EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 791-805
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Asymptotic Approximations for generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi Polynomials in terms of Hyperbolic Functions

Cristina B. Corcino[1,2,*], Roberto B. Corcino[1,2]

[1] *Research Institute for Computational Mathematics and Physics, Cebu Normal University, 6000 Cebu City, Philippines*
[2] *Mathematics Department, Cebu Normal University, 6000 Cebu City, Philippines*

Abstract. Asymptotic approximation formulas for polynomials of the type Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi with integer order and real parameters are obtained via hyperbolic functions. The derivation of the formulas is done using the principle of saddle point and expansion of appropriate hyperbolic function about a saddle point.

2020 Mathematics Subject Classifications: 11B68, 42A16, 11M35

Key Words and Phrases: Approximations, Bernoulli polynomials, Euler polynomials, Genocchi polynomials

1. Introduction

Let $\alpha \in \mathbb{Z}^+$, $\lambda \in \mathbb{C}\backslash\{0\}$, $a, b, c \in \mathbb{R}^+$, $b \neq 1$, $c \neq 1$, $a \neq b$ and $x \in \mathbb{R}$. The generalized Apostol-Bernoulli, Euler and Genocchi polynomials with parameters α, λ, a, b, c, are given by means of the following generating functions (see [1]).

$$\left(\frac{t}{\lambda b^t - a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \qquad \left|t\ln\frac{b}{a}\right| < 2\pi, \tag{1.1}$$

$$\left(\frac{2}{\lambda b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \qquad \left|t\ln\frac{b}{a}\right| < \pi, \tag{1.2}$$

and

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^\alpha c^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \qquad \left|t\ln\frac{b}{a}\right| < \pi. \tag{1.3}$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4703

Email addresses: corcinoc@cnu.edu.ph (C. Corcino), corcinor@cnu.edu.ph (R. Corcino)

The polynomials defined above will also be referred to as Apostol-Bernoulli type, Apostol-Euler type and Apostol-Genocchi type polynomials in the discussion below. When $\alpha = 1, \lambda = 1$, $b = c = e$ and $a = 1$, these polynomials will reduce to the classical Bernoulli, Euler and Genocchi polynomials.

Asymptotic approximations for higher order Genocchi polynomials using residues were done in [2] and [3]. Approximations for the Bernoulli and Euler polynomials using hyperbolic functions were obtained in [4] and approximations for Genocchi polynomials in terms of hyperbolic functions were obtained in [5]. At the time of the search, there were no approximations for the generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials found in the literature.

In this paper asymptotic approximations for these polynomials will be derived using the method of [4]. In particular, the following results in [4] will be utilized.

Lemma 1.1. *For $z \in \mathbb{C}\backslash\{0\}$, the function $\Phi_k(n, z)$ defined by*

$$\Phi_k(n,z) = \frac{n!}{(nz)^n}\frac{1}{2\pi i}\int_C (w - z^{-1})^k e^{nzw}\frac{dw}{w^{n+1}}\ , \tag{1.4}$$

where C is a circle with center at the origin and radius ϵ_1, can be represented in the form

$$\Phi_k(n,z) = \frac{p_k(n)}{(nz)^k} \tag{1.5}$$

with

$$p_0(n) = 1,\ \ p_1(n) = 0,\ \ p_2(n) = -n,\ \ p_3(n) = 2n, \tag{1.6}$$

and the remaining polynomials are given by the recurrence

$$p_k(n) = (1 - k)p_{k-1}(n) + np_{k-2}(n). \tag{1.7}$$

Lemma 1.2. *For fixed $z \neq 0$, the sequence $\Phi_k(n, z)$ is an asymptotic sequence for $n \to +\infty$ that satisfies $\Phi_k(n, z) = O(n^{[\frac{k}{2}]-k})$.*

Theorem 1.3. *Let $f(w)$ be a meromorphic function with simple poles $w_1, w_2, ...$ and analytic at the origin. Let the contour C be a circle whose center is at the origin and which contains no poles of $f(w)$ inside. The polynomials $P_n(nz)$ defined by*

$$P_n(nz) = \frac{n!}{2\pi i}\int_C f(w)e^{nwz}\frac{dw}{w^{n+1}} \tag{1.8}$$

may be expanded as the infinite sum

$$P_n(nz) = (nz)^n\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}, \tag{1.9}$$

valid for $z \in \mathbb{C}\backslash\{0\}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = 1, 2, ...$ where $p_k(n)$ are the polynomials given in Lemma 1.1.

2. Proof of Theorem 1.3

The following proof of Theorem 1.3 is an expository of the proof presented in [4]. This is being provided to aid the derivation of the asymptotic formulas in Section 3.

Proof. Write

$$P_n(nz) = \frac{n!}{2\pi i}\int_C f(w)e^{nwz-n\log w}\frac{dw}{w}. \tag{2.1}$$

The key observation used for obtaining approximations of $P_n(nz)$ for large n and fixed z is that the main contribution of the integrand to the integral originates at the saddle point of the argument of the exponential (for a discussion of saddle point method see [6]), that is, at $w = z^{-1}$. If z^{-1} is not a pole of $f(w)$, then $f(w)$ can be expanded around z^{-1} as follows:

$$f(w) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})}{k!}(w-z^{-1})^k, \quad \left|w-z^{-1}\right| < r, \tag{2.2}$$

where r is the distance from the z^{-1} to the nearest singularity of $f(w)$. The radius ϵ_1 of the contour C in the definition of $P_n(z)$ can be chosen as close to 0 as necessary. Then for $w \in C(C : |w| = \epsilon_1)$, the above series is absolutely convergent if $\left|z^{-1}\right| < \left|z^{-1} - w_k\right|$ for all $k = 1, 2, \cdots$.

Substituting the expansion to $f(w)$ yields

$$P_n(nz) = \frac{n!}{2\pi i}\int_C \sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})}{k!}(w-z^{-1})^k e^{nwz}\frac{dw}{w^{n+1}}, \tag{2.3}$$

where

$$f^{(k)}(z^{-1}) = \frac{k!}{2\pi}\int_{C'}\frac{f(t)dt}{(t-z^{-1})^{k+1}}, \tag{2.4}$$

and C' is a circle around z^{-1} whose radius $R \equiv \left|t - z^{-1}\right| < \left|z^{-1} - w_k\right|$ for all k. That is,

$$R \equiv \min\left|z^{-1} - w_k\right| - \epsilon_2 \text{ for some } \epsilon_2 > 0.$$

Since $f(t)$ is bounded on C' there exists M_1 such that $|f(t)| < M_1$ for $t \in C'$. Therefore,

$$\begin{aligned}\left|f^{(k)}(z^{-1})\right| &\leq \frac{k!}{2\pi}\int_{C'}\frac{|f(t)|}{\left|t-z^{-1}\right|^{k+1}}|dt| \\ &\leq \frac{k!}{2\pi}\frac{M_1}{R^{k+1}}2\pi R = M_1\frac{k!}{R^k}. \end{aligned}\tag{2.5}$$

Let $a = \max_{w\in C}\frac{\left|w-z^{-1}\right|}{R}$. Note that a depends only on z, ϵ, and R and we can make $a < 1$. Then

$$|P_n(nz)| \leq M_1\sum_{k=0}^{\infty}\frac{n!}{2\pi}\int_C\left(\frac{\left|w-z^{-1}\right|}{R}\right)^k\left|e^{nwz}\right|\frac{|dw|}{|w^{n+1}|} \tag{2.6}$$

The function e^{nwz} is bounded on C for finite n and fixed z. Thus, there exists M_2 such that $|e^{nwz}| \leq M_2$, for $w \in C$. Hence,

$$\begin{aligned} |P_n(nz)| &\leq M_1\, M_2 \sum_{k=0}^{\infty} \frac{n!}{2\pi} \int_C a^k \frac{|dw|}{|w^{n+1}|} \\ &= M_1\, M_2 \sum_{k=0}^{\infty} a^k \frac{n!}{2\pi} \frac{1}{\epsilon_1^{n+1}} 2\pi\, \epsilon_1 \\ &= M_1\, M_2\, \frac{n!}{\epsilon_1^n} \sum_{k=0}^{\infty} a^k < \infty\ . \end{aligned} \tag{2.7}$$

Evaluating the integral in (2.3),

$$\begin{aligned} P_n(nz) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} \frac{n!}{2\pi i} \int_C (w - z^{-1})^k e^{nwz} \frac{dw}{w^{n+1}} \\ &= (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} \frac{n!}{(nz)^n} \frac{1}{2\pi i} \int_C (w - z^{-1})^k e^{nwz} \frac{dw}{w^{n+1}} \\ &= (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!}\, \Phi_k(n, z), \end{aligned} \tag{2.8}$$

where

$$\Phi_k(n, z) = \frac{n!}{(nz)^n} \frac{1}{2\pi i} \int_C (w - z^{-1})^k e^{nwz} \frac{dw}{w^{n+1}}. \tag{2.9}$$

From Lemma 1.1 and Lemma 1.2 , the functions $\Phi_k(n, z)$ are polynomials in n divided by powers of nz and constitute an asymptotic sequence for $n \to +\infty$. The desired asymptotic sequence is

$$P_n(nz) = (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k}, \tag{2.10}$$

where $p_k(n)$ are defined in Lemma 1.1.

Remark 2.1. *As can be seen in the proof of Theorem 1.3, the results of the theorem still hold for $f(t)$ having poles w_1, w_2, $\ldots$ of order greater than 1.*

3. The Asymptotic Approximations

The following are the main results of the study. In the discussion below, $\delta = \log \lambda$, $\lambda \in \mathbb{C}\backslash\{0\}$ where the logarithm is taken to be the principal branch and $\rho = (\delta + \mu \ln(ba^{-1}))/2$.

Theorem 3.1. *(Apostol-Bernoulli type polynomials of order 1)*
Let $a, b, c \in \mathbb{R}^+\backslash\{1\}, a \neq b$ and $\mu = (x \ln c)^{-1}$. For $x \in \mathbb{C}\backslash\{0\}$, such that $|\mu| < |\mu \pm \frac{\delta}{\ln(ba^{-1})}|$,

the following formula holds,

$$B_n(nx;\lambda;a,b,c)=\frac{(nx\ln c)^n}{2\sqrt{\lambda}}\frac{\mu(ab)^{\frac{-\mu}{2}}}{\sinh\rho}\left\{1-\frac{A}{2n(x\ln c)^2}+O(n^{-2})\right\}, \tag{3.1}$$

where,

$$\begin{aligned}A=&\left(\frac{\ln(ab)}{2}-\frac{1}{\mu}+\frac{\ln(ba^{-1})}{2}\coth\rho\right)\left(\frac{\ln(ab)}{2}+\frac{\ln(ba^{-1})}{2}\coth\rho\right)-\frac{\ln(ab)}{2\mu}\\&+\frac{\ln(ba^{-1})}{2}\left(csch^2\rho\ \frac{\ln(ba^{-1})}{2}-\frac{\coth\rho}{\mu}\right).\end{aligned} \tag{3.2}$$

Proof. Taking $\alpha=1$, (1.1) reduces to

$$\left(\frac{t}{\lambda b^t-a^t}\right)c^{xt}=\sum_{n=0}^{\infty}B_n(x;\lambda;a,b,c)\frac{t^n}{n!}.$$

Applying the Cauchy Integral Formula (for a discussion about Cauchy Integral Formula, see [7], [8]),

$$\frac{B_n(x;\lambda;a,b,c)}{n!}=\frac{1}{2\pi i}\int_C\frac{tc^{xt}}{\lambda b^t-a^t}\frac{dt}{t^{n+1}}, \tag{3.3}$$

where C is a circle with center at the origin and radius $<\left|\frac{\delta}{\ln(ba^{-1})}\right|$. Note that $-\delta/\ln(ba^{-1})$ is the simple pole of the integrand of (3.3) different from zero and nearest to the origin as can be seen in the computation of the singularities below.

Rewriting

$$\begin{aligned}\lambda b^t-a^t&=e^{\delta}e^{t\ln b}-e^{t\ln a}\\&=\left(e^{\delta+t\ln b}-e^{t\ln a}\right)\frac{e^{-t\ln a}}{e^{-t\ln a}}\\&=\left(e^{\delta+t(\ln ba^{-1})}-1\right)e^{t\ln a}\\&=\left[2e^{\frac{\delta+t\ln(ba^{-1})}{2}}\sinh\left(\frac{\delta+t\ln(ba^{-1})}{2}\right)\right]e^{t\ln a}.\end{aligned}$$

Then (3.3) becomes

$$\frac{B_n(x;\lambda;a,b,c)}{n!}=\frac{\frac{1}{2}\lambda^{-1/2}}{2\pi i}\int_C\frac{t(ab)^{\frac{-t}{2}}}{\sinh\left(\frac{\delta+t\ln(ba^{-1})}{2}\right)}c^{xt}\frac{dt}{t^{n+1}},$$

from which,

$$\frac{B_n(nx;\lambda;a,b,c)}{n!} = \frac{\frac{1}{2}\lambda^{-\frac{1}{2}}}{2\pi i}\int_C g(t)e^{tnx\ln c - n\log t}\frac{dt}{t}, \tag{3.4}$$

where

$$g(t) = \frac{t(ab)^{\frac{-t}{2}}}{\sinh\left(\frac{\delta + t\ln(ba^{-1})}{2}\right)}. \tag{3.5}$$

The saddle-point at which the major contribution to the integral in (3.4) occurs is $\mu = (x\ln c)^{-1}$. The singularities of $g(t)$ are computed as follows:

$$\begin{aligned}\sinh\left(\frac{\delta + t\ln(ba^{-1})}{2}\right) \Leftrightarrow \frac{\delta + t\ln(ba^{-1})}{2} &= k\pi i, k \in \mathbb{Z}\\ \delta + t\ln(ba^{-1}) &= 2k\pi i\\ t\ln(ba^{-1}) &= 2k\pi i - \delta\\ t_k := t &= \frac{2k\pi i - \delta}{\ln(ba^{-1})}, k \in \mathbb{Z}.\end{aligned}$$

Assume that $\mu = (x\ln c)^{-1}$ is not a singularity of g. Then $g(t)$ can be expanded about μ. That is,

$$g(t) = \sum_{k=0}^{\infty}\frac{g^{(k)}(\mu)}{k!}(t-\mu)^k, \quad |t-\mu| < r$$

where r is the distance from μ to the nearest singularity of $g(t)$. The derivatives of $g(t)$ for $k = 1, 2$ evaluated at $t = \mu$ are given below:

$$g'(\mu) = \left(1 + \frac{-\mu\ln(ab)}{2} - \mu\frac{\ln(ba^{-1})}{2}\coth\rho\right)\frac{e^{\frac{-\mu}{2}\ln(ab)}}{\sinh\rho}, \tag{3.6}$$

$$\begin{aligned}g''(\mu) = \Bigg\{&\left(\frac{\ln(ab)}{2} - \frac{1}{\mu} + \frac{\ln(ba^{-1})}{2}\coth\rho\right)\left(\frac{\ln(ab)}{2} + \frac{\ln(ba^{-1})}{2}\coth\rho\right)\\ &-\frac{\ln(ab)}{2\mu} + \frac{[\ln(ba^{-1})]^2}{4}csch^2\rho - \frac{\ln(ba^{-1})}{2\mu}\coth\rho\Bigg\} \times \frac{\mu\, e^{\frac{-\mu}{2}\ln(ab)}}{\sinh\rho}.\end{aligned} \tag{3.7}$$

Using (3.4) and applying Theorem 1.3,

$$\begin{aligned}B_n(nx;\lambda;a,b,c) &= \frac{(nx\ln c)^n}{2\sqrt{\lambda}}\left\{g(\mu) - \frac{g''(\mu)}{2n(x\ln c)^2} + O(n^{-2})\right\}\\ &= \frac{(nx\ln c)^n}{2\sqrt{\lambda}}\left\{\frac{\mu(ab)^{\frac{-\mu}{2}}}{\sinh\rho} - \frac{\mu(ab)^{\frac{-\mu}{2}}}{\sinh\rho}\;\frac{A}{2n(x\ln c)^2} + O(n^{-2})\right\}\end{aligned}$$

$$= \frac{(nx\ln c)^n}{2\sqrt{\lambda}}\ \frac{\mu(ab)^{\frac{-\mu}{2}}}{\sinh\rho}\left\{1-\frac{A}{2n(x\ln c)^2}+O(n^{-2})\right\},$$

where A is as given in (3.2).

Asymptotic formula for the Apostol-Bernoulli type polynomials of order $\alpha > 1$ is given in the next theorem.

Theorem 3.2. *(Apostol-Bernoulli type polynomials of order $\alpha \geq 2$)*
Let $a, b, c \in \mathbb{R}^+\backslash\{1\}, a \neq b$ and $\mu = (x\ln c)^{-1}$. For $x \in \mathbb{C}\backslash\{0\}$ such that $|\mu| < \left|\mu \pm \frac{\delta}{\ln(ba^{-1})}\right|$, $\delta = \log\lambda$, $n \geq \alpha$, the following holds,

$$B_n^{(\alpha)}(nx;\lambda;a,b,c) = \frac{(nx\ln c)^n}{2^\alpha\lambda^{\frac{\alpha}{2}}}\left(\frac{\mu(ab)^{-\frac{\mu}{2}}}{\sinh\rho}\right)^\alpha\left\{1-\frac{\alpha(A+(\alpha-1)J^2)}{2n(x\ln c)^2}+O(n^{-2})\right\}, \quad (3.8)$$

where A is given in (3.2) *and J is given by*

$$J = -\frac{\ln(ab)}{2}+\frac{1}{\mu}-\coth\rho\frac{\ln(ba^{-1})}{2}. \quad (3.9)$$

Proof. Applying the Cauchy Integral Formula to (1.1) yields

$$\frac{B_n^{(\alpha)}(x;\lambda;a,b,c)}{n!} = \frac{1}{2\pi i}\int_C\left(\frac{t}{\lambda b^t-a^t}\right)^\alpha c^{xt}\frac{dt}{t^{n+1}},$$

where C is a circle around the origin with radius $< \left|\frac{\delta}{\ln(ba^{-1})}\right|$.

Writing

$$\begin{aligned}
\left(\frac{t}{\lambda b^t-a^t}\right)^\alpha &= \frac{t^\alpha a^{-\alpha t}}{(e^\delta(ba^{-1})^t-1)^\alpha} = \frac{t^\alpha a^{-\alpha t}}{[\exp(t\ln(ba^{-1})+\delta)-1]^\alpha}\\
&= \frac{t^\alpha a^{-\alpha t}}{\left[2\exp\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)\sinh\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)\right]^\alpha}\\
&= \frac{t^\alpha (ab)^{\frac{-\alpha}{2}t}}{2^\alpha\lambda^{\frac{\alpha}{2}}\sinh^\alpha\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{B_n^{(\alpha)}(nx;\lambda;a,b,c)}{n!} &= \left(2^{-\alpha}\lambda^{\frac{-\alpha}{2}}\right)\frac{1}{2\pi i}\int_C\frac{t^\alpha(ab)^{-\frac{\alpha}{2}t}c^{nxt}}{\sinh^\alpha\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)}\frac{dt}{t^{n+1}}\\
&= 2^{-\alpha}\lambda^{\frac{-\alpha}{2}}\frac{1}{2\pi i}\int_C g_\alpha(t)c^{nxt}\frac{dt}{t^{n+1}},
\end{aligned}$$

where,

$$g_\alpha(t) = [g(t)]^\alpha = \left(\frac{t(ab)^{\frac{-t}{2}}}{\sinh\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)} \right)^\alpha . \tag{3.10}$$

The saddle-point is still $\mu = (x\ln c)^{-1}$. The function $g_\alpha(t)$ has poles of order α at $t_k = \frac{2k\pi i-\delta}{\ln(ba^{-1})}$, $k \in \mathbb{Z}$. Assuming that $\mu = (x\ln c)^{-1}$ is not a singularity of $g_\alpha(t)$. Then $g_\alpha(t)$ can be expanded about μ. That is,

$$g_\alpha(t) = \sum_{k=0}^{\infty} \frac{g_\alpha^{(k)}(\mu)}{k!}(t-\mu)^k, \qquad |t-\mu| < r$$

where r is the distance from μ to the nearest singularity of $g_\alpha(t)$.

The derivatives for $k = 1, 2$ are

$$g_\alpha'(t) = \alpha[g(t)]^{\alpha-1}g'(t), \tag{3.11}$$

$$g_\alpha''(t) = \alpha\left\{g(t)^{\alpha-1}g''(t) + (\alpha-1)g(t)^{\alpha-2}[g'(t)]^2\right\}, \tag{3.12}$$

where $g(t)$ is defined in (3.5). Evaluating at $t = \mu$,

$$g_\alpha'(\mu) = \alpha\left(\frac{\mu(ab)^{-\frac{\mu}{2}}}{\sinh\rho}\right)^\alpha \left(-\frac{\ln(ab)}{2} + \frac{1}{\mu} - \coth\rho\frac{\ln(ba^{-1})}{2}\right), \tag{3.13}$$

$$g_\alpha''(\mu) = \alpha\left\{\left(\frac{\mu(ab)^{-\frac{\mu}{2}}}{\sinh\rho}\right)^{\alpha-1} \frac{\mu e^{-\frac{\mu\ln(ab)}{2}}}{\sinh\rho} A + (\alpha-1)\left(\frac{\mu(ab)^{-\frac{\mu}{2}}}{\sinh\rho}\right)^{\alpha-2}\left(\frac{\mu e^{-\frac{\mu\ln(ab)}{2}}}{\sinh\rho}J\right)^2\right\} \tag{3.14}$$

$$= \alpha\left(\frac{\mu(ab)^{-\frac{\mu}{2}}}{\sinh\rho}\right)^\alpha\left\{A + (\alpha-1)\left(-\frac{\ln(ab)}{2} + \frac{1}{\mu} - \coth\rho\frac{\ln(ba^{-1})}{2}\right)^2\right\}. \tag{3.15}$$

It follows from Theorem 1.3 that

$$\begin{aligned}\frac{B_n^{(\alpha)}(nx;\lambda;a,b,c)}{2^{-\alpha}\lambda^{\frac{-\alpha}{2}}} &= (nx\ln c)^n \sum_{k=0}^{\infty} \frac{g_\alpha^{(k)}(n)}{k!}\frac{p_k(n)}{(nx\ln c)^k} \\ &= (nx\ln c)^n\left\{g_\alpha(\mu) + \frac{g_\alpha''(\mu)}{2(nx\ln c)^2}p_2(n) + O(n^{-2})\right\}.\end{aligned}$$

Then,

$$B_n^{(\alpha)}(nx;\lambda;a,b,c) = 2^{-\alpha}\lambda^{-\frac{\alpha}{2}}(nx\ln c)^n\left\{\left(\frac{\mu(ab)^{\frac{-\mu}{2}}}{\sinh\rho}\right)^\alpha + \frac{g_\alpha''(\mu)}{2n(x\ln c)^2} + O(n^{-2})\right\}$$

$$= 2^{-\alpha}\lambda^{-\frac{\alpha}{2}}(nx\ln c)^n\left(\frac{\mu(ab)^{\frac{-\mu}{2}}}{\sinh\rho}\right)^{\alpha}\left\{1-\frac{\alpha A+\alpha(\alpha-1)J^2}{2n(x\ln c)^2}+O(n^{-2})\right\},$$

where A is given in(3.2) and

$$J = -\frac{\ln(ab)}{2}+\frac{1}{\mu}-\coth\rho\frac{\ln(ba^{-1})}{2}.$$

Theorem 3.3. *(Apostol-Euler type polynomials of order 1)*
Let $a, b, c \in \mathbb{R}^+\backslash\{1\}, a \neq b$ and $\mu = (x\ln c)^{-1}$. For $x \in \mathbb{C}\backslash\{0\}$ such that $|\mu| < |\mu \pm \frac{\pi i-\delta}{\ln(ba^{-1})}|$, the following holds,

$$E_n(nx;\lambda;a,b,c) = \frac{(nx\ln c)^n(ab)^{\frac{-\mu}{2}}\lambda^{\frac{-1}{2}}}{\cosh\rho}\left\{1-\frac{F}{2n(x\ln c)^2}+O(n^{-2})\right\} \tag{3.16}$$

where

$$F = \left(\frac{\ln(ab)}{2}+\frac{\ln(ba^{-1})}{2}\tanh\rho\right)^2-\frac{\ln^2(ba^{-1})}{4}\operatorname{sech}^2\rho. \tag{3.17}$$

Proof. Taking $\alpha = 1$, (1.2) reduces to

$$\left(\frac{2}{\lambda b^t+a^t}\right)c^{xt} = \sum_{n=0}^{\infty}E_n(x;\lambda;a,b,c)\frac{t^n}{n!}, \qquad \left|t\ln\frac{b}{a}\right| < \pi.$$

Applying the Cauchy Integral Formula,

$$\frac{E_n(x;\lambda;a,b,c)}{n!} = \frac{1}{2\pi i}\int_C\frac{2c^{xt}}{[\lambda b^t+a^t]}\frac{dt}{t^{n+1}},$$

where C is a circle around the origin with radius $< \left|\frac{\pi i-\delta}{\ln(ba^{-1})}\right|$.

Writing

$$\frac{1}{\lambda b^t+a^t} = \frac{a^{-t}}{e^{\delta}(ba^{-1})^t+1},$$

then

$$\frac{E_n(x;\lambda;a,b,c)}{n!} = \frac{1}{2\pi i}\int_C\frac{2(a^{-1}c^x)^t}{[e^{\delta}(ba^{-1})^t+1]}\frac{dt}{t^{n+1}}.$$

With

$$\begin{aligned} e^{\delta}(ba^{-1})^t + 1 &= e^{\delta} e^{t\ln(ba^{-1})} + 1 \\ &= 2\exp\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)\cosh\left(\frac{t\ln(ba^{-1})+\delta}{2}\right), \end{aligned}$$

yields

$$\begin{aligned} \frac{E_n(x;\lambda;a,b,c)}{n!} &= \frac{e^{-\frac{\delta}{2}}}{2\pi i}\int_C \frac{((ab)^{-\frac{1}{2}}c^x)^t}{\cosh(\frac{t\ln(ba^{-1})+\delta}{2})}\frac{dt}{t^{n+1}} \\ &= \frac{(\lambda)^{-\frac{1}{2}}}{2\pi i}\int_C \frac{(ab)^{-\frac{1}{2}t}c^{xt}}{\cosh(\frac{t\ln(ba^{-1})+\delta}{2})}\frac{dt}{t^{n+1}} \\ &= \frac{\lambda^{-\frac{1}{2}}}{2\pi i}\int_C f(t)c^{xt}\frac{dt}{t^{n+1}} \qquad (3.18) \end{aligned}$$

where

$$f(t) = \frac{(ab)^{-\frac{1}{2}t}}{\cosh(\frac{t\ln(ba^{-1})+\delta}{2})}. \qquad (3.19)$$

Taking $x \mapsto nx$ and writing $c^{xt} = e^{tx\ln c}$, (3.18) will take the form

$$\frac{E_n(nx;\lambda,a,b,c)}{\lambda^{-\frac{1}{2}}} = \frac{n!}{2\pi i}\int_C f(t)e^{t(nx\ln c)}\frac{dt}{t^{n+1}},$$

which is of the form (1.8) where $z = x\ln c$.

The saddle-point occurs at

$$\begin{aligned} \frac{d}{dt}(nxt\ln c - n\log t) &= 0 \\ x\ln c - \frac{n}{t} &= 0 \\ \Leftrightarrow t = (x\ln c)^{-1} &= z^{-1} := \mu \end{aligned}$$

The function $f(t)$ is defined at $t = 0$ with $f(0) = \frac{1}{cosh\frac{\delta}{2}}$. Also $f'(0)$ is defined. The singularities of $f(t)$ are the zeros of $\cosh\frac{t\ln(ba^{-1})+\delta}{2}$, which are computed by solving for t such that

$$\cosh\left(\frac{t\ln(ba^{-1})+\delta}{2}\right) = 0.$$

Let $w = \frac{t\ln(ba^{-1})+\delta}{2}$. Then

$$\cosh w = 0 \Leftrightarrow w = \left(k+\frac{1}{2}\right)\pi i, \quad (k \in \mathbb{Z}).$$

That is,

$$\begin{aligned}\frac{t\ln(ba^{-1})+\delta}{2} &= (k+\frac{1}{2})\pi i\\ t\ln(ba^{-1}) &= (2k+1)\pi i - \delta\\ t_k := t &= \frac{(2k+1)\pi i - \delta}{\ln(ba^{-1})}, k \in \mathbb{Z}.\end{aligned}$$

The t_k are simple poles of $f(t)$.

Assume that $\mu = (x\ln c)^{-1}$ is not a singularity of $f(t)$. Then $f(t)$ can be expanded about $t = \mu$ as follows:

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\mu)}{k!}(t-\mu)^k, \quad |t-\mu| < r$$

where r is the distance from μ to the nearest singularity of $f(t)$.

The first few derivatives of f at $t = \mu$ are given below:

$$f'(\mu) = \left(-\frac{\ln(ab)}{2} - \frac{\ln(ba^{-1})}{2}\ \tanh\rho\right)\frac{(ab)^{-\frac{t}{2}}}{\cosh\rho}, \tag{3.20}$$

$$f''(\mu) = \left\{\left[\frac{\ln(ab)}{2} + \frac{\ln(ba^{-1})}{2}\tanh\rho\right]^2 - \frac{\ln^2(ba^{-1})}{4}\ \text{sech}^2\,\rho\right\}\frac{(ab)^{-\frac{t}{2}}}{\cosh\rho}. \tag{3.21}$$

Applying Theorem 1.3, the result follows.

For the Apostol-Euler type polynomials of order $\alpha > 1$ see the following theorem.

Theorem 3.4. *(Apostol-Euler type polynomials of order $\alpha \geq 2$) Let $a, b, c \in \mathbb{R}^+\backslash\{1\}, \alpha \in \mathbb{Z}^+, a \neq b$ and $\mu = (x\ln c)^{-1}$. For $x \in \mathbb{C}$ such that $\left|\mu \pm \frac{\pi i-\delta}{\ln(ba^{-1})}\right|$ and $n \geq \alpha$, the following formula holds,*

$$E_n^{(\alpha)}(nx;\lambda;a,b,c) = \frac{(nx\ln c)^n}{\lambda^{\frac{\alpha}{2}}}\left(\frac{(ab)^{-\frac{\mu}{2}}}{\cosh\rho}\right)^{\alpha}\left\{1 - \frac{\alpha F + \alpha(\alpha-1)H^2}{2n(x\ln c)^2} + O(n^{-2})\right\}, \tag{3.22}$$

where

$$H = \frac{-\ln(ab)}{2} - \frac{\ln(ba^{-1})}{2}\tanh\rho,$$

and F is given in Theorem 3.3.

Proof. Applying the Cauchy - Integral Formula to (1.2) yields

$$\frac{E_n^{(\alpha)}(x;\lambda;a,b,c)}{n!} = \frac{2^\alpha}{2\pi i}\int_C \frac{c^{xt}}{(\lambda b^t + a^t)^\alpha}\frac{dt}{t^{n+1}} \ ,$$

where C is a circle centered at zero with radius $< \left|\frac{\pi i - \delta}{\ln(ba^{-1})}\right|$.

Write

$$\begin{aligned}
\frac{1}{\lambda b^t + a^t} &= \frac{a^{-t}}{e^\delta (ba^{-1})^t + 1}, \\
\frac{1}{(\lambda b^t + a^t)^\alpha} &= \frac{a^{-\alpha t}}{[e^\delta (ba^{-1})^t + 1]^\alpha}, \\
e^\delta (ba^{-1})^t + 1 &= 2\exp\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)\cosh\left(\frac{t\ln(ba^{-1})+\delta}{2}\right), \\
\left[e^\delta (ba^{-1})^t + 1\right]^\alpha &= \left[2\exp\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)\right]^\alpha \left[\cosh\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)\right]^\alpha \\
&= 2^\alpha \lambda^{\frac{\alpha}{2}} e^{t\frac{\alpha}{2}\ln(ba^{-1})}\left[\cosh\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)\right]^\alpha .
\end{aligned}$$

Then,

$$\frac{E_n^{(\alpha)}(x;\lambda;a,b,c)}{n!} = \frac{\lambda^{-\frac{\alpha}{2}}}{2\pi i}\int_C \frac{(ab)^{-\frac{\alpha}{2}t}}{\left[\cosh\left(\frac{t\ln(ba^{-1})+\delta}{2}\right)\right]^\alpha}\ c^{xt}\frac{dt}{t^{n+1}} \ ,$$

Let

$$h(t) = \frac{(ab)^{-\frac{\alpha}{2}t}}{\left[\cosh(\frac{t\ln(ba^{-1})+\delta}{2})\right]^\alpha} = [f(t)]^\alpha \ .$$

Then

$$\frac{E_n^{(\alpha)}(nx;\lambda;a,b,c)}{n!} = \frac{\lambda^{-\frac{\alpha}{2}}}{2\pi i}\int_C h(t)\ c^{nxt}\frac{dt}{t^{n+1}} \ , \tag{3.23}$$

still with saddle point at $\mu = (x\ln c)^{-1}$. The poles of $h(t)$ are at $t = 0$ of order $n+1$ and at $t_k = \frac{(2k+1)\pi i - \delta}{\ln(ba^{-1})}$, $k \in \mathbb{Z}$ each of order α. Assuming that μ is not a singularity of $h(t)$, $h(t)$ can be expanded about μ given by

$$h(t) = \sum_{k=0}^{\infty}\frac{h^{(k)}(\mu)}{k!}(t-\mu)^k.$$

It follows from Theorem 1.3 that

$$\frac{E_n^{(\alpha)}(nx;\lambda;a,b,c)}{\lambda^{-\frac{\alpha}{2}}} = (nx\ln c)^n \sum_{k=0}^{\infty}\frac{h^{(k)}(\mu)}{k!}\frac{p_k(n)}{(nx\ln c)^k}$$

$$= (nx \ln c)^n \left\{ h(\mu) - \frac{h''(\mu)}{n(x\ln c)^2} + O(n^{-2}) \right\}. \tag{3.24}$$

Computing the derivatives $h^{(k)}(t)$, $k = 0, 1, 2$ with $h^0(t) = h(t)$ and evaluating at $t = \mu$ will give

$$\begin{aligned} h(\mu) &= [f(\mu)]^\alpha = \frac{(ab)^{-\frac{\alpha}{2}\mu}}{\cosh^\alpha \rho} \\ h'(\mu) &= \alpha\, [f(\mu)]^{\alpha-1} f'(\mu) \\ h''(\mu) &= \alpha \left\{ [f(\mu)]^{\alpha-1} f''(\mu) + (\alpha-1)[f(\mu)]^{\alpha-2}[f']^2 \right\}, \end{aligned}$$

where $f(\mu)$ is obtained from (3.19) , $f'(\mu)$, and $f''(\mu)$ are given in (3.20), and (3.21), respectively. Substitution to (3.24) will give the desired result.

To obtain an asymptotic formula for the Apostol-Genocchi type polynomials the following lemma will be used.

Lemma 3.5. *Let* $a, b, c \in \mathbb{R}^+\backslash\{1\}$, $\alpha \in \mathbb{Z}^+$, $\lambda \in \mathbb{C}\backslash\{1\}$, $a \neq b$. *For* $x \in \mathbb{C}$,

$$G^{(\alpha)}_{n+\alpha}(x; \lambda; a, b, c) = (n+\alpha)_\alpha E^{(\alpha)}_n(x; \lambda; a, b, c),$$

where

$$(n)_\alpha = n(n-1)(n-2)....(n-(\alpha-1).$$

Proof. Dividing both sides of (1.3) by t^α yields,

$$\begin{aligned} \left(\frac{2}{\lambda b^t + a^t}\right)^\alpha c^{xt} &= \sum_{n=0}^{\infty} G^{(\alpha)}_n(x; \lambda; a, b, c) \frac{t^{n-\alpha}}{n!} \\ &= \sum_{n=\alpha}^{\infty} \frac{(n-\alpha)!}{n!} G^{(\alpha)}_n(x; \lambda; a, b, c) \frac{t^{n-\alpha}}{(n-\alpha)!} \\ &= \sum_{n=\alpha}^{\infty} \frac{G^{(\alpha)}_n(x; \lambda; a, b, c)}{(n)_\alpha} \frac{t^{n-\alpha}}{(n-\alpha)!} \end{aligned}$$

Let $s = n - \alpha$. Then $n = s + \alpha$ and

$$\begin{aligned} \left(\frac{2}{\lambda b^t + a^t}\right)^\alpha c^{xt} &= \sum_{s=0}^{\infty} \frac{G^{(\alpha)}_{s+\alpha}(x; \lambda; a.b.c)}{(s+\alpha)_\alpha} \frac{t^s}{s!} \\ &= \sum_{n=0}^{\infty} \frac{G^{(\alpha)}_{n+\alpha}(x; \lambda; a.b.c)}{(n+\alpha)_\alpha} \frac{t^n}{n!} \\ \sum_{n=0}^{\infty} E^{(\alpha)}_n(x; \lambda; a, b, c) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \frac{G^{(\alpha)}_{n+\alpha}(x; \lambda; a.b.c)}{(n+\alpha)_\alpha} \frac{t^n}{n!} \end{aligned}$$

Comparing coefficients yields

$$G_{n+\alpha}^{(\alpha)}(x;\lambda;a.b.c) = (n+\alpha)_\alpha E_n^{(\alpha)}(x;\lambda;a,b,c). \tag{3.25}$$

Taking $\alpha = 1$, it follows from Lemma 3.5 that

$$G_{n+1}(x;\lambda;a.b.c) = (n+1)E_n(x;\lambda;a,b,c). \tag{3.26}$$

Corollary 3.6. *Let* $a,b,c \in \mathbb{R}\backslash\{1\}$, $a \neq b$ *and* $\mu = (x\ln c)^{-1}$. *For* λ, $x \in \mathbb{C}\backslash\{0\}$, $\lambda \neq 1$ *such that* $|\mu| < \left|\mu \pm \frac{\pi i - \delta}{\ln(ba^{-1})}\right|$,

$$G_{n+1}(nx;\lambda;a,b,c) = (n+1)\frac{(nx\ln c)^n (ab)^{\frac{-\mu}{2}}\lambda^{\frac{-1}{2}}}{\cosh\rho}\left\{1 - \frac{F}{2n(x\ln c)^2} + O(n^{-2})\right\}, \tag{3.27}$$

where F *is given in Theorem 3.3.*

Proof. This follows from (3.26) and Theorem 3.3. □

Corollary 3.7. *Let* $a,b,c \in \mathbb{R}\backslash\{1\}$, $\alpha \in \mathbb{Z}^+$, $a \neq b$ *and* $\mu = (x\ln c)^{-1}$.
For λ, $x \in \mathbb{C}\backslash\{0\}$

$$G_{n+\alpha}^{(\alpha)}(nx;\lambda;a,b,c) = (n+\alpha)_\alpha \frac{(nx\ln c)^n}{\lambda^{\frac{\alpha}{2}}}\left(\frac{(ab)^{-\frac{\mu}{2}}}{\cosh\rho}\right)^\alpha \left\{1 - \frac{\alpha F - \alpha(\alpha-1)H^2}{2n(x\ln c)^2} + O(n^{-2})\right\}, \tag{3.28}$$

where

$$H = \frac{-\ln(ab)}{2} - \frac{\ln(ba^{-1})}{2}\tanh\rho,$$

and F *is given in Theorem 3.3.*

Proof. This follows from Lemma 3.5 and Theorem 3.4. □

4. Conclusion and Recommendation

The formulas obtained in the paper are valid for nonzero complex numbers x such that the distance of $(x\ln c)^{-1}$ from the origin is smaller than its distance to the pole of the generating function nearest to the origin. This validity can be enlarged by isolating the contribution of the poles. This method was done in [4], [5]. The authors recommend to obtain approximation formulas with enlarged region of validity for the polynomials studied here.

Acknowledgements

This research is funded by Cebu Normal University through its Center for Research and Development (CRD).

References

[1] W.A. Khan, D. Srivastava, *On the Generalized Apostol-Type Frobenius-Genocchi Polynomials*, Filomat 33: 7(2019), 1967-1977.

[2] C.B. Corcino, R.B. Corcino, *Asymptotics of Genocchi Polynomials and Higher Order Genocchi Polynomials using Residues*, Afr. Mat., 31 (2020) pp. 781-792.

[3] C.B. Corcino, *Asymptotic Approximations of Apostol-Genocchi Numbers and Polynomials*, Eur. J. Pure Appl. Math., 14:3 (2021) pp. 666-684.

[4] J.L. Lopez and N.M. Temme, *Uniform Approximations of Bernoulli and Euler Polynomials in Terms of Hyperbolic Functions*, Stud. Appl. Math. 103(1999), no.3, 241-258.

[5] C.B. Corcino, R.B. Corcino, J.M. Ontolan, W.D. Castaneda, *Approximations of Genocchi Polynomials in Terms of Hyperbolic Functions*, Journal of Mathematical Analysis (2019) Vol. 10 Issue 3, 76-88.

[6] R. Wong, *Asymptotic Approximations of Integrals* Academic Press, New York, 1989.

[7] R.V. Churchill and J.W. Brown, Complex Variables and Applications, McGraw-Hill Book Company, 4th ed., 1984.

[8] L.L. Pennisi, Elements of Complex Variables, Holt, Rinehart and Winston, 2nd ed., 1976.