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# Construction of Fourier Series Expansion of Apostol-Frobenius-Type Tangent and Genocchi Polynomials of Higher-order

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**Abstract.** In this study, the Fourier series expansions of the Apostol-Frobenius type of Tangent and Genocchi polynomials of higher order are derived using the Cauchy residue theorem. Some novel and intriguing results are obtained by applying the Fourier series expansion of these types of polynomials.

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**Key Words and Phrases**: Cauchy residue theorem; Fourier series, Tangent polynomials, Bernoulli polynomials, Genocchi polynomials

## 1. Introduction

There are numerous well-known special functions, numbers, and polynomials, such the Bernoulli, Tangent, and Genocchi numbers and polynomials, and derivative polynomials, that are well studied in the current literature due to their broad applications ranging from number theory and combinatorics to other fields of applied mathematics[5, 6, 13]. In the literature, other variants and extensions of these functions, numbers, and polynomials have appeared. Some versions have been created by combining two or three special functions, integers, or polynomials. Poly-Bernoulli numbers and polynomials, for example, were created by combining the notions of polylogarithm and Bernoulli numbers and polynomials[9, 14].

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The principles of Apostol, Frobenius, Genocchi, and Euler polynomials were combined to create the Apostol-Genocchi polynomials, Frobenius-Euler polynomials, Frobenius-Genocchi polynomials, and Apostol-Frobenius-type poly-Genocchi polynomials in the paper of Ryoo et.al.[10–12]. Another intriguing combination of special polynomials may be created by combining the principles of Apostol and Frobenius polynomials with Tangent, Bernoulli, and Genocchi polynomials, which will be the topic direction of this paper. This will be carried out using the Fourier series expansion method[7]. Currently, there was no literature or related works that mentioned the Fourier series expansion of Apostol-Frobenius: Tangent, Bernoulli, and Genocchi polynomials that were accessible at the time of the research.

Fourier series is widely known as an expansion of a periodic function $f(x)$ in terms of an infinite sum of sine and cosine functions. It makes use of the orthogonality relationships of the sine and cosine functions. Fourier series is expressed as[1]

$$s(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi}{T}nx\right) + b_n \sin\left(\frac{2\pi}{T}nx\right)\right] \tag{1}$$

where T is the function's period. The above expression can also be cast into its exponential form as follows:

$$s(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i2\pi nx/T} \tag{2}$$

where the coefficients $c_n$ are computed as

$$c_n = \frac{1}{T}\int_0^T e^{-i2\pi nx/T} \cdot s(t)dt \tag{3}$$

The Fourier expansion of several well-known polynomials has recently piqued the curiosity of mathematicians. The Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials are given by Lou (2009). Using the Lipschitz summation formula, Lou derives the Fourier expansions and integral representation for Genocchi polynomials the same year. Araci-Acikgoz (2018) made a significant finding about the Fourier expansion of the Apostol Frobenius-Euler and Genocchi polynomials. With this motivation, we are interested in determining the Fourier series expansions of higher order Apostol-Frobenius-type Tangent and Genocchi polynomials using the Cauchy residue theorem and a complex integral over a contour[2], which they found to be particularly useful.

## 2. Main Results

In this section, we use the Cauchy Residue theorem and Bayad's method[2] in evaluating the complex integral over a circle $C$ to obtain the Fourier series expansion for the Frobenius type of Apostol-Tangent and Apostol-Genocchi polynomials.

### 2.1. Fourier expansion and Integral representation of Apostol-Frobenius-Tangent Polynomials:

The Tangent polynomials with complex argument $x$ are defined as coefficients of the following generating function[8]

$$\sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!} = \left(\frac{2}{e^{2t}+1}\right)e^{xt} \tag{4}$$

where $T_n(0) = T_n$. The Apostol-Frobenius-type Tangent polynomials, which are another variation of the Tangent polynomials, are defined as follows:

$$\sum_{n=0}^{\infty} T_n(x;u,\lambda)\frac{t^n}{n!} = \frac{1-u}{\lambda e^{2t}-u}e^{xt} \tag{5}$$

where $u, \lambda \in \mathbb{C}$ with $u \neq 1, \lambda \neq 1$ and $u \neq \lambda$. By Cauchy Integral formula, we have

$$\frac{T_n(x;u,\lambda)}{n!} = \frac{1}{2\pi i}\int_C \frac{1-u}{\lambda e^{2t}-u}e^{xt}\frac{dt}{t^{n+1}} \tag{6}$$

Consider now the function inside the integral

$$f(t) = \frac{1-u}{\lambda e^{2t}-u}\frac{e^{xt}}{t^{n+1}} \tag{7}$$

which has a pole at $t = 0$ of order $n + 1$. We then find the other poles of the function $f_n(t)$ as follows:

$$\begin{aligned}
\lambda e^{2t} - u &= 0\\
\lambda e^{2t} &= \frac{u}{\lambda}\\
2t &= \log\frac{u}{\lambda} + 2k\pi i\\
t_k := t &= \log\left(\frac{u}{\lambda}\right)^{1/2} + k\pi i, \qquad \text{for } k \in \mathbb{Z}
\end{aligned}$$

By Cauchy Residue Theorem, we have

$$\frac{1}{2\pi i}\int_{C_N} f_n(t)dt = \text{Res}(f(t), t = 0) + \sum_{k\in\mathbb{Z}}(f(t), t = t_k) \tag{8}$$

In Eq. (8), we integrate $f_n(t)$ around the circle with radius $(N + \epsilon)\pi$ where $\epsilon \in \mathbb{R}$. That is, $\epsilon\pi i \pm \log\left(\frac{u}{\lambda}\right)^{1/2} \neq 0 \pmod{2\pi i}$. This radius guarantees that the circle $C_N$ does not pass through any of the poles $t_k$. The following lemma contains the limit of the integral in Eq. (8) as $N \to \infty$

**Lemma 1.** *Let* $u, \lambda \in \mathbb{C}\{0,1\}$ *with* $|\lambda| \neq |u|$. *For* $0 < x \leq 1$

$$\lim_{N\to\infty} \int_{C_N} f_n(t)dt = \int_{C_N} \frac{1-u}{\lambda e^{2t}-u} e^{xt} \frac{dt}{t^{n+1}} = 0$$

*where* $C_N = \{t : |t| = (N+\epsilon)\pi$ *and* $\epsilon \in \mathbb{R}, \left(\epsilon\pi i \pm \log(u/\lambda)^{1/2}\right) \neq 0\}$

*Proof.* We first take the modulus of the integral given as

$$\left|\int_{C_N} \frac{1-u}{\lambda e^{2t}-u} e^{xt} \frac{dt}{t^{n+1}}\right| \leq \int_{C_N} \frac{|1-u||e^{xt}|}{|\lambda e^{2t}-u||t^{n+1}||dt|} \leq \int_{C_N} \frac{|e^{xt}|}{|\lambda e^{2t}-u|}|dt|$$

Consider the function in the last integral

$$\begin{aligned} \frac{|e^{xt}|}{|\lambda e^{2t}-u|} &= \frac{|e^{xt}|}{|-u|\left|-\frac{u}{\lambda}e^{2t}+1\right|} \\ &\leq \frac{e^{\mathrm{Re}(t)}}{|\alpha||u||e^{2t}+\frac{1}{\alpha}|}, \quad \text{where } \alpha = -\frac{\lambda}{u} \\ &\leq \frac{1}{|-\lambda|} \end{aligned}$$

So that,

$$\left|\int_{C_N} \frac{1-u}{\lambda e^{2t}-u} e^{xt} \frac{dt}{t^{n+1}}\right| \leq \frac{1}{|-\lambda|} \int_{C_N} \frac{|dt|}{|t^{n+1}|} = \frac{2^{n+1}}{|-\lambda|((2N+\epsilon)\pi)^{n+1}}$$

As $N \to \infty$ for $n \geq 1$

$$\int_{C_N} \frac{1-u}{\lambda e^{2t}-u} e^{xt} \frac{dt}{t^{n+1}} \to 0$$

Using Lemma 1, Eq. (8) becomes

$$\mathrm{Res}(f_n(t), t=0) = -\sum_{k\in\mathbb{Z}} \mathrm{Res}(f_n(t), t=t_k)$$

We then compute the $\mathrm{Res}(f_n(t), t=0,)$ and $\sum_{k\in\mathbb{Z}} \mathrm{Res}(f_n(t), t=t_k)$ to obtain the Fourier series expansion of Apostol-Frobenius-Tangent polynomials. The following theorem explicitly shows the Fourier series representation of the said polynomials

**Theorem 1.** *Let* $u, \lambda \in \mathbb{C}\{0,1\}$ *with* $|\lambda| \neq |u|$. *For* $0 < x \leq 1$

$$T_n(x;u,\lambda) = \frac{n!}{2}\frac{u-1}{u}\left(\frac{u}{\lambda}\right)^{\frac{1}{2}x} \sum_{k\in\mathbb{Z}} \frac{e^{i\pi kx}}{2\lambda\left[\log\left(\frac{u}{\lambda}\right)^{1/2}+k\pi i\right]^{n+1}} \tag{9}$$

*Proof.* We compute $\text{Res}(f_n(t), t = 0)$ and $\sum_{k\in\mathbb{Z}}\text{Res}(f_n(t), t = t_k)$ as follows:

$$\begin{aligned}\text{Res}(f_n(t), t = 0) &= \lim_{t\to 0}\frac{1}{n!}\frac{d^n}{dt^n}(t-0)^{n+1}\frac{1}{t^{n+1}}\sum_{m=0}^{\infty}T_m(x;,u,\lambda)\frac{t^m}{m!}\\ &= \lim_{t\to 0}\frac{1}{n!}\frac{d^n}{dt^n}\sum_{m=0}^{\infty}T_m(x;u,\lambda)\\ &= \lim_{t\to 0}\frac{1}{n!}\sum_{m=0}^{\infty}T_m(x;u,\lambda)\frac{t^{m-n}}{(m-n)!}\\ &= \frac{T_m(x;u,\lambda)}{n!} \qquad (10)\end{aligned}$$

and

$$\begin{aligned}\text{Res}(f_n(t), t = t_k) &= \lim_{t\to t_k}(t-t_k)\frac{1-u}{\lambda e^{2t}-u}e^{xt}\frac{1}{t^{n+1}}\\ &= \frac{(1-u)}{t_k^{n+1}}e^{xt_k}\lim_{t\to t_k}\frac{t-t_k}{\lambda e^{2t}-u}\\ &= \frac{1-u}{t_k^{n+1}}e^{xt_k}\lim_{t\to t_k}\frac{1}{2\lambda e^{2t}}\\ &= \frac{(1-u)e^{(x-2)t_k}}{2\lambda t_k^{n+1}}, \quad \text{where } t_k = \log\left(\frac{u}{\lambda}\right)^{1/2} + k\pi i\\ &= \frac{(1-u)\exp\left\{(x-2)\left[\log(u/\lambda)^{1/2}+k\pi i\right]\right\}}{2\lambda\left[\log\left(\frac{u}{\lambda}\right)^{1/2}+k\pi i\right]^{n+1}}\\ &= \frac{(1-u)e^{x\log(u/\lambda)^{1/2}}e^{k\pi ix}e^{-2\log(u/\lambda)^{1/2}}e^{-2k\pi i}}{2\lambda\left[\log\left(\frac{u}{\lambda}\right)^{1/2}+k\pi i\right]^{n+1}}\\ &= \frac{(1-u)(u/\lambda)^{\frac{1}{2}x-1}e^{k\pi ix}}{2\lambda\left[\log\left(\frac{u}{\lambda}\right)^{1/2}+k\pi i\right]^{n+1}} \qquad (11)\end{aligned}$$

Combining the results of the residues equations (10) and (11) and substitute it to Eq. (8), we get

$$\frac{T_n(x;u,\lambda)}{n!} = -\sum_{k\in\mathbb{Z}}\frac{(1-u)\left(\frac{u}{\lambda}\right)^{\frac{1}{2}x-1}e^{k\pi ix}}{2\lambda\left[\log\left(\frac{u}{\lambda}\right)^{1/2}+k\pi i\right]^{n+1}}$$

Simplifying the above expression we then have

$$T_n(x;u,\lambda) = \frac{n!}{2}\frac{u-1}{u}\left(\frac{u}{\lambda}\right)^{\frac{1}{2}x}\sum_{k\in\mathbb{Z}}\frac{e^{i\pi kx}}{\left[\log\left(\frac{u}{\lambda}\right)^{1/2}+k\pi i\right]^{n+1}}$$

When $u = -1$, the above expression will then be $T_n(x; -1, \lambda) = T(x; \lambda)$. That is

$$\begin{aligned} T_n(x; u = -1, \lambda) &= \frac{n!}{2}\left(\frac{-1}{\lambda}\right)^{\frac{1}{2}x} \sum_{k\in\mathbb{Z}} \frac{e^{ik\pi x}\cdot 2^{n+1}}{\left[\log\left(\frac{-1}{\lambda}\right) + 2k\pi i\right]^{n+1}} \\ &= \frac{n!}{2}\frac{e^{-i\pi k\frac{x}{2}}}{\lambda^{\frac{1}{2}x}} \sum_{k\in\mathbb{Z}} \frac{e^{i\pi kx}\cdot 2^{n+1}}{[(2k-1)\pi - \log\lambda]^{n+1}} \\ &= \frac{n!}{2}\frac{1}{(\lambda)^{\frac{1}{2}x}} \sum_{k\in\mathbb{Z}} \frac{e^{i\pi kx}e^{-i\pi\frac{x}{2}}\cdot 2^{n+1}}{\left[\log\left(\frac{-1}{\lambda}\right) + 2k\pi i\right]^{n+1}} \\ T(x; \lambda) &= \frac{n!}{2}\frac{1}{(\lambda)^{\frac{1}{2}x}} \sum_{k\in\mathbb{Z}} \frac{e^{\frac{x}{2}(2k-1)\pi i}\cdot 2^{n+1}}{[(2k-1)\pi i - \log\lambda]^{n+1}} \end{aligned}$$

Note that, the above expression is the known Apostol-Tangent polynomial as shown in the paper of Corcino et al.[3]

Now, consider an integral formulation of the polynomials Apostol-Frobenius-Tangent.

**Theorem 2.** *For* $n \in \mathbb{N}$ *(set of natural numbers),* $0 < x < 1$, $\xi < \frac{1}{2}$, $\xi \in \mathbb{R}$, *we have*

$$\begin{aligned} T_n\left(x; u; -ue^{2\xi\pi i}\right) &= \left(\frac{u-1}{u}\right) 2^{n-1} e^{-\xi\pi i x} \\ &\quad \times \left[\int_0^\infty \frac{M(n;x;v)\cosh(2\xi\pi v) + iN(n;x;v)\sinh(2\xi\pi v)}{\cosh(2\pi vs.) + \cos(\pi x)} v^n dv\right] \end{aligned} \tag{12}$$

*where*

$$M(n;x;v) = e^{\pi v}\cos\left(-\frac{\pi}{2}x + \frac{(n+1)\pi}{2}\right) - e^{-\pi v}\cos\left(\frac{\pi}{2}x + \frac{(n+1)\pi}{2}\right) \tag{13}$$

$$N(nn;x;v) = e^{\pi v}\sin\left(-\frac{\pi}{2}x + \frac{(n+1)\pi}{2}\right) - e^{-\pi v}\sin\left(\frac{\pi}{2}x + \frac{(n+1)\pi}{2}\right) \tag{14}$$

*Proof.* From Eq.(9), we let $\lambda = -ue^{2\xi\pi i}$ and $k \mapsto -k$

$$\begin{aligned} T(x; u; -ue^{2\xi\pi i}) &= \frac{u-1}{u} 2^n \left(-e^{-2\xi\pi i}\right)^{\frac{1}{2}x} n! \sum_{k\in\mathbb{Z}} \frac{e^{-i\pi kx}}{[-2k\pi i + \log(e^{-\pi i}\cdot e^{-2\xi\pi i})]^{n+1}} \\ &= \frac{u-1}{u} 2^n \left(e^{-\pi i - 2\xi\pi i}\right)^{\frac{1}{2}x} n! \sum_{k\in\mathbb{Z}} \frac{e^{-i\pi kx}}{[-2k\pi i - \pi i - 2\xi\pi i]^{n+1}} \\ &= \frac{u-1}{u} 2^n \left(e^{-(\frac{1}{2}+\xi)\pi i x}\right) n! \sum_{k\in\mathbb{Z}} \frac{e^{-i\pi kx}}{(-\pi i)^{n+1}(2k + 2\xi + 1)^{n+1}} \\ &= \frac{u-1}{u} 2^n \frac{e^{-(\frac{1}{2}+\xi)\pi i x}}{(-\pi i)^{n+1}} n! \sum_{k\in\mathbb{Z}} \frac{e^{-i\pi kx}}{(2k + 2\xi + 1)^{n+1}} \end{aligned}$$

$$= \frac{u-1}{u}2^n\frac{e^{-(\frac{1}{2}+\xi)\pi ix}}{(-\pi i)^{n+1}}n!\left[\sum_{k=0}^{\infty}\frac{e^{-\pi kx}}{(2k+2\xi+1)^{n+1}}+\sum_{k=1}^{\infty}\frac{e^{i\pi kx}}{(-2k+2\xi+1)^{n+1}}\right]$$

$$= \frac{u-1}{u}2^n\frac{e^{-(\frac{1}{2}+\xi)\pi ix}}{(-\pi i)^{n+1}}n!\left[\sum_{k=0}^{\infty}\frac{e^{-\pi kx}}{(2k+2\xi+1)^{n+1}}+(-1)^{n+1}\sum_{k=1}^{\infty}\frac{e^{i\pi kx}}{(2k-2\xi-1)^{n+1}}\right]$$

$$= \frac{u-1}{u}2^n\frac{e^{-(\frac{1}{2}+\xi)\pi ix}}{(-\pi i)^{n+1}}\left[\sum_{k=0}^{\infty}e^{-i\pi kx}\frac{n!}{(2k+2\xi+1)^{n+1}}\right.$$
$$\left.+(-1)^{n+1}\sum_{k=1}^{\infty}e^{i\pi kx}\frac{n!}{(2k-2\xi-1)^{n+1}}\right]$$

where

$$(2k+2\xi+1)>0 \text{ if } |\xi|<\frac{1}{2}, \xi\in\mathbb{R} \text{ and } k\geq 0,$$
$$(2k-2\xi-1)>0 \text{ if } |\xi|<\frac{1}{2}, \xi\in\mathbb{R} \text{ and } k\geq 0.$$

We then apply the integral formula given as

$$\int_0^{\infty}t^ne^{-at}dt=\frac{n!}{a^{n+1}}\quad (n=0,1,...;\Re(a)>0)$$

So that,

$$T(x;u;-ue^{2\xi\pi i}) = \frac{u-1}{u}2^n\frac{e^{-(\frac{1}{2}+\xi)\pi ix}}{(-\pi i)^{n+1}}\left[\sum_{k=0}^{\infty}e^{-i\pi kx}\int_0^{\infty}t^ne^{-(2k+2\xi+1)t}dt\right.$$
$$\left.+(-1)^{n+1}\sum_{k=1}^{\infty}e^{i\pi kx}\int_0^{\infty}t^ne^{-(2k-2\xi-1)t}dt\right]$$

$$= \frac{u-1}{u}2^n\frac{e^{-(\frac{1}{2}+\xi)\pi ix}}{(-\pi i)^{n+1}}\left[\int_0^{\infty}t^ne^{-(2\xi+1)t}\sum_{k=0}^{\infty}e^{-(i\pi x+2t)k}dt\right.$$
$$\left.+(-1)^{n+1}\int_0^{\infty}e^{(2\xi+1)t}\sum_{k=1}^{\infty}e^{(i\pi x-2t)k}t^ndt\right]$$

$$= \frac{u-1}{u}2^n\frac{e^{-(\frac{1}{2}+\xi)\pi ix}}{(-\pi i)^{n+1}}\left[\int_0^{\infty}t^ne^{-(2\xi+1)t}\frac{1}{1-e^{-(i\pi x+2t)}}dt\right.$$
$$+(-1)^{n+1}\int_0^{\infty}e^{(2\xi+1)t}\frac{e^{i\pi x-2t}}{1-e^{i\pi x-2t}}t^ndt$$

$$= \frac{u-1}{u}2^n\frac{e^{-(\frac{1}{2}+\xi)\pi ix}}{(-\pi i)^{n+1}}\left[\int_0^{\infty}t^ne^{-(2\xi+1)t}\frac{e^{i\pi x}}{e^{i\pi x}-e^{-2t}}dt\right.$$
$$+(-1)^{n+1}\int_0^{\infty}e^{(2\xi+1)t}\frac{e^{i\pi x}}{e^{2t-e^{i\pi x}}}t^ndt$$

$$= \frac{u-1}{u} 2^n \frac{e^{-\xi\pi ix}}{(-\pi i)^{n+1}} \left[ \int_0^\infty \frac{e^{\frac{1}{2}\pi ix}}{e^{\pi ix} - e^{-2t}} e^{-(2\xi+1)t} t^n dt \right.$$
$$\left. + (-1)^{n+1} \int_0^\infty \frac{e^{\frac{1}{2}\pi ix}}{e^{2t} - e^{i\pi x}} e^{(2\xi+1)t} t^n dt \right]$$
$$= \frac{u-1}{u} 2^n \frac{e^{-\xi\pi ix}}{(-\pi i)^{n+1}} \left[ \int_0^\infty \frac{e^{-(2\xi+1)t}}{e^{\frac{1}{2}\pi ix} - e^{-2t-\frac{1}{2}\pi ix}} t^n dt \right.$$
$$\left. + (-1)^{n+1} \int_0^\infty \frac{e^{(2\xi+1)t}}{e^{2t-\frac{1}{2}\pi ix} - e^{\frac{1}{2}\pi ix}} t^n dt \right]$$
$$= \frac{u-1}{u} 2^n \frac{e^{-\xi\pi ix}}{(-\pi i)^{n+1}} \left[ \int_0^\infty \frac{e^{-(2\xi+1)t}}{e^{\frac{1}{2}\pi ix} - e^{-2t-\frac{1}{2}\pi ix}} \cdot \frac{\left(e^{2t} - e^{\pi ix}\right) e^{-\frac{1}{2}\pi ix}}{e^{2t-\frac{1}{2}\pi ix} - e^{\frac{1}{2}\pi ix}} t^n dt \right.$$
$$\left. + (-1)^{n+1} \int_0^\infty \frac{e^{(2\xi+1)t}}{e^{2t-\frac{1}{2}\pi ix} - e^{\frac{1}{2}\pi ix}} \cdot \frac{\left(e^{\pi ix} - e^{-2t}\right) e^{-\frac{1}{2}\pi ix}}{e^{\frac{1}{2}\pi ix} - e^{-2t-\frac{1}{2}\pi ix}} t^n dt \right]$$
$$= \frac{u-1}{u} 2^{n-1} \frac{e^{-\xi\pi ix}}{(-\pi i)^{n+1}} \left[ \int_0^\infty \frac{e^{-(2\xi+1)t}(e^{2t} - e^{\pi ix})e^{-\frac{1}{2}\pi ix}}{\cosh(2t) - \cos(\pi x)} t^n dt \right.$$
$$\left. + (-1)^{n+1} \int_0^\infty \frac{e^{(2\xi+1)t}(e^{\pi ix} - e^{2t})e^{-\frac{1}{2}\pi ix}}{\cosh(2t) - \cos(\pi x)} t^n dt \right]$$

We then make the substitution $t = \pi v$, $(-1/i)^{n+1} = e^{(n+1)\pi i/2}$ and $(-1)^{n+1} = e^{-(n+1)\pi i}$, we find that

$$T(x; u; -ue^{2\xi\pi i}) = \frac{u-1}{u} 2^{n-1} \frac{e^{-\xi\pi ix}}{(-\pi i)^{n+1}} \left[ \int_0^\infty \frac{e^{-(2\xi+1)\pi v}(e^{2\pi v} - e^{\pi ix})e^{-\frac{1}{2}\pi ix}}{\cosh(2\pi vs.) - \cos(\pi x)} (\pi v)^n \pi dv \right.$$
$$\left. + (-1)^{n+1} \int_0^\infty \frac{e^{(2\xi+1)\pi v}(e^{\pi ix} - e^{2\pi v})e^{-\frac{1}{2}\pi ix}}{\cosh(2\pi vs) - \cos(\pi x)} (\pi v)^n \pi dv \right]$$
$$= \frac{u-1}{u} 2^{n-1} \frac{e^{-\xi\pi ix}}{\pi^{n+1}} \left[ \int_0^\infty \frac{e^{(n+1)\pi i/2}(e^{2\pi v} - e^{\pi ix})e^{-(2\xi+1)\pi v} e^{-\frac{1}{2}\pi ix}}{\cosh(2\pi vs) - \cos(\pi x)} \pi^{n+1} v^n dv \right.$$
$$\left. + (-1)^{n+1} \int_0^\infty \frac{e^{-(n+1)\pi i/2}(e^{\pi x} - e^{2\pi v})e^{(2\xi+1)\pi v} e^{-\frac{1}{2}\pi ix}}{\cosh(2\pi vs.) - \cos(\pi x)} \pi^{n+1} v^n dv \right]$$
$$= \frac{u-1}{u} 2^{n-1} e^{-\xi\pi ix} \left[ \int_0^\infty \frac{e^{\pi v} e^{i\left[-\frac{\pi}{2}x + \frac{(n+1)\pi}{2}\right]} e^{-2\xi\pi v} - e^{-\pi v} e^{i\left[\frac{\pi}{2}x + \frac{(n+1)\pi}{2}\right]} e^{-2\xi\pi v}}{\cosh(2\pi vs.) - \cos(\pi x)} v^n dv \right.$$
$$\left. + \int_0^\infty \frac{e^{\pi v} e^{i\left[\frac{\pi}{2}x - \frac{(n+1)\pi}{2}\right]} e^{2\xi\pi v} - e^{-\pi v} e^{i\left[-\frac{\pi}{2}x - \frac{(n+1)\pi}{2}\right]} e^{2\xi\pi v}}{\cosh(2\pi vs) - \cos(\pi x)} v^n dv \right]$$

Let $A=-\frac{\pi}{2}x+\frac{(n+1)\pi}{2}$ and $B=\frac{\pi}{2}x+\frac{(n+1)\pi}{2}$. So that,

$$\begin{aligned}
T(x;u;-ue^{2\xi\pi i}) &= \frac{u-1}{u}2^{n-1}e^{-\xi\pi ix}\left[\int_0^\infty \frac{e^{\pi v}e^{iA}e^{-2\xi\pi v}-e^{-\pi v}e^{iB}e^{-2\xi\pi v}}{\cosh(2\pi vs.)-\cos(\pi x)}v^n dv\right.\\
&\qquad\left.+\int_0^\infty \frac{e^{\pi v}e^{-iA}e^{2\xi\pi v}-e^{-\pi v}e^{-iB}e^{2\xi\pi v}}{\cosh(2\pi vs.)-\cos(\pi x)}v^n dv\right]\\
&= \frac{u-1}{u}2^{n-1}e^{-\xi\pi ix}\left[\int_0^\infty \frac{(e^{\pi v}\cos A-e^{-\pi v}\cos B)(e^{-2\xi\pi v}+e^{2\xi\pi v})}{\cosh(2\pi vs.)-\cos(\pi x)}v^n dv\right.\\
&\qquad\left.+i\int_0^\infty \frac{(e^{\pi v}\sin A-e^{-\pi v}\sin B)(e^{2\xi\pi v}-e^{2\xi\pi v})}{\cosh(2\pi vs.)-\cos(\pi x)}v^n dv\right]\\
&= \frac{u-1}{u}2^{n-1}e^{-\xi\pi ix}\left[\int_0^\infty \frac{(e^{\pi v}\cos A-e^{-\pi v}\cos B)(\cosh(2\xi\pi v))}{\cosh(2\pi vs.)-\cos(\pi x)}v^n dv\right.\\
&\qquad\left.+i\int_0^\infty \frac{(e^{\pi v}\sin A-e^{-\pi v}\sin B)(\sinh(2\xi\pi v))}{\cosh(2\pi vs.)-\cos(\pi x)}v^n dv\right]
\end{aligned}$$

Simplifying the above equation gives us the integral representation of the Apostol-Frobenius-Tangent polynomials

$$T(x;u;-ue^{2\xi\pi i}) = \frac{u-1}{u}2^{n-1}e^{-\xi\pi ix}\left[\int_0^\infty \frac{M(n;x;v)\cosh(2\xi\pi v)+iN(n;x;v)\sinh(2\xi\pi v)}{\cosh(2\pi vs.)-\cos(\pi x)}v^n dv\right] \tag{15}$$

where

$$\begin{aligned}
M(n;x;v) &= e^{\pi v}\cos\left(-\frac{\pi}{2}x+\frac{(n+1)\pi}{2}\right)-e^{-\pi v}\cos\left(\frac{\pi}{2}x+\frac{(n+1)\pi}{2}\right)\\
N(nn;x;v) &= e^{\pi v}\sin\left(-\frac{\pi}{2}x+\frac{(n+1)\pi}{2}\right)-e^{-\pi v}\sin\left(\frac{\pi}{2}x+\frac{(n+1)\pi}{2}\right)
\end{aligned}$$

## 2.2. Fourier expansion of Apostol-Frobenius-Tangent Polynomials of Higher-order

The Apostol-Frobenius-Tangent polynomials of higher order, denoted by $T_n^{(r)}(x;u,\lambda)$, are defined as coefficients of the following generating function

$$\sum_{n=0}^{\infty} T_n^{(r)}(x;u,\lambda)\frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^{2t}-u}\right)^r e^{xt} \tag{16}$$

where $r\geq 1$, $u,\lambda\in\mathbb{C}$ with $u\neq 1,\lambda\neq 1$ and $u\neq\lambda$.

In this section, we derive the Fourier expansion for Apostol-Frobenius-Tangent polynomials of higher order as shown in the following theorem.

**Theorem 3.** *For* $0 \leq x \leq 1$,

$$T_n^{(r)}(x;u,\lambda) = -n! \sum_{k\in\mathbb{Z}} \left(\frac{1-u}{2u}\right)^r \sum_{j=0}^{r-1} (-1)^{j-1} 2^j \binom{n+r-1-j}{r-1-j} \frac{B_l^{(r)}\left(\frac{x}{2}\right)}{j!} \frac{\left(\frac{u}{\lambda}\right)^{\frac{x}{2}} e^{xk\pi i}}{\left[\log\left(\frac{u}{\lambda}\right)^{\frac{1}{2}} + k\pi i\right]^{r+n-j}} \tag{17}$$

*where* $B_n^{(r)}\left(\frac{x}{2}\right)$ *is the Bernoulli polynomials of order* $r$ *defined by*

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}$$

*Proof.* By Cauchy Residue Theorem,

$$\frac{1}{2\pi i} \int_{C_N} f(t)dt = \text{Res}(f(t), t=0) + \sum_{k\in\mathbb{Z}} \text{Res}(f(t), t=t_k)$$

where

$$t_k = \log\left(\frac{u}{\lambda}\right)^{\frac{1}{2}} + k\pi i, \qquad k \in \mathbb{Z}$$
$$f(t) = \left(\frac{1-u}{\lambda e^{2t}-u}\right)^r \frac{e^{xt}}{t^{n+1}}$$

Consider the left-hand side of the equation in the Cauchy Residue Theorem as $N \to \infty$, we have

$$\lim_{N\to\infty} \int_{C_N} f(t)dt = \lim_{N\to\infty} \int_{C_N} \left(\frac{1-u}{\lambda e^{2t}-u}\right)^r e^{xt} \frac{dt}{t^{n+1}} = 0$$

We use the same proof as in the case when $r = 1$ as shown in Lemma 1. We shall evaluate the first term of the right-hand side of the equation in the Cauchy Residue Theorem as $t = 0$, we have

$$\begin{aligned}\text{Res}(f(t), t=0) &= \lim_{t\to 0} \frac{1}{n!} \frac{d^n}{dt^n} (t-0)^{n+1} \frac{1}{t^{n+1}} \sum_{m=0}^{\infty} T_m^{(r)}(x;u,\lambda) \frac{t^m}{m!} \\ &= \frac{1}{n!} T_n^{(r)}(x;u,\lambda).\end{aligned}$$

Now we will evaluate the second term of the right-hand side of the equation of the Cauchy Residue theorem as $t = t_k$. That is, for $r \geq 2$

$$\text{Res}(f(t), t=t_k) = \frac{1}{(r-1)!} \lim_{t\to t_k} \frac{d^{r-1}}{dt^{r-1}} (t-t_k)^r \left(\frac{1-u}{\lambda e^{2t}-u}\right)^r \frac{e^{xt}}{t^{n+1}}$$

Consider the function

$$(t-t_k)^r \left(\frac{1-u}{\lambda e^{2t}-u}\right)^r \frac{e^{xt}}{t^{n+1}} = (t-t_k)^r \frac{(1-u)^r}{\left(\frac{\lambda}{u} e^{2t}-1\right)^r} \frac{1}{u^r} \frac{e^{xt}}{t^{n+1}}$$

$$
\begin{aligned}
&= (t-t_k)^r \frac{(1-u)^r u^{-r}}{\left(\frac{\lambda}{u}e^{2t}-1\right)^r} e^{xt} t^{-(n+1)} \\
&= (t-t_k)^r \frac{(1-u)^r u^{-r}}{\left(\frac{\lambda}{u}e^{2t}-1\right)^r} e^{xt} t^{-(n+1)} \\
&= (t-t_k)^r \frac{(1-u)^r u^{-r}}{\left(\frac{\lambda}{u}e^{2(t-t_k)}\frac{u}{\lambda}-1\right)^r} e^{xt} t^{-(n+1)} \qquad \text{since } \frac{u}{\lambda}e^{-2t_k}=1 \\
&= \frac{(1-u)^r}{(2u)^r} \frac{[2(t-t_k)]^r}{(e^{2(t-t_k)}-1)^r} \frac{e^{xt}}{t^{n+1}} \\
&= \frac{(1-u)^r}{(2u)^r} \sum_{n=0}^{\infty} B_n^{(r)} \frac{[2(t-t_k)]^r}{n!} t^{-(n+1)} e^{xt}
\end{aligned}
$$

where

$$
\left(\frac{w}{e^w-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{w^n}{n!}
$$

To get the derivative, we employ the Leibniz Rule which gives us

$$
\begin{aligned}
&\frac{d^{r-1}}{dt^{r-1}} \left\{ (t-t_k)^r \left(\frac{1-u}{\lambda e^{2t}-u}\right)^r \frac{e^{xt}}{t^{n+1}} \right\} \\
&\quad = \left(\frac{1-u}{2u}\right)^r \frac{d^{r-1}}{dt^{r-1}} \left\{ (t-t_k)^r \left[\sum_{n=0}^{\infty} B_n^{(r)} \frac{[2(t-t_k)]^n}{n!}\right] e^{xt} t^{-(n+1)} \right\} \\
&\quad = \left(\frac{1-u}{2u}\right)^r \frac{d^{r-1}}{dt^{r-1}} \left\{ \left( \left[ B_n^{(r)} \frac{[2(t-t_k)]^n}{n!} \right] e^{xt} \right) t^{-(n+1)} \right\} \\
&\quad = \left(\frac{1-u}{2u}\right)^r \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{d^{r-1-j}}{dt^{r-1-j}} t^{-(n+1)} \frac{d^j}{dt^j} \left( \underbrace{e^{xt} \sum_{n=0}^{\infty} B_n^{(r)} 2^n \frac{(t-t_k)^n}{n!}}_{H(t)} \right)
\end{aligned}
$$

Consider now,

$$
\begin{aligned}
\frac{d^j}{dt^j}(H(t)) &= \sum_{l=0}^{j} \binom{j}{l} x^{j-l} e^{xt} \sum_{n=0}^{\infty} B_n^{(r)} \frac{2^n (n)_l}{n!} (t-t_k)^{n-l} \\
&= e^{xt} \sum_{l=0}^{j} \binom{j}{l} x^{j-l} \sum_{n=0}^{\infty} B_n^{(r)} 2^n \frac{(t-t_k)^{n-l}}{(n-l)!}
\end{aligned}
$$

So that, the derivative becomes

$$
\frac{d^{r-1}}{dt^{r-1}} \left\{ (t-t_k)^r \left(\frac{1-u}{\lambda e^{2t}-u}\right)^r \frac{e^{xt}}{t^{n+1}} \right\} = \left(\frac{1-u}{2u}\right)^r \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{d^{r-1-j}}{dt^{r-1-j}} t^{-(n+1)}
$$

$$\times e^{xt}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}\sum_{n=0}^{\infty}B_n^{(r)}2^n\frac{(t-t_k)^{n-l}}{(n-l)!}$$

Thus,

$$\begin{aligned}\operatorname{Res}(f(t),t=t_k) &= \frac{1}{(r-1)!}\lim_{t\to t_k}\frac{d^{r-1}}{dt^{r-1}}(t-t_k)^r\left(\frac{1-u}{\lambda e^{2t}-u}\right)^r\frac{e^{xt}}{t^{n+1}}\\ &= \frac{1}{(r-1)!}\lim_{t\to t_k}\left(\frac{1-u}{2u}\right)^r\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{d^{r-1-j}}{dt^{r-1-j}}t^{-(n+1)}\\ &\quad\times e^{xt}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}\sum_{n=0}^{\infty}B_n^{(r)}2^n\frac{(t-t_k)^{n-l}}{(n-l)!}\end{aligned}$$

Note that $B_n^{(r)}\frac{(t-t_k)^{n-l}}{(n-l)!}\to 0$ as $t\to t_k$ except when $n=l$. So that

$$\begin{aligned}\operatorname{Res}(f(t),t=t_k) &= \frac{1}{(r-1)!}\left(\frac{1-u}{2u}\right)^r\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}(n+r-1-j)_{r-1-j}t_k^{-(n+r-j)}\\ &\quad\times e^{xt_k}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}2^lB_l^{(r)}\\ &= \frac{1}{(r-1)!}\frac{(1-u)^r}{(2u)^r}\sum_{j=0}^{r-1}\frac{(r-1)!}{j!(r-1-j)!}(-1)^{r-1-j}(n+r-1-j)_{r-1-j}t_k^{-(n+r-j)}\\ &\quad\times e^{xt_k}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}2^lB_l^{(r)}\\ &= \left(\frac{u-1}{2u}\right)^r\sum_{j=0}^{r-1}\binom{n+r-1-j}{r-1-j}(-1)^{-j-1}\frac{t^{j-n-r}}{j!}e^{xt_k}2^j\sum_{l=0}^{j}\binom{j}{l}\frac{x^{j-l}}{2^{j-l}}B_n^{(r)}.\end{aligned}$$

We use the identity that $B_l^{(r)}=\sum_{j=0}^{j}\binom{j}{l}B_l^{(r)}\left(\frac{x}{2}\right)^{j-l}$. Thus,

$$\operatorname{Res}(f(t),t=t_k)=\left(\frac{u-1}{2u}\right)^r\sum_{j=0}^{r-1}\binom{n+r-1-j}{r-1-j}(-1)^{j-1}\frac{B_l^{(r)}\left(\frac{x}{2}\right)}{j!}\frac{e^{xt_k}}{t_k^{r+n-j}}$$

Substituting $t_k=\log\left(\frac{u}{\lambda}\right)^{\frac{1}{2}}+k\pi i$, we get

$$\operatorname{Res}(f(t),t=t_k)=\left(\frac{u-1}{2u}\right)^r\sum_{j=0}^{r-1}\binom{n+r-1-j}{r-1-j}(-1)^{j-1}\frac{B_l^{(r)}\left(\frac{x}{2}\right)}{j!}\frac{e^{x\left(\log\left(\frac{u}{\lambda}\right)^{\frac{1}{2}}+k\pi i\right)^{\frac{1}{2}}}}{\left[\log\left(\frac{u}{\lambda}\right)^{\frac{1}{2}}+k\pi i\right]^{r+n-j}}$$

This gives

$$T_n^{(r)}(x;u,\lambda) = -n!\sum_{k\in\mathbb{Z}}\left(\frac{u-1}{2u}\right)^r\sum_{j=0}^{r-1}(-1)^{j-1}2^j\binom{n+r-1-j}{r-1-j}\frac{B_l^{(r)}\left(\frac{x}{2}\right)}{j!}\frac{\left(\frac{u}{\lambda}\right)^{x/2}e^{xk\pi i}}{\left[\log\left(\frac{u}{\lambda}\right)^{\frac{1}{2}}+k\pi i\right]^{r+n-j}}$$

where

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}$$

### 2.3. Fourier expansion of Apostol-Frobenius-Genocchi Polynomials of Higher Order

The Genocchi polynomials can be defined as [4]

$$\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)e^{xt} \tag{18}$$

The Apostol-Frobenius-Genocchi polynomials which are certain variation of the Genocchi polynomials are defined by Araci and Acikgoz [1] as coefficients of the following generating function:

$$\sum_{n=0}^{\infty}G_n(x;u,\lambda)\frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t-u}e^{xt} \tag{19}$$

where $u,\lambda\in\mathbb{C}$ with $u\neq 1, \lambda\neq 1$ and $u\neq\lambda$. By Cauchy Integral formula, we observe that

$$\frac{G_n(x;u,\lambda)}{n!} = \frac{1}{2\pi i}\int_C\frac{(1-u)t}{\lambda e^t-u}e^{xt}\frac{dt}{t^{n+1}}$$

If we consider the function

$$f(t) = \frac{(1-u)}{\lambda e^t-u}\frac{e^{xt}}{t^n},$$

then it has a pole at $t=0$ of order $n$. The other poles are found to be at

$$\begin{aligned}\lambda e^t-u &= 0\\ \lambda e^t &= u\\ e^t &= \frac{u}{\lambda}\\ t_k := t &= \log\left(\frac{u}{\lambda}\right)+2k\pi i\end{aligned}$$

By Cauchy Residue Theorem, we have

$$\frac{1}{2\pi i}\int_{C_N}f_n(t)dt = \mathrm{Res}(f_n(t),t=0)+\sum_{k\in\mathbb{Z}}\mathrm{Res}(f_n(t),t=t_k) \tag{20}$$

In Eq. (20), we take the limit of the integral, $\int_{C_N}f_n(t)dt$ as $N\to\infty$ as explicitly shown in the following lemma

**Lemma 2.** *Let* $u, \lambda \in \mathbb{C}\{0,1\}$ *with* $|\lambda| \neq |u|$. *For* $0 < x \leq 1$

$$\int_{C_N} \frac{(1-u)te^{xt}}{(\lambda e^t - u)t^{n+1}} dt = 0$$

*Proof.* Consider

$$\left| \int_{C_N} \frac{(1-u)te^{xt}}{\lambda e^t - u} \frac{dt}{t^{n+1}} \right| \leq \int_{C_N} \frac{|1-u||e^{xt}||dt|}{|\lambda e^t - u||t^n|}$$
$$= \int_{C_N} \frac{|1-u||e^{xt}||dt|}{|\alpha u| \left| e^t + \frac{1}{\alpha} \right| |t^n|}, \quad \text{where } \alpha = -\frac{\lambda}{u}$$
$$\int_{C_N} \frac{|1-u||dt|}{|-\lambda||t^n|}$$

So that

$$\left| \int_{C_N} \frac{(1-u)te^{xt}}{\lambda e^t - u} \frac{dt}{t^{n+1}} \right| \leq \frac{|1-u|}{|-\lambda|} \int_{C_N} \frac{|dt|}{|t^n|} = \frac{1}{((2N+\epsilon)\pi)^n}$$

As $N \to \infty$

$$\int_{C_N} \frac{(1-u)te^{xt}}{\lambda e^t - u} \frac{dt}{t^{n+1}} dt \to 0$$

Using Lemma 2, Eq. (20) becomes

$$\text{Res}(f_n(t), t = 0) = -\sum_{k \in \mathbb{Z}} \text{Res}(f_n(t), t = t_k)$$

With these, Araci and Acikgoz [1] obtained the following Fourier series expansion by calculating the residues of the function $f_n(t)$ at $t = 0$ and $t = t_k$, respectively:

$$G_n(x; u, \lambda) = n! \frac{1-u}{u} \left( \frac{u}{\lambda} \right)^x \sum_{k \in \mathbb{Z}} e^{i2\pi kx} \left[ \log(u/\lambda) + 2k\pi i \right]^{-n} \tag{21}$$

where $u, \lambda \in \mathbb{C}\{0,1\}$ with $|\lambda| \neq |u|$ and $0 < x \leq 1$. Note that, when we take $u = -1$, we have

$$G(x; u = -1, \lambda) = n! \frac{-1-1}{-1} \left( \frac{-1}{\lambda} \right)^x \sum_{k \in \mathbb{Z}} e^{2k\pi ix} \left[ \log \left( \frac{-1}{\lambda} \right) + 2k\pi i \right]^{-n}$$
$$= 2n! \frac{(-1)^x}{\lambda^x} \sum_{k \in \mathbb{Z}} e^{2k\pi ix} [\log(-1) - \log \lambda + 2k\pi i]^{-n}$$
$$= 2n! \frac{e^{\pi ix}}{\lambda^x} \sum_{k \in \mathbb{Z}} e^{2k\pi ix} [\pi i - \log \lambda + 2k\pi i]^{-n}$$
$$G(x; \lambda) = \frac{2n!}{\lambda^x} \sum_{k \in \mathbb{Z}} \frac{e^{(2k+1)\pi ix}}{[-\log \lambda + (2k+1)\pi i]^n} \tag{22}$$

Note that Eq.(22) is the Apostol-Genocchi polynomials obtained in the paper of Corcino, et.al.[2]. That is, $G(x; u = -1, \lambda) = G(x; \lambda)$.

The Apostol-Frobenius-Genocchi polynomials of higher order, denoted by $G_n^{(r)}(x; u, \lambda)$, are defined as coefficients of the following generating function:

$$\sum_{n=0}^{\infty} G_n^{(r)}(x; u, \lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^r e^{xt} \tag{23}$$

where $r \geq 1$, $u, \lambda \in \mathbb{C}$ with $u \neq 1, \lambda \neq 1$ and $u \neq \lambda$.

The following theorem contains the Fourier expansion of these polynomials.

**Theorem 4.** *For* $0 \leq x \leq 1$

$$G_n^{(r)}(x; u, \lambda) = -n! \left(\frac{u-1}{u}\right)^r \sum_{k\in\mathbb{Z}} \sum_{j=0}^{r-1} (-1)^{j-1} \binom{n-1-j}{r-1-j} \frac{B_l^{(r)}(x)}{j!} \frac{\left(\frac{u}{\lambda}\right)^x e^{2xk\pi i}}{\left[\log\left(\frac{u}{\lambda}\right) + 2k\pi i\right]^{n-j}} \tag{24}$$

*where* $B_n^{(r)}(x)$ *is the Bernoulli polynomials of order* $r$.

*Proof.* Using the Cauchy Residue Theorem (CRT),

$$\frac{1}{2\pi i}\int_{C_N} f(t)dt = \text{Res}(f(t), t = 0) + \sum_{k\in\mathbb{Z}} \text{Res}(f(t), t = t_k)$$

where

$$\begin{aligned} t_k &= \log\left(\frac{u}{\lambda}\right) + 2k\pi i, \qquad k \in \mathbb{Z} \\ f(t) &= \left(\frac{(1-u)t}{\lambda e^t - u}\right)^r \frac{e^{xt}}{t^{n+1}} \\ &= \left(\frac{1-u}{\lambda e^t - u}\right)^r \frac{e^{xt}}{t^{n+1-r}} \end{aligned}$$

Consider the left-hand side of the equation in the CRT as $N \to \infty$, we have

$$\lim_{N\to\infty}\int_{C_N} f(t)dt = \lim_{N\to\infty}\int_{C_N} \left(\frac{(1-u)}{\lambda e^t - u}\right)^r e^{xt}\frac{dt}{t^{n+1-r}} = 0,$$

which can easily be shown using the same proof as in the case when $r = 1$ as shown in Lemma 2. We shall evaluate the first term of the right-hand side of the equation in the Cauchy Residue Theorem as $t = 0$, we have

$$\begin{aligned} \text{Res}(f(t), t = 0) &= \lim_{t\to 0}\frac{1}{n!}\frac{d^n}{dt^n}(t-0)^{n+1}\frac{1}{t^{n+1}}\sum_{m=0}^{\infty} G_m^{(r)}(x; u, \lambda)\frac{t^m}{m!} \\ &= \lim_{t\to 0}\frac{1}{n!}\frac{d^n}{dt^n}\sum_{m=0}^{\infty} G_m^{(r)}(x; u, \lambda)\frac{t^m}{m!} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{1}{n!}\sum_{m=0}^{\infty} G_m^{(r)}(x;u,\lambda)\frac{(m)_n}{m!}t^{m-n}\\
&= \lim_{t\to 0} \frac{1}{n!}\sum_{m=0}^{\infty} G_m^{(r)}(x;u,\lambda)\frac{t^{m-n}}{(m-n)!}\\
&= \frac{1}{n!}G_n^{(r)}(x;u,\lambda)
\end{aligned}
$$

Now we will evaluate the second term of the right-hand side of the equation of the Cauchy Residue theorem as $t = t_k$. That is, for $r \geq 2$

$$
\text{Res}(f(t), t = t_k) = \frac{1}{(r-1)!}\lim_{t\to t_k}\frac{d^{r-1}}{dt^{r-1}}(t-t_k)^r\left(\frac{1-u}{\lambda e^t - u}\right)^r \frac{e^{xt_k}}{t^{n-r+1}}
$$

Consider now the function

$$
\begin{aligned}
(t-t_k)^r\left(\frac{1-u}{\lambda e^t-u}\right)^r\frac{e^{xt_k}}{t^{n-r+1}} &= \frac{(1-u)^r}{u^r}\left(\frac{t-t_k}{\frac{\lambda}{u}e^t-1}\right)^r\frac{e^{xt_k}}{t^{n-r+1}}\\
&= \frac{(1-u)^r}{u^r}\left(\frac{t-t_k}{\frac{\lambda}{u}e^{t-t_k}\frac{u}{\lambda}-1}\right)^r\frac{e^{xt_k}}{t^{n-r+1}}, \qquad \text{since } \frac{u}{\lambda}e^{-t_k}=1\\
&= \left(\frac{1-u}{u}\right)^r\left(\frac{t-t_k}{e^{t-t_k}-1}\right)^r\frac{e^{xt_k}}{t^{n-r+1}}\\
&= \left(\frac{1-u}{u}\right)^r\sum_{n=0}^{\infty}B_n^{(r)}\frac{(t-t_k)^n}{n!}\frac{e^{xt_k}}{t^{n-r+1}}
\end{aligned}
$$

where

$$
\left(\frac{w}{e^w-1}\right)^r = \sum_{n=0}^{\infty}B_n^{(r)}\frac{w^n}{n!}
$$

Using the Leibniz Rule, the derivative part for our residue at $t = t_k$, we get

$$
\begin{aligned}
\frac{d^{r-1}}{dt^{r-1}}\left(\frac{t-t_k}{e^{t-t_k}-1}\right)^r\frac{e^{xt_k}}{t^{n-r+1}} &= \frac{d^{r-1}}{dt^{r-1}}\left\{\left[\sum_{n=0}^{\infty}B_n^{(r)}\frac{(t-t_k)^n}{n!}\right]\frac{e^{xt_k}}{t^{n-r+1}}\right\}\\
&= \frac{d^{r-1}}{dt^{r-1}}\left\{\left[\sum_{n=0}^{\infty}B_n^{(r)}\frac{(t-t_k)^n}{n!}\right]e^{xt_k}\right\}t^{-(n-r+1)}
\end{aligned}
$$

Performing the Leibniz derivative rule on the above equation, we get

$$
\frac{d^{r-1}}{dt^{r-1}}\left\{\left[\sum_{n=0}^{\infty}B_n^{(r)}\frac{(t-t_k)^n}{n!}\right]e^{xt_k}\right\}t^{-(n-r+1)} = \sum_{j=0}^{r-1}\binom{r-1}{j}\frac{d^{r-1-j}}{dt^{r-1-j}}t^{-(n-r+1)}
$$

$$\times \frac{d^j}{dt^j}\left(e^{xt_k}\underbrace{\sum_{n=0}^{\infty} B_n^{(r)}\frac{(t-t_k)^n}{n!}}_{H(t)}\right)$$

Now, consider the derivative

$$\begin{aligned}\frac{d^j}{dt^j}(e^{xt_k}H(t)) &= \sum_{l=0}^{j}\binom{j}{l}x^{j-l}e^{xt_k}\sum_{n=0}^{\infty}\frac{B_n^{(r)}}{n!}(n)_l(t-t_k)^{n-l}\\ &= e^{xt_k}\sum_{l=0}^{l}\binom{j}{l}x^{j-l}\sum_{n=0}^{\infty}B_n^{(r)}\frac{(t-t_k)^n}{(n-l)!}\end{aligned}$$

So that,

$$\begin{aligned}\mathrm{Res}(f(t),t=t_k) = \frac{1}{(r-1)!}\left(\frac{1-u}{u}\right)^r \lim_{t\to t_k}\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{d^{r-1-j}}{dt^{r-1-j}}t^{-(n-r+1)}e^{xt_k}\\ \times \sum_{l=0}^{j}\binom{j}{l}x^{j-l}\sum_{n=0}^{\infty}B_n^{(r)}\frac{(t-t_k)^n}{(n-l)!}\end{aligned}$$

Note that $B_n^{(r)}\frac{(t-t_k)^{n-l}}{(n-l)!}\to 0$ as $t\to t_k$ except when $n=l$

$$\begin{aligned}\mathrm{Res}(f(t),t=t_k) &= \frac{1}{(r-1)!}\left(\frac{1-u}{u}\right)^r\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}(n-1-j)_{r-1-j}t_k^{-(n-j)}\\ &\quad\times e^{xt_k}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}B_l^{(r)}\\ &= \left(\frac{1-u}{u}\right)^r\sum_{j=0}^{r-1}\frac{(-1)^{r-1-j}}{j!(r-1-j)!}(n-1-j)_{r-1-j}t_k^{-(n-j)}\\ &\quad\times e^{xt_k}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}B_l^{(r)}\\ &= \left(\frac{1-u}{u}\right)^r\sum_{j=0}^{r-1}\binom{n-1-j}{r-1-j}(-1)^{-j-1}\frac{t_k^{j-n}}{j!}\\ &\quad\times e^{xt_k}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}B_l^{(r)}\end{aligned}$$

Recall that $B_l^{(r)}(x)=\sum_{l=0}^{j}\binom{j}{l}B_l^{(r)}(x)^{j-l}$. Thus,

$$\mathrm{Res}(f(t),t=t_k) = \left(\frac{1-u}{u}\right)^r\sum_{j=0}^{r-1}\binom{n-1-j}{r-1-j}(-1)^{-j-1}\frac{B_l^{(r)}}{j!}\frac{e^{xt_k}}{t^{n-j}}$$

Substituting $t_k = \log \frac{u}{\lambda} + 2k\pi i$, we get

$$\operatorname{Res}(f(t), t = t_k) = \left(\frac{1-u}{u}\right)^r \sum_{j=0}^{r-1} \binom{n-1-j}{r-1-j} (-1)^{-j-1} \frac{B_l^{(r)}}{j!} \frac{e^{x(\log \frac{u}{\lambda} + 2k\pi i)}}{\left[\log \frac{u}{\lambda} + 2k\pi i\right]^{n-j}}$$

This gives,

$$G_n^{(r)}(x; u, \lambda) = -n! \sum_{k \in \mathbb{Z}} \left(\frac{1-u}{u}\right)^r \sum_{j=0}^{r-1} \binom{n-1-j}{r-1-j} (-1)^{-j-1} \frac{B_l^{(r)}}{j!} \frac{\left(\frac{u}{\lambda}\right)^x e^{2kx\pi i}}{\left[\log \frac{u}{\lambda} + 2k\pi i\right]^{n-j}}.$$

## 3. Conclusion

The researchers were able to obtain the Fourier series expansion of the Apostol-Frobenius type of: Tangent and Genocchi polynomials of higher order. Taking into consideration all of the generating function's residues, together with the Cauchy Residue theorem, proved to be a useful strategy for deriving the Fourier series of these polynomials of higher order. For future study, it will be interesting to derive the integral representations of these higher order polynomials.

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## References

[1] S. Araci and M. Acikgoz. Construction of fourier expansion of apostol frobenius-euler polynomials and its applications. *Advances in Difference Equations*, 2018(1):1–14., 2018.

[2] A. Bayad and Y. Hamahata. Polylogarithms and poly-bernoulli polynomials. *Kyushu J. Math*, 65:15–24., 2011.

[3] B.A. Damgo C. B. Corcino, R. B. Corcino and J.A.A. Ca nete. Integral representation and explicit formula at rational arguments for apostol-tangent polynomials. *Symmetry*, 14(1):Article 35, 2021.

[4] A.F. Horadam. *Applications of Fibonacci Numbers*, chapter Genocchi Polynomials, pages 145–166. Springer, 1991.

[5] T. Kim. Euler numbers and polynomials associated with zeta functions. *Abstract and Applied Analysis*, 2008:Article ID 581582, 11 pages, doi:10.1155/2008/581582, 2008.

[6] T. Kim. Note on the euler numbers and polynomials. *Adv. Stud. Contemp. Math.*, 17(2):109–116, 2008.

[7] Fixed Point. Real world application of fourier series. https://math.stackexchange.com/q/579695, 11 2013.

[8] C. S. Ryoo. A note on the tangent numbers and polynomials. *Adv. Studies Theor. Phys*, 7(9):447–454, 2013.

[9] C.S. Ryoo. On the twisted q-tangent numbers and polynomials. *Appl. Math. Sci.*, 7(99):4935–4941.

[10] C.S. Ryoo. A note on the symmetric properties for the tangent polynomials. *Int. J. Math. Anal.*, 7(52):2575–2581., 2013.

[11] C.S. Ryoo. A numerical investigation on the zeros of the tangent polynomials. *J. Appl. Math. Info.*, 32(3-4):315–322., 2014.

[12] C.S. Ryoo. Explicit identities for the generalized tangent polynomials. *Nonlinear Analysis and Differential Equations*, 6(1):43–51, 2018.

[13] E. Sen S. Araci, M. Acikgoz. Some new formulae for genocchi numbers and polynomials involving bernoulli and euler polynomials. *International Journal of Mathematics and Mathematical Sciences*, 2014:Article ID 760613., 2014.

[14] Y. Simsek. On twisted generalized euler numbers. *Bull. Korean Math. Soc.*, 41(2):299–306., 2004.