**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 16, No. 2, 2023, 724-735
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Non-existence of Positive Integer Solutions of the Diophantine Equation $p^x + (p+2q)^y = z^2$, where $p$, $q$ and $p+2q$ are Prime Numbers

Suton Tadee[1], Apirat Siraworakun[1,*]

[1] *Department of Mathematics, Faculty of Science and Technology, Thepsatri Rajabhat University, Lopburi 15000, Thailand*

**Abstract.** The Diophantine equation $p^x + (p+2q)^y = z^2$, where $p$, $q$ and $p+2q$ are prime numbers, is studied widely. Many authors give $q$ as an explicit prime number and investigate the positive integer solutions and some conditions for non-existence of positive integer solutions. In this work, we gather some conditions for odd prime numbers $p$ and $q$ for showing that the Diophantine equation $p^x + (p+2q)^y = z^2$ has no positive integer solution. Moreover, many examples of Diophantine equations with no positive integer solution are illustrated.

**2020 Mathematics Subject Classifications**: 11D61

**Key Words and Phrases**: Diophantine Equation, Legendre Symbol, The Chinese Remainder Theorem

## 1. Introduction

Studying non-negative integer solutions of the Diophantine equation $p^x + q^y = z^2$, where $p$ and $q$ are prime numbers, has been done in numerous ways. One of them is that $p$ and $q$ are given as explicit prime numbers. For example, in [4] and [5], Kumar, Gupta and Kishan showed that the Diophantine equations $61^x + 67^y = z^2$, $67^x + 73^y = z^2$, $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$ have no non-negative integer solution and Burshtein [3] revealed that the Diophantine equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ have no positive integer solutions $(x, y, z)$.

Many researchers studied the Diophantine equation by considering $q = p+k$, where $k$ is an even number. In [2], Burshtein investigated the solutions of the Diophantine equation $p^x + (p+6)^y = z^2$, where $p$ and $p+6$ are primes and $x + y = 2, 3, 4$. Gupta, Kumar and Kishan [6] studied the Diophantine equation $p^x + (p+6)^y = z^2$, where $p$ and $p+6$ are sexy primes with $p = 6n+1$ and $n$ is a natural number. Burshtein [1] showed that

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4702

*Email addresses:* suton.t@lawasri.tru.ac.th (S. Tadee), apirat.si@lawasri.tru.ac.th (A. Siraworakun)

 

the Diophantine equation $p^x + (p+4)^y = z^2$, where $p > 3$ and $p+4$ are primes, has no positive integer solutions $(x, y, z)$. In addition, Rao [10] studied the Diophantine equation $3^x + 7^y = z^2$. Neres [9] investigated the Diophantine equation $p^x + (p+8)^y = z^2$, where $p > 3$ and $p+8$ are primes. Moreover, Tadee ([13], [14]) has given the solutions of the Diophantine equations $p^x + (p+10)^y = z^2$ and $p^x + (p+14)^y = z^2$, where $p$, $p+10$ and $p+14$ are primes.

In [7], Mina and Bacani use the concepts of Legendre symbol and Jacobi symbol to find some condition for non-existence of solutions of the Diophantine equations of the form $p^x + q^y = z^{2n}$. Two years later, the solutions of the Diophantine equation $p^x + (p+4k)^y = z^2$, where $k$ is a natural number and $p$, $p+4k$ are prime numbers, were investigated [8].

The goal of this article is to give some conditions on primes $p$ and $q$ to show that the Diophantine equation $p^x + (p+2q)^y = z^2$, where $p$, $q$ and $p+2q$ are prime numbers, has no positive integer solution. Moreover, the forms of odd prime numbers $p$, when $q$ is a prime number, are investigated and many examples of Diophantine equations with no positive integer solution are demonstrated.

## 2. Preliminaries

First, we recall some elementary definitions and theorems in number theory. See [11] for instance.

**Definition 1.** *Let $n$ be a positive integer. The Euler phi-function $\phi(n)$ is defined to be the number of positive integers not exceeding $n$ that are relatively prime to $n$.*

**Definition 2.** *Let $a$ and $n$ be relatively prime integers with $a \neq 0$ and $n > 0$. The least positive integer $x$ such that $a^x \equiv 1 \pmod{n}$ is called the order of $a$ modulo $n$ and is denoted by $\mathrm{ord}_n a$.*

**Theorem 1.** *(Fermat's Little Theorem). If $p$ is a prime number and $a$ is an integer with $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.*

**Definition 3.** *Let $r$ and $n$ be relatively prime integers with $n > 0$. The integer $r$ is called a primitive root modulo $n$ if $\mathrm{ord}_n r = \phi(n)$.*

**Theorem 2.** *Every prime number has a primitive root.*

The concepts of quadratic residue and Legendre symbol have important roles in this paper.

**Definition 4.** *Let $a$ and $m$ be positive integers with $(a, m) = 1$. We say that $a$ is a quadratic residue of $m$ if the congruence $x^2 \equiv a \pmod{m}$ has a solution. Otherwise, $a$ is a quadratic nonresidue of $m$.*

**Definition 5.** *Let $p$ be an odd prime number and $a$ be an integer with $p \nmid a$. The Legendre symbol $(\frac{a}{p})$ is defined by*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

Some properties of Legendre symbol are given in Theorems 3 and 4.

**Theorem 3.** *Let $p$ be an odd prime number and $a, b$ be integers with $p \nmid a$ and $p \nmid b$. The following statements hold.*

*(i) If $a \equiv b \pmod{p}$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.*

*(ii) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$;*

*(iii) $\left(\frac{a^2}{p}\right) = 1$.*

**Theorem 4.** *(The Law of Quadratic Reciprocity). Let $p$ and $q$ be distinct odd prime numbers. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}.$$

The following theorem shows the form of odd prime number $p$ in the Legendre symbol $\left(\frac{2}{p}\right)$.

**Theorem 5.** *Let $p$ be an odd prime number. Then*

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8} \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

**Theorem 6.** *Let $p$ and $q$ be distinct odd prime numbers.*

*(i) If $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$, then $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$.*

*(ii) If $p \equiv 3 \pmod{4}$ and $q \equiv 3 \pmod{4}$, then $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$.*

**Theorem 7.** *Let $a_1, a_2, a_3, \ldots, a_n$ be integers and $m_1, m_2, m_3, \ldots, m_n$ be positive integers. Then, the system of congruences*

$$\begin{aligned} x \equiv\ & a_1 \pmod{m_1} \\ x \equiv\ & a_2 \pmod{m_2} \\ x \equiv\ & a_3 \pmod{m_3} \\ & \vdots \\ x \equiv\ & a_n \pmod{m_n} \end{aligned}$$

*has a solution if and only if $(m_i, m_j) | (a_i - a_j)$ for all pairs of integer $(i, j)$.*

In 2004, Siraworakun investigated the forms of odd prime numbers $p$ in Legendre symbol $\left(\frac{q}{p}\right)$, where $q$ is an odd prime number, in his unplublished senoir project. We review some important results in Theorem 8 - 11.

**Theorem 8.** *Let $p$ be an odd prime number. Then there is no a primitive root modulo $p$ in the form $n^2$, where $n$ is a natural number with $n < p$.*

*Proof.* Assume that there is a primitive root $n_0^2$ modulo $p$, where $n_0$ is a natural number with $n_0 < p$. Then $(n_0^2)^{p-1} \equiv 1 \pmod{p}$. So $(n_0^{p-1})^2 \equiv 1 \pmod{p}$. Thus, $n_0^{p-1} \equiv 1 \pmod{p}$ or $n_0^{p-1} \equiv -1 \pmod{p}$. If $n_0^{p-1} \equiv 1 \pmod{p}$, then $(n_0^2)^{\frac{p-1}{2}} \equiv 1 \pmod{p}$. It contradicts to the order of $n_0^2$ modulo $p$. Hence, $n_0^{p-1} \equiv -1 \pmod{p}$. Since $n_0 < p$, it contradicts to Fermat 's Little Theorem. Therefore, there is no a primitive root modulo $p$ in the form $n^2$, where $n$ is a natural number with $n < p$.

**Theorem 9.** *Let $p$ be an odd prime number and $r$ be a primitive root modulo $p$. Then $r^2, r^4, r^6, \ldots, r^{p-1}$ are quadratic residues of $p$ and $r^1, r^3, r^5, \ldots, r^{p-2}$ are quadratic non-residues of $p$.*

*Proof.* It is obvious that $r^2, r^4, r^6, \ldots, r^{p-1}$ are quadratic residues of $p$. Since $r$ is a primitive root modulo $p$ and Theorem 8, there is no a natural number $n_0$ with $n_0 < p$ such that $n_0^2 \equiv r \pmod{p}$. Then $\left(\frac{r}{p}\right) = -1$. By Theorem 3$(ii), (iii)$, we have $\left(\frac{r^3}{p}\right) = \left(\frac{r^5}{p}\right) = \cdots = \left(\frac{r^{p-2}}{p}\right) = -1$ . Hence, $r^1, r^3, r^5, \ldots, r^{p-2}$ are quadratic nonresidues of $p$.

To find the forms of odd prime numbers $p$ in Legendre symbol $\left(\frac{q}{p}\right)$, where $q$ is an odd prime number, we use the Chinese Remainder Theorem for solving the system of congruences.

**Theorem 10.** *Let $p$ and $q$ be distinct odd prime numbers with $q \equiv 1 \pmod{4}$. Then*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q + r^{S_1}q + r^{S_1} \pmod{2q} \\ -1 & \text{if } p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q} \end{cases},$$

*where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and $r$ is a primitive root modulo $q$.*

*Proof.* Since $q \equiv 1 \pmod{4}$, we have $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$ by Theorem 6$(i)$. Let $r$ be a primitive root modulo $q$. By Theorem 9, we obtain that

$$\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if } p \equiv r^{S_1} \pmod{q} \\ -1 & \text{if } p \equiv r^{S_2} \pmod{q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$ and $S_2 \in \{1, 3, 5, \ldots, q-2\}$.

Case 1. $\left(\frac{q}{p}\right) = 1$. Then $\left(\frac{p}{q}\right) = 1$. Thus, $p \equiv r^{S_1} \pmod{q}$. Since $p \equiv 1 \pmod{2}$ and by the Chinese Remainder Theorem, we obtain that $p \equiv q + r^{S_1}q + r^{S_1} \pmod{2q}$.

Case 2. $\left(\frac{q}{p}\right) = -1$. Then $\left(\frac{p}{q}\right) = -1$. Thus, $p \equiv r^{S_2} \pmod q$. Since $p \equiv 1 \pmod 2$ and by the Chinese Remainder Theorem, we obtain that $p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q}$.

**Theorem 11.** *Let $p$ and $q$ be distinct odd prime numbers with $q \equiv 3 \pmod 4$. Then*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 3q + 4n_0 r^{S_1}, \; -3q + 4n_0 r^{S_2} \pmod{4q} \\ -1 & \text{if } p \equiv 3q + 4n_0 r^{S_2}, \; -3q + 4n_0 r^{S_1} \pmod{4q} \end{cases},$$

*where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$, $r$ is a primitive root modulo $q$ and $n_0 = \dfrac{q+1}{4}$.*

*Proof.* Let $r$ be a primitive root modulo $q$. By Theorems 6 and 9, we obtain that

$$\left(\frac{q}{p}\right) = \begin{cases} \left(\frac{p}{q}\right) & \text{if } p \equiv 1 \pmod 4 \\ -\left(\frac{p}{q}\right) & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

and

$$\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if } p \equiv r^{S_1} \pmod q \\ -1 & \text{if } p \equiv r^{S_2} \pmod q \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$ and $S_2 \in \{1, 3, 5, \ldots, q-2\}$. Since $q \equiv 3 \pmod 4$, we choose an integer $n_0 = \dfrac{q+1}{4}$. Then $q = 4n_0 - 1$. Thus $3q \equiv 1 \pmod 4$ and $4n_0 \equiv 1 \pmod q$. In the following cases, the systems of congruences are solved by the Chinese Remainder Theorem.

Case 1. $\left(\frac{q}{p}\right) = 1$.

Case 1.1 $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$ and $\left(\frac{p}{q}\right) = 1$. Then $p \equiv 1 \pmod 4$ and $p \equiv r^{S_1} \pmod q$. Thus, $p \equiv 3q + 4n_0 r^{S_1} \pmod{4q}$.

Case 1.2 $\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right)$ and $\left(\frac{p}{q}\right) = -1$. Then $p \equiv 3 \pmod 4$ and $p \equiv r^{S_2} \pmod q$. Thus, $p \equiv -3q + 4n_0 r^{S_2} \pmod{4q}$.

Case 2. $\left(\frac{q}{p}\right) = -1$.

Case 2.1 $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$ and $\left(\frac{p}{q}\right) = -1$. Then $p \equiv 1 \pmod 4$ and $p \equiv r^{S_2} \pmod q$. Thus, $p \equiv 3q + 4n_0 r^{S_2} \pmod{4q}$.

Case 2.2 $\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right)$ and $\left(\frac{p}{q}\right) = 1$. Then $p \equiv 3 \pmod 4$ and $p \equiv r^{S_1} \pmod q$ . Thus, $p \equiv -3q + 4n_0 r^{S_1} \pmod{4q}$.

Moreover, Siraworakun has given the forms of prime numbers $p$ in Legendre symbol $\left(\frac{2q}{p}\right)$, where $q$ is a prime number.

**Theorem 12.** *Let $p$ and $q$ be distinct odd prime numbers with $q \equiv 1 \pmod 4$. Then*

$$\left(\frac{2q}{p}\right) = 1 \quad if \quad p \equiv q^2 + 8n_1 r^{S_1}, \ -q^2 + 8n_1 r^{S_1}, \ 3q^2 + 8n_1 r^{S_2}, -3q^2 + 8n_1 r^{S_2} \pmod{8q},$$

$$\left(\frac{2q}{p}\right) = -1 \quad if \quad p \equiv q^2 + 8n_1 r^{S_2}, -q^2 + 8n_1 r^{S_2}, 3q^2 + 8n_1 r^{S_1}, -3q^2 + 8n_1 r^{S_1} \pmod{8q},$$

*where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$, $r$ is a primitive root modulo $q$ and if $\frac{q-1}{4}$ is an even number, then $n_1 = \frac{-q+1}{8}$, and if otherwise, then $n_1 = \frac{3q+1}{8}$.*

*Proof.* By Theorem 3$(ii)$, we have $\left(\frac{2q}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{q}{p}\right)$. Let $r$ be a primitive root modulo $q$. By Theorems 5 and 10, we know that

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8 \\ -1 & \text{if } p \equiv \pm 3 \pmod 8 \end{cases}$$

and

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q + r^{S_1} q + r^{S_1} \pmod{2q} \\ -1 & \text{if } p \equiv q + r^{S_2} q + r^{S_2} \pmod{2q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$ and $S_2 \in \{1, 3, 5, \ldots, q-2\}$. Since $q \equiv 1 \pmod 4$, we have $q = 4k+1$ for some integer $k$. Then $q^2 \equiv 1 \pmod 8$. If $k$ is even, then $k = -2l$ for some integer $l$. It leads to $8l - 1 = -q$. Otherwise, $q = 8s+5$ for some integer $s$, so $8(3s+2) - 1 = 3q$. Choose $n_1 = \frac{-q+1}{8}$, when $k$ is even and otherwise, $n_1 = \frac{3q+1}{8}$. Thus, $8n_1 \equiv 1 \pmod q$. In the following cases, we use the Chinese Remainder Theorem for solving the systems of congruences.

Case 1. $\left(\frac{2q}{p}\right) = 1$.

Case 1.1 $\left(\frac{2}{p}\right) = 1$ and $\left(\frac{q}{p}\right) = 1$. Then $p \equiv \pm 1 \pmod 8$ and $p \equiv q + r^{S_1} q + r^{S_1} \pmod{2q}$. So $p \equiv r^{S_1} \pmod q$. It implies that $p \equiv q^2 + 8n_1 r^{S_1} \pmod{8q}$ or $p \equiv -q^2 + 8n_1 r^{S_1} \pmod{8q}$.

Case 1.2 $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{q}{p}\right) = -1$. Then $p \equiv \pm 3 \pmod 8$ and $p \equiv q + r^{S_2} q + r^{S_2} \pmod{2q}$. So $p \equiv r^{S_2} \pmod q$. It implies that $p \equiv 3q^2 + 8n_1 r^{S_2} \pmod{8q}$ or $p \equiv -3q^2 + 8n_1 r^{S_2} \pmod{8q}$.

Case 2. $\left(\frac{2q}{p}\right) = -1$.

Case 2.1 $\left(\frac{2}{p}\right) = 1$ and $\left(\frac{q}{p}\right) = -1$. Then $p \equiv \pm 1 \pmod 8$ and $p \equiv q + r^{S_2} q + r^{S_2} \pmod{2q}$. So $p \equiv r^{S_2} \pmod q$. It implies that $p \equiv q^2 + 8n_1 r^{S_2} \pmod{8q}$ or $p \equiv -q^2 + 8n_1 r^{S_2} \pmod{8q}$.

Case 2.2 $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{q}{p}\right) = 1$. Then $p \equiv \pm 3 \pmod 8$ and $p \equiv q + r^{S_1} q + r^{S_1} \pmod{2q}$. So $p \equiv r^{S_1} \pmod q$. It implies that $p \equiv 3q^2 + 8n_1 r^{S_1} \pmod{8q}$ or $p \equiv -3q^2 + 8n_1 r^{S_1} \pmod{8q}$.

**Theorem 13.** *Let $p$ and $q$ be distinct odd prime numbers with $q \equiv 3 \pmod 4$. Then*

$$\left(\frac{2q}{p}\right) = 1 \quad if \quad p \equiv \; q^2 + 32n_0n_1r^{S_1}, -q^2 + 32n_0n_1r^{S_2}, 3q^2 + 32n_0n_1r^{S_1}, -3q^2 + 32n_0n_1r^{S_2} \pmod{8q},$$

$$\left(\frac{2q}{p}\right) = -1 \quad if \quad p \equiv \; q^2 + 32n_0n_1r^{S_2}, -q^2 + 32n_0n_1r^{S_1}, 3q^2 + 32n_0n_1r^{S_2}, -3q^2 + 32n_0n_1r^{S_1} \pmod{8q},$$

*where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$, $r$ is a primitive root modulo $q$, $n_0 = \dfrac{q+1}{4}$ and if $\dfrac{q-3}{4}$ is an even number, then $n_1 = \dfrac{5q+1}{8}$, and if otherwise, then $n_1 = \dfrac{q+1}{8}$.*

*Proof.* By Theorem 3$(ii)$, we have $\left(\frac{2q}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{q}{p}\right)$. Let $r$ be a primitive root modulo $q$. By Theorems 5 and 11, we know that

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8 \\ -1 & \text{if } p \equiv \pm 3 \pmod 8 \end{cases}$$

and

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 3q + 4n_0r^{S_1}, \; -3q + 4n_0r^{S_2} \pmod{4q} \\ -1 & \text{if } p \equiv 3q + 4n_0r^{S_2}, \; -3q + 4n_0r^{S_1} \pmod{4q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$, $r$ is a primitive root modulo $q$ and $n_0 = \dfrac{q+1}{4}$. Since $q \equiv 3 \pmod 4$, we have $q = 4k+3$ for some integer $k$. Then $q^2 \equiv 1 \pmod 8$. If $k$ is even, then $k = 2l$ for some integer $l$. It leads to $8(5l+2) - 1 = 5q$. Otherwise, $q = 8s + 7$ for some integer $s$, so $8(s+1) - 1 = q$. Choose $n_1 = \dfrac{5q+1}{8}$, when $k$ is even and otherwise, $n_1 = \dfrac{q+1}{8}$. Thus, $8n_1 \equiv 1 \pmod q$. In the following cases, the systems of congruences are solved by the Chinese Remainder Theorem and Theorem 7.

Case 1. $\left(\frac{2q}{p}\right) = 1$.

Case 1.1 $\left(\frac{2}{p}\right) = 1$ and $\left(\frac{q}{p}\right) = 1$. Then $p \equiv \pm 1 \pmod 8$ and $p \equiv 3q + 4n_0r^{S_1}, -3q + 4n_0r^{S_2} \pmod{4q}$. So $p \equiv 4n_0r^{S_1}, 4n_0r^{S_2} \pmod q$. Since $(8, 4q) = 4$ and $q \equiv 3 \pmod 4$, we get that $4 \nmid (-3q+4n_0r^{S_2})-1$ and $4 \nmid (3q+4n_0r^{S_1})+1$. Hence, the system of congruences $p \equiv 1 \pmod 8$ and $p \equiv -3q + 4n_0r^{S_1} \pmod{4q}$ and the system of congruences $p \equiv -1 \pmod 8$ and $p \equiv 3q + 4n_0r^{S_1} \pmod{4q}$ have no solution. Thus, $p \equiv q^2 + 32n_0n_1r^{S_1} \pmod{8q}$ or $p \equiv -q^2 + 32n_0n_1r^{S_2} \pmod{8q}$.

Case 1.2 $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{q}{p}\right) = -1$. Then $p \equiv \pm 3 \pmod 8$ and $p \equiv 3q+4n_0r^{S_2}, -3q+4n_0r^{S_1} \pmod{4q}$. So $p \equiv 4n_0r^{S_2}, 4n_0r^{S_1} \pmod q$. Since $(8, 4q) = 4$ and $q \equiv 3 \pmod 4$, we get that $4 \nmid (3q+4n_0r^{S_2})-3$ and $4 \nmid (-3q+4n_0r^{S_1})+3$. Hence, the system of congruences $p \equiv 3 \pmod 8$ and $p \equiv 3q + 4n_0r^{S_2} \pmod{4q}$ and the system of congruences $p \equiv -3 \pmod 8$ and $p \equiv -3q + 4n_0r^{S_1} \pmod{4q}$ have no solution. Thus, $p \equiv 3q^2 + 32n_0n_1r^{S_1} \pmod{8q}$ or $p \equiv -3q^2 + 32n_0n_1r^{S_2} \pmod{8q}$.

Case 2. $\left(\frac{2q}{p}\right) = -1$.

Case 2.1 $\left(\frac{2}{p}\right) = 1$ and $\left(\frac{q}{p}\right) = -1$. Then $p \equiv \pm 1 \pmod 8$ and $p \equiv 3q + 4n_0r^{S_2}, -3q + 4n_0r^{S_1} \pmod{4q}$. So $p \equiv 4n_0r^{S_2}, 4n_0r^{S_1} \pmod q$. Since $(8, 4q) = 4$ and $q \equiv 3 \pmod 4$, we get that $4 \nmid (-3q+4n_0r^{S_1})-1$ and $4 \nmid (3q+4n_0r^{S_2})+1$. Hence, the system of congruences $p \equiv 1 \pmod 8$ and $p \equiv -3q + 4n_0r^{S_1} \pmod{4q}$ and the system of congruences $p \equiv -1 \pmod 8$ and $p \equiv 3q + 4n_0r^{S_1} \pmod{4q}$ have no solution. Thus, $p \equiv q^2 + 32n_0n_1r^{S_2} \pmod{8q}$ or $p \equiv -q^2 + 32n_0n_1r^{S_1} \pmod{8q}$.

Case 2.2 $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{q}{p}\right) = 1$. Then $p \equiv \pm 3 \pmod 8$ and $p \equiv 3q + 4n_0r^{S_1}, -3q + 4n_0r^{S_2} \pmod{4q}$. So $p \equiv 4n_0r^{S_1}, 4n_0r^{S_2} \pmod q$. Since $(8, 4q) = 4$ and $q \equiv 3 \pmod 4$, we get that $4 \nmid (3q+4n_0r^{S_1})-3$ and $4 \nmid (-3q+4n_0r^{S_1})+3$. Hence, the system of congruences $p \equiv 3 \pmod 8$ and $p \equiv 3q + 4n_0r^{S_1} \pmod{4q}$ and the system of congruences $p \equiv -3 \pmod 8$ and $p \equiv -3q + 4n_0r^{S_2} \pmod{4q}$ have no solution. Thus, $p \equiv 3q^2 + 32n_0n_1r^{S_2} \pmod{8q}$ or $p \equiv -3q^2 + 32n_0n_1r^{S_1} \pmod{8q}$.

## 3. Main Results

In this section, we study the Diophantine equation $p^x + (p + 2q)^y = z^2$, where $p$, $q$ and $p + 2q$ are prime numbers. Thus, $p$ is an odd prime number with $(p, q) = 1$. For the case $q = 2$, Burshtein [1] showed that the Diophantine equation $p^x + (p + 4)^y = z^2$, where $p > 3$ and $p + 4$ are primes, has no non-negative solution. Moreover, Rao [10] investigated the same equation, when $q = 2$ and $p = 3$. From now on, $x, y$ are positive integers and $p, q$ are distinct odd prime numbers.

**Lemma 1.** *Let $x$ be an even number. If the Diophantine equation $p^x + (p+2q)^y = z^2$ has a positive integer solution, then $2q \equiv 1 \pmod p$.*

*Proof.* Assume that the Diophantine equation $p^x + (p + 2q)^y = z^2$ has a positive integer solution. Since $x$ is even, there exists a positive integer $k$ such that $x = 2k$. Thus, $(p + 2q)^y = z^2 - p^{2k} = (z - p^k)(z + p^k)$. Since $p + 2q$ is a prime number, we have $z - p^k = (p + 2q)^u$ and $z + p^k = (p + 2q)^{y-u}$, where $u$ is a non-negative integer. So $y > 2u$ and $2p^k = (p + 2q)^u((p + 2q)^{y-2u} - 1)$. Since $p$ and $p + 2q$ are prime numbers, we obtain that $u = 0$ and so $2p^k = (p + 2q)^y - 1 = (p + 2q - 1)((p + 2q)^{y-1} + (p + 2q)^{y-2} + \cdots + 1)$. Hence $p|(p + 2q - 1)$. Therefore $2q \equiv 1 \pmod p$.

**Lemma 2.** *Let $x$ be an odd number. If the Diophantine equation $p^x + (p+2q)^y = z^2$ has a positive integer solution, then $\left(\frac{2q}{p}\right) = 1$.*

*Proof.* Assume that the Diophantine equation $p^x + (p + 2q)^y = z^2$ has a positive integer solution. Then $p^x \equiv z^2 \pmod{p + 2q}$. By Division Algorithm, we can write $x = (p + 2q - 1)m + l$, where $m$ and $l$ are integers with $0 \le l < p + 2q - 1$. Since $x$ is odd, we obtain $l$ is odd. By Theorem 1, we obtain that $p^{p+2q-1} \equiv 1 \pmod{p + 2q}$. So $p^{(p+2q-1)m+l} \equiv p^l \pmod{p+2q}$. Then $p^x \equiv p^l \pmod{p+2q}$. Thus, $z^2 \equiv p^l \pmod{p+2q}$.

Hence, $\left(\frac{p^l}{p+2q}\right) = 1$. Since $l$ is an odd number, we obtain $\left(\frac{p}{p+2q}\right) = 1$ by Theorem 3(ii). By Theorem 4, we obtain $\left(\frac{p}{p+2q}\right)\left(\frac{p+2q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{p+2q-1}{2}\right)} = 1$. Thus, $\left(\frac{p+2q}{p}\right) = 1$. By Theorem 3(i), we have $\left(\frac{2q}{p}\right) = \left(\frac{p+2q}{p}\right)$. Then $\left(\frac{2q}{p}\right) = 1$.

From above lemmas, we have the following result.

**Theorem 14.** *Let $p$ and $q$ be distinct prime numbers with $2q \not\equiv 1 \pmod p$ and $\left(\frac{2q}{p}\right) = -1$. Then the Diophantine equation $p^x + (p+2q)^y = z^2$ has no positive integer solution.*

By applying Theorems 12 and 14, the forms of odd prime number $p$ are identified, when $q \equiv 1 \pmod 4$.

**Theorem 15.** *Let $q$ be a prime number such that $q \equiv 1 \pmod 4$. If $p$ is a prime number with $2q \not\equiv 1 \pmod p$ and satisfies any of the following conditions:*

*(i)* $p \equiv q^2 + 8n_1r^{S_2} \pmod{8q}$,

*(ii)* $p \equiv -q^2 + 8n_1r^{S_2} \pmod{8q}$,

*(iii)* $p \equiv 3q^2 + 8n_1r^{S_1} \pmod{8q}$, *or*

*(iv)* $p \equiv -3q^2 + 8n_1r^{S_1} \pmod{8q}$,

*where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$, $r$ is a primitive root modulo $q$, and if $\dfrac{q-1}{4}$ is an even number, then $n_1 = \dfrac{-q+1}{8}$, and if otherwise, then $n_1 = \dfrac{3q+1}{8}$. Then, the Diophantine equation $p^x + (p+2q)^y = z^2$ has no positive integer solution.*

**Example 1.** *Let $q = 17$ and $r = 3$. Then $r$ is a primitive root of $q$ and $n_1 = \dfrac{-17+1}{8} = -2$ since $\dfrac{17-1}{4}$ is an even number. Consider a prime number $p$ that satisfies any of the following congruences:*

*(i)* $p \equiv 289 + (-16)\cdot(3)^{S_2} \pmod{136}$,

*(ii)* $p \equiv -289 + (-16)\cdot(3)^{S_2} \pmod{136}$,

*(iii)* $p \equiv 867 + (-16)\cdot(3)^{S_1} \pmod{136}$, *or*

*(iv)* $p \equiv -867 + (-16)\cdot(3)^{S_1} \pmod{136}$,

*where $S_1 \in \{2, 4, 6, 8, 10, 12, 14, 16\}$ and $S_2 \in \{1, 3, 5, 7, 9, 11, 13, 15\}$. Thus,*

$$p \equiv \pm 7, \pm 13, \pm 19, \pm 21, \pm 23, \pm 31, \pm 35, \pm 39, \pm 41, \pm 43, \pm 53, \pm 57, \pm 59, \pm 63, \pm 65, \pm 67 \pmod{136}.$$

*By Theorem 15, we obtain that the Diophantine equation $p^x + (p+34)^y = z^2$ has no positive integer solution. For example, $7^x + 41^y = z^2$, $13^x + 47^y = z^2$, $19^x + 47^y = z^2$, $53^x + 87^y = z^2$ and $67^x + 101^y = z^2$.*

**Example 2.** *Let $q = 5$ and $r = 2$. Then $r$ is a primitive root of $q$ and $n_1 = \dfrac{3(5)+1}{8} = 2$ since $\dfrac{5-1}{4}$ is an odd number. Consider a prime number $p$ that satisfies any of the following congruences:*

*(i)* $p \equiv \ 25 + (16) \cdot (2)^{S_2} \pmod{40}$,

*(ii)* $p \equiv \ -25 + (16) \cdot (2)^{S_2} \pmod{40}$,

*(iii)* $p \equiv \ 75 + (16) \cdot (2)^{S_1} \pmod{40}$, *or*

*(iv)* $p \equiv \ -75 + (16) \cdot (2)^{S_1} \pmod{40}$,

*where $S_1 \in \{2, 4\}$ and $S_2 \in \{1, 3\}$. Therefore, $p \equiv \pm 7, \pm 11, \pm 17, \pm 19 \pmod{40}$. By Theorem 15, we obtain that the Diophantine equation $p^x + (p + 10)^y = z^2$ has no positive integer solution. For example, $7^x + 17^y = z^2$, $19^x + 29^y = z^2$ (Burshtein [3]), $61^x + 71^y = z^2$ (Kumar [5]), $73^x + 83^y = z^2$ and $97^x + 107^y = z^2$.*

From Theorems 13 and 14, the forms of odd prime number $p$ can be obtained, when $q \equiv 3 \pmod 4$.

**Theorem 16.** *Let $p$ and $q$ be distinct prime numbers such that $q \equiv 3 \pmod 4$ and $2q \not\equiv 1 \pmod p$. If $p$ satisfies any of the following conditions:*

*(i)* $p \equiv \ q^2 + 32n_0n_1r^{S_2} \pmod{8q}$,

*(ii)* $p \equiv \ -q^2 + 32n_0n_1r^{S_1} \pmod{8q}$,

*(iii)* $p \equiv \ 3q^2 + 32n_0n_1r^{S_2} \pmod{8q}$, *or*

*(iv)* $p \equiv \ -3q^2 + 32n_0n_1r^{S_1} \pmod{8q}$,

*where $S_1 \in \{2, 4, 6, \dots, q-1\}$, $S_2 \in \{1, 3, 5, \dots, q-2\}$, $r$ is a primitive root modulo $q$, $n_0 = \dfrac{q+1}{4}$ and if $\dfrac{q-3}{4}$ is an even number, then $n_1 = \dfrac{5q+1}{8}$, and if otherwise, then $n_1 = \dfrac{q+1}{8}$. Then, the Diophantine equation $p^x + (p + 2q)^y = z^2$ has no positive integer solution.*

**Example 3.** *Let $q = 3$ and $r = 2$. Then $r$ is a primitive root of $q$, $S_1 = 2$, $S_2 = 1$, $n_0 = \dfrac{3+1}{4} = 1$ and $n_1 = \dfrac{5(3)+1}{8} = 2$ since $\dfrac{3-3}{4}$ is an even number. Consider a prime number $p$ that satisfies any of the following congruences:*

*(i)* $p \equiv \ 9 + (64) \cdot (2)^1 \pmod{24}$,

*(ii)* $p \equiv \ -9 + (64) \cdot (2)^2 \pmod{24}$,

*(iii)* $p \equiv \ 27 + (64) \cdot (2)^1 \pmod{24}$, *or*

*(iv)* $p \equiv \; -27 + (64)\cdot(2)^2 \pmod{24}$.

*Thus,* $p \equiv \pm 7, \pm 11 \pmod{24}$. *By Theorem 16, we obtain that the Diophantine equation* $p^x + (p+6)^y = z^2$ *has no positive integer solution. For example,* $7^x + 13^y = z^2$, $11^x + 17^y = z^2$, $13^x + 19^y = z^2$, $17^x + 23^y = z^2$ *and* $61^x + 67^y = z^2$ *(Kumar [4]).*

**Example 4.** *Let* $q = 7$ *and* $r = 3$. *Then* $r$ *is a primitive root of* $q$, $n_0 = \frac{7+1}{4} = 2$ *and* $n_1 = \dfrac{7+1}{8} = 1$ *since* $\dfrac{7-3}{4}$ *is an odd number. Consider a prime number* $p$ *that satisfies any of the following congruences:*

*(i)* $p \equiv \; 49 + 64(3^{S_2}) \pmod{56}$,

*(ii)* $p \equiv \; -49 + 64(3^{S_1}) \pmod{56}$,

*(iii)* $p \equiv \; 147 + 64(3^{S_2}) \pmod{56}$, *or*

*(iv)* $p \equiv \; -147 + 64(3^{S_1}) \pmod{56}$,

*where* $S_1 \in \{2,4,6\}$ *and* $S_2 \in \{1,3,5\}$. *Thus,* $p \equiv \pm 3, \pm 15, \pm 17, \pm 19, \pm 23, \pm 27 \pmod{56}$. *By Theorem 16, we obtain that the Diophantine equation* $p^x + (p+14)^y = z^2$ *has no positive integer solution. For example,* $3^x + 17^y = z^2$ *(Sroysang [12]),* $17^x + 31^y = z^2$, $23^x + 37^y = z^2$, $29^x + 43^y = z^2$, *and* $53^x + 67^y = z^2$.

## Acknowledgements

This work was supported by Research and Development Institute and Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

## References

[1] N Burshtein. The diophantine equation $p^x + (p+4)^y = z^2$ when $p > 3$ and $p+4$ are primes is insolvable in positive integers $x, y, z$. *Annals of Pure and Applied Mathematics*, 16(2):283–286, 2018.

[2] N Burshtein. Solutions of the diophantine equation $p^x + (p+6)^y = z^2$ when $p, p+6$ are primes and $x + y = 2, 3, 4$. *Annals of Pure and Applied Mathematics*, 17(1):101–106, 2018.

[3] N Burshtein. The diophantine equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ are insolvable in positive integers $x, y, z$. *Annals of Pure and Applied Mathematics*, 22(2):119–123, 2020.

[4] S Kumar S Gupta and H Kishan. On the non-linear diophantine equation $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. *Annals of Pure and Applied Mathematics*, 18(1):91–94, 2018.

[5] S Kumar S Gupta and H Kishan. On the non-linear diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. *Annals of Pure and Applied Mathematics*, 18(2):185–188, 2018.

[6] S Gupta S Kumar and H Kishan. On the non-linear diophantine equation $p^x + (p + 6)^y = z^2$. *Annals of Pure and Applied Mathematics*, 18(1):125–128, 2018.

[7] R J S Mina and J B Bacani. Non-existence of solutions of diophantine equations of the form $p^x + q^y = z^{2n}$. *Mathematics and Statistics*, 7(3):78–81, 2019.

[8] R J S Mina and J B Bacani. On the solutions of the diophantine equation $p^x + (p + 4k)^y = z^2$ for prime pairs $p$ and $p + 4k$. *European Journal of Pure and Applied Mathematics*, 14(2):471–479, 2021.

[9] F Neres. On the solvability of the diophantine equation $p^x + (p+8)^y = z^2$ when $p > 3$ and $p + 8$ are primes. *Annals of Pure and Applied Mathematics*, 18(1):9–13, 2018.

[10] C G Rao. On the diophantine equation $3^x + 7^y = z^2$. *EPRA International Journal of Research and Development*, 3(6):93–95, 2018.

[11] K H Rosen. *Elementary Number theory.* Clays Ltd, Great Britain, 2014.

[12] B Sroysang. On the diphantine equation $3^x + 17^y = z^2$. *International Journal of Pure and Applied Mathematics*, 89(1):111–114, 2013.

[13] S Tadee. On the diophantine equation $p^x+(p+10)^y = z^2$ when $p$ and $p+10$ are primes. *Udon Thani Rajabhat University Journal of Sciences and Technology*, 10(2):155–162, 2022.

[14] S Tadee. On the diophantine equation $p^x + (p+14)^y = z^2$, where $p$, $p+14$ are primes. *Annals of Pure and Applied Mathematics*, 26(2):125–130, 2022.