}\|w(u) - w(v)\|$	$D_x IW_{ve}(G; x, y)\|_{x=y=1}$
Sigma index	$\sigma W_{ve}(G)$	$\Sigma_{uv\in E(G)}(w(u) - w(v))^2$	$D^2_x IW_{ve}(G; x, y)\|_{x=y=1}$

Where the operator D_x and D_y on $W_{ve}(G; x, y)$ are defined as:

$D_xW_{ve}(G; x, y) = x\frac{\partial W_{ve}(G; x, y)}{\partial x}, D_yW_{ve}(G; x, y) = y\frac{\partial W_{ve}(G; x, y)}{\partial y}, JW_{ve}(G; x, y) = W_{ve}(G; x, x)$ and $IW_{ve}(G; x, y) = W_{ve}(G; x, x^{-1})$.

Theorem 2.1: Let G be any graph with order $p = |V(G)|$ and size $q = |E(G)|$. Then

1. $RW^1_{ve}(G) = W^1_{ve}(G) - 2q$.
2. $RW^2_{ve}(G) = W^2_{ve}(G) - W^1_{ve}(G) + q$.
3. $FW_{ve}(G) = HypW_{ve}(G) - 2W^2_{ve}(G)$.
4. $\sigma W_{ve}(G) = HypW_{ve}(G) - 4W^2_{ve}(G)$.
5. $\sigma W_ve(G) = 2FW_{ve}(G) - HypW_{ve}(G)$.

Proof:

1. From the definition of the reduced first Zagreb, we have:

$$RW_{ve}^1(G) = \Sigma_{uv\in E(G)}(w(u) + w(v) - 2)$$
$$= \Sigma_{uv\in E(G)}(w(u) + w(v)) - \Sigma_{uv\in E(G)}2$$
$$= W_{ve}^1(G) - 2q.$$

2. From the definition of the reduced second Zagreb, we have:

$$RW_{ve}^2(G) = \Sigma_{uv\in E(G)}(w(u) - 1)(w(v) - 1)$$
$$= \Sigma_{uv\in E(G)}w(u)w(v) - \Sigma_{uv\in E(G)}(w(u) + w(v)) + \Sigma_{uv\in E(G)}1$$
$$= R_{ve}^2(G) - R_{ve}^1(G) + q.$$

3. From the definition of the Forgotten index, we have:

$$FW_{ve}(G) = \Sigma_{uv\in E(G)}(w(u)^2 + w(v)^2)$$
$$= \Sigma_{uv\in E(G)}(w(u) + w(v))^2 - 2\Sigma_{uv\in E(G)}w(u)w(v)$$
$$= HypW_{ve}(G) - 2W_{ve}^2(G).$$

4. From the definition of the sigma index, we have:

$$\sigma W_{ve}(G) = \Sigma_{uv\in E(G)}(w(u) - w(v))^2$$
$$= \Sigma_{uv\in E(G)}((w(u))^2 + (w(v))^2) - 2\Sigma_{uv\in E(G)}w(u)w(v)$$
$$= FW_{ve}(G) - 2W_{ve}^2(G)$$
$$= HypW_{ve}(G) - 4W_{ve}^2(G).$$

5. From 3 and 4, we have:

$$\sigma W_{ve}(G) = 2FW_{ve}(G) - HypW_{ve}(G).$$

.

In the next theorem, the upper and lower boundaries will be found using the vertex−degree−based topological indices τ_u, $u \in V(G)$.

Theorem 2.2: Let G be any graph with order p and size q. Then

1. $2q \leq M_{ve}^1(G) \leq 2q^2$.
2. $q \leq M_{ve}^2(G) \leq q^3$.
3. $0 \leq RM_{ve}^1(G) \leq 2q(q-1)$.
4. $0 \leq RM_{ve}^2(G) \leq q(q-1)^2$.
5. $4q \leq HypM_{ve}(G) \leq 4q^3$.

6. $2q \leq FM_{ve}(G) \leq 2q^3$.

Proof:

Since $1 \leq \tau_u \leq q$, for all u in G, then

1. $2 \leq \tau_u + \tau_v \leq 2q$, this implies that

$$\Sigma_{uv\in E(G)}2 \leq \Sigma_{uv\in E(G)}(\tau_u + \tau_v) \leq \Sigma_{uv\in E(G)}2q, \text{ then}$$
$$2q \leq M^1_{ve}(G) \leq 2q^2.$$

2. $1 \leq \tau_u\tau_v \leq q^2$, this implies that

$$\Sigma_{uv\in E(G)}1 \leq \Sigma_{uv\in E(G)}(\tau_u\tau_v) \leq \Sigma_{uv\in E(G)}q^2, \text{ then}$$
$$q \leq M^2_{ve}(G) \leq q^3.$$

3. $0 \leq \tau_u + \tau_v - 2 \leq 2(q-1)$, this implies that

$$\Sigma_{uv\in E(G)}0 \leq \Sigma_{uv\in E(G)}(\tau_u + \tau_v - 2) \leq \Sigma_{uv\in E(G)}2(q-1), \text{ then}$$
$$0 \leq RM^1_{ve}(G) \leq 2q(q-1).$$

4. $0 \leq (\tau_u - 1)(\tau_v - 1) \leq (q-1)^2$, this implies that

$$\Sigma_{uv\in E(G)}0 \leq \Sigma_{uv\in E(G)}(\tau_u - 1)(\tau_v - 1) \leq \Sigma_{uv\in E(G)}(q-1)^2, \text{ then}$$
$$0 \leq RM^2_{ve}(G) \leq q(q-1)^2.$$

5. $4 \leq (\tau_u + \tau_v)^2 \leq 4q^2$, this implies that

$$\Sigma_{uv\in E(G)}4 \leq \Sigma_{uv\in E(G)}(\tau_u + \tau_v)^2 \leq \Sigma_{uv\in E(G)}4q^2, \text{ then}$$
$$4q \leq HypM_{ve}(G) \leq 4q^3.$$

6. $2 \leq (\tau_u)^2 + (\tau_v)^2 \leq 2q^2$, this implies that

$$\Sigma_{uv\in E(G)}2 \leq \Sigma_{uv\in E(G)}((\tau_u)^2 + (\tau_v)^2) \leq \Sigma_{uv\in E(G)}2q^2, \text{ then}$$
$$2q \leq FM_{ve}(G) \leq 2q^3.$$

Remark: We note that equality exists for all topological indices in Theorem 2.2, if $G = K_2$.

3. $M_{ve}-$Polynomials of $r-$Regular Graphs

In the next theorems, we discuss the general $M_{ve}-$polynomial of $r-$regular simple graph of size q.

Theorem 3.1: Let G is an $r-$regular simple graph G of size q, then

1. $M(G;x,y) = qx^r y^r$.
2. $NM(G;x,y) = qx^{r^2} y^{r^2}$.
3. If the graph G without triangle cycles, then $M_{ve}(G;x,y) = qx^{r^2} y^{r^2}$.

Proof:

1. Let G is $r-$regular graph with q edges, then every edge in G is an incident on two vertices which have r degree. Hence, $M(G;x,y) = qx^r y^r$.

2. Since every vertex u in $r-$regular graph G is adjacent r vertices, then $\delta_u = r^2$. Hence, $NM(G;x,y) = qx^{r^2} y^{r^2}$.

3. If G is an $r-$regular simple graph without triangle cycles, then $\tau_u = \tau_v = r^2$, for every edge $e = uv$ where $u, v \in V(G)$. Hence $M_{ve}(G;x,y) = qx^{r^2} y^{r^2}$.

Example 3.2: Let H_i be $3-$regular graph for all $i = 1, 2, 3$. See Figure 1.

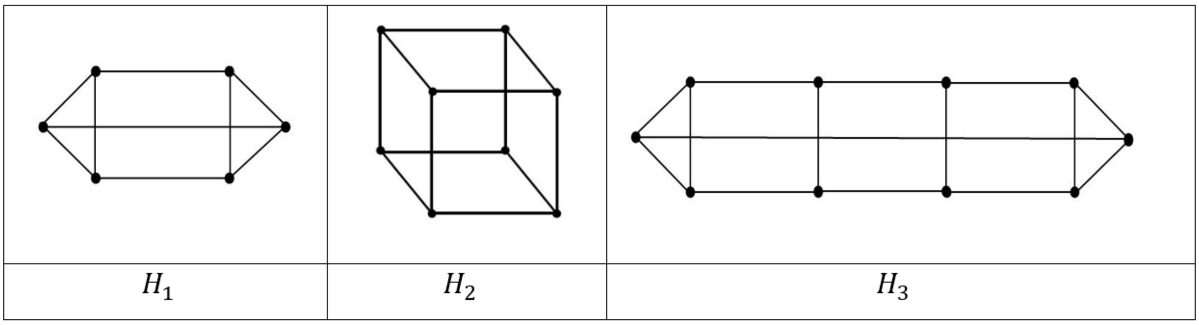

Figure 1: 3−regular graphs.

We note that:
$M(H_1;x,y) = 9x^3y^3, NM(H_1;x,y) = 9x^9y^9$ and $M_{ve}(H_1;x,y) = 9x^8y^8$.
$M(H_2;x,y) = 12x^3y^3$ and $NM(H_2;x,y) = M_{ve}(H_2;x,y) = 12x^9y^9$.
$M(H_3;x,y) = 15x^3y^3, NM(H_3;x,y) = 15x^9y^9$ and
$M_{ve}(H_3;x,y) = 7x^8y^8 + 4x^8y^9 + 4x^9y^9$.

It is very difficult to find a general formula for $M_{ve}-$polynomial of a regular graph, so some conditions were given on the regular graphs in order to obtain $M_{ve}-$polynomial.

Theorem 3.3: Let G is an $r-$regular simple graph such that every vertex in G lies on only one triangle cycle, then

$$M_{ve}(G;x,y) = qx^{r^2-1}y^{r^2-1}, \text{ where } q = |E(G)|.$$

Proof:

Let $e = uv$ be any edge in G where $u, v \in V(G)$, then the two vertices u and v are adjacent with third vertex, then $\tau_u = \tau_v = r^2 - 1$. Hence $M_{ve}(G;x,y) = qx^{r^2-1}y^{r^2-1}$, where $q = |E(G)|$.

Theorem 3.4: Let G is a $r-$regular simple graph such that any vertex in G lies at most on one triangular cycle. If h be the number of edges lies between any two vertices belong to triangular cycle (or cycles) and k be the number of edges which one of whose ends, but not both lies on a triangular cycle. Then

$$M_{ve}(G;x,y) = hx^{r^2-1}y^{r^2-1} + kx^{r^2-1}y^{r^2} + (q-h-k)x^{r^2}y^{r^2},$$

where $q = |E(G)|$.

Proof: Obvious.

Definition 3.5: Let G be a simple graph and a vertex $u \in V(G)$, we can rewrite δ_u as:

$$\delta_u = \Sigma_{z \in N(u)} d_z = d_u + 2\varepsilon_1 + \varepsilon_2,$$

where ε_1 be the number of edges of which both its ends belong to a triangular cycle and ε_2 be the number of edges which lies on the vertices are neighbors of a vertex u.

Theorem 3.6: Let G be a any graph without triangular cycle, then,

$$NM(G;x,y) = M_{ve}(G;x,y).$$

Proof:

For edge $e = uv$ where $u, v \in V(G)$, then $m'_{\delta_u\delta_v}x^{\delta_u}y^{\delta_v} = m'_{\delta_u\delta_v}x^{d_u+2\varepsilon_1+\varepsilon_2}y^{d_v+2\varepsilon_1+\varepsilon_2}$, by Definition 3.5. Since G is a graph without triangular cycles, then $\varepsilon_1 = 0$. Hence $m'_{\delta_u\delta_v}x^{\delta_u}y^{\delta_v} = m'_{\delta_u\delta_v}x^{d_u+\varepsilon_2}y^{d_v+\varepsilon_2} = c_{\delta_u\delta_v}x^{\tau_u}y^{\tau_v}$, by definition vertex – edge degree of the graph. Hence,

$$NM(G;x,y) = M_{ve}(G;x,y).$$

4. $M_{ve}-$Polynomials of $2-$Ary Tree Graph

Definition 4.1: [16] A rooted tree G is an acyclic connected graph with a special node that is called the root of the tree and every edge directly or indirectly originates from the root. An ordered rooted tree is a rooted tree where the children of each internal vertex is ordered. If every internal vertex of a rooted tree has not more than m children, it is called an $m-$ary tree. In this section, we determine the $M_{ve}-$polynomial of special case of $m-$ary tree is $2-$ary tree of n levels denoted by $\aleph_n$ and as shown in Figure 2.

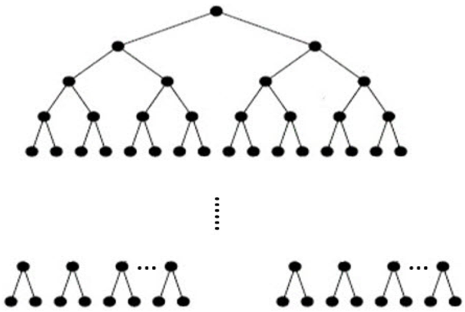

Figure 2: 2−Ary Tree graph $\aleph_n$.

Some Properties of a 2−Ary Tree Graph $\aleph_n$:

- At each level of i, the number of vertices are 2^i, for $0 \leq i \leq n$.
- The order and the size are $p(\aleph_n) = 2^{n+1} - 1$ and $q(\aleph_n) = 2^{n+1} - 2$, respectively.
- The rooted vertex of degree 2, the degrees of vertices at each level of i, $1 \leq i \leq n-1$ are 3 which represent the maximum degree " $\triangle\ (\aleph_n) = 3$" and the degrees of the last level are 1 which represent the minimum degree "$\delta(\aleph_n) = 1$".
- The maximum and minimum $ve-$degree are $\triangle_{ve}\ (\aleph_n) = 9$ and $\delta_{ve}(\aleph_n) = 3$, respectively.

Theorem 4.2: Let $\aleph_n$ be the $2-$ary tree graph of order $2^{n+1} - 1$, $n \geq 4$. Then,

$$M_{ve}(\aleph_n; x, y) = 2x^6y^8 + 4x^8y^9 + 2^n x^3y^5 + 2^{n-1}x^5y^9 + (2^{n-1} - 8)x^9y^9.$$

Proof:

From the Definition 4.1 and Figure 2 of $2-$ary tree graph $\aleph_n$, we can observe that the vertices are divided into five partitions:

$$|V_1| = |v \in V(\aleph_n) : \tau_v = 3| = 2^n,$$
$$|V_2| = |v \in V(\aleph_n) : \tau_v = 5| = 2^{n-1},$$
$$|V_3| = |v \in V(\aleph_n) : \tau_v = 6| = 2^0,$$
$$|V_4| = |v \in V(\aleph_n) : \tau_v = 8| = 2^1,$$
$$|V_5| = |v \in V(\aleph_n) : \tau_v = 9| = 2^{n-1} - 4.$$

The edges set of $2-ary$ tree graph $\aleph_n$ can be partitioned as $|E_{3,5}| = |uv \in E(\aleph_n) : \tau_u = 3 \text{ and } \tau_v = 5| = 2^n$, $|E_{5,9}| = |uv \in E(\aleph_n) : \tau_u = 5 \text{ and } \tau_v = 9| = 2^{n-1}, |E_{6,8}| = |uv \in E(\aleph_n) : \tau_u = 6 \text{ and } \tau_v = 8| = 2, |E_{8,9}| = |uv \in E(\aleph_n) : \tau_u = 8 \text{ and } \tau_v = 9| = 2^2, |E_{9,9}| = |uv \in E(\aleph_n) : \tau_u = \tau_v = 9| = |E(\aleph_n)| - |E_{3,5}| - |E_{5,9}| - |E_{6,8}| - |E_{8,9}| = 2^{n-1} - 8$.

Thus, the $M_{ve}-$polynomial of 2−ary tree graph $\aleph_n$ is

$$M_{ve}(\aleph_n; x, y) = \Sigma_{i\leq j} c_i j x^i y^j$$

$$= \Sigma_{3\leq 5} c_{35} x^3y^5 + \Sigma_{5\leq 9} c_{59} x^5y^9 + \Sigma_{6\leq 8} c_{68} x^6y^8 + \Sigma_{8\leq 9} c_{89} x^8y^9 + \Sigma_{9\leq 9} c_{99} x^9y^9$$

$$= \Sigma_{uv\in E_{3,5}} c_{35} x^3y^5 + \Sigma_{uv\in E_{5,9}} c_{59} x^5y^9 + \Sigma_{uv\in E_{6,8}} c_{68} x^6y^8 + \Sigma_{uv\in E_{8,9}} c_{89} x^8y^9 + \Sigma_{uv\in E_{9,9}} c_{99} x^9y^9.$$

$$= |E_{3,5}|x^3y^5 + |E_{5,9}|x^5y^9 + |E_{6,8}|x^6y^8 + |E_{8,9}|x^8y^9 + |E_{9,9}|x^9y^9$$

$$= 2^n x^3y^5 + 2^{n-1}x^5y^9 + 2x^6y^8 + 4x^8y^9 + (2^{n-1} - 8)x^9y^9.$$

Corollary 4.3: Let $\aleph_n$ be the $2-ary$ tree graph of order $2^{n+1} - 1$, $n \geq 4$. Then

1. $M^1_{ve}(\aleph_n) = 8(2^{n+1} + 2^n - 6)$.
2. $M^2_{ve}(\aleph_n) = 3(42 \times 2^{n-1} + 5 \times 2^n - 88)$.
3. $RM^1_{ve}(\aleph_n) = (2^{n+4} + 2^{n+3} - 2^{n+2} - 44)$.
4. $RM^2_{ve}(\aleph_n) = 2(2^{n+2} + 48 \times 2^{n-1} - 109)$.
5. $HypM_{ve}(\aleph_n) = 4(2^{n+4} + 130 \times 2^{n-1} - 261)$.
6. $FM_{ve}(\aleph_n) = 2(17 \times 2^n + 134 \times 2^{n-1} - 258)$.
7. $AlbM_{ve}(\aleph_n) = 4(2^n + 2)$.
8. $\sigma M_{ve}(\aleph_n) = 4(2^{n+1} + 2^n + 3)$.

Remark 4.4: From Theorem 3.5, we get

$$M_{ve}(\aleph_n; x, y) = NM(\aleph_n; x, y) = 2x^6y^8 + 4x^8y^9 + 2^n x^3y^5 + 2^{n-1}x^5y^9 + (2^{n-1} - 8)x^9y^9.$$

5. Conclusion:

In this paper, we have given a new polynomial based on the terms of a vertex–edge degree of a graph G. From this polynomial we proved many properties and proved that it's equal to the neighbor polynomial when G there is a graph without a triangular cycle. We also discover a vertex−edge degree polynomial for an r−regular graph under certain conditions.

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