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On Divisibility Property of Type 2 (p, q)-Analogue of r-Whitney Numbers of the Second Kind

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Abstract. In this paper, the divisibility property of the type 2 (p, q)-analogue of the r-Whitney numbers of the second kind is established. More precisely, a congruence relation modulo pq for this (p, q)-analogue is derived.

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1. Introduction

The r-Whitney numbers of the second kind were introduced by Mezo [18] as coefficients of the following generating function:

$$(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n, k) x^{\underline{k}},$$

where $x^{\underline{k}} = x(x - 1) \dots (x - k + 1)$. These numbers satisfy the following properties:

1. the exponential generating function

$$\sum_{n=0}^{\infty} W_{m,r}(n, k) \frac{z^n}{n!} = \frac{e^{rz}}{k!} \left(\frac{e^{mz} - 1}{m} \right)^k,$$

2. the explicit formula

$$W_{m,r}(n, k) = \frac{1}{m^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (mi + r)^n,$$

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3. the triangular recurrence relation

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km+r)W_{m,r}(n-1,k).$$

These properties are exactly the same properties that the (r,β)-Stirling numbers in [7] have possessed. This implies that the r-Whitney numbers of the second kind and the (r,β)-Stirling numbers are equivalent. More properties of these numbers can be found in [2, 4, 5, 7, 18].

One of the early studies on q-analogue of Stirling numbers of the second kind was introduced by Carlitz in [1] in connection with a problem in abelian groups. This is known as q-Stirling numbers of the second kind and is defined in terms of the following recurrence relation

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q S_q[n-1,k], \quad [k]_q = \frac{1-q^k}{1-q}$$

such that, when $q \to 1$, this gives the triangular recurrence relation for the classical Stirling numbers of the second kind $S(n,k)$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Another version of definition of this q-analogue was adapted in [17] as follows

$$S_q[n,k] = q^{k-1}S_q[n-1,k-1] + [k]_q S_q[n-1,k]. \tag{1}$$

Through this definition, the Hankel transform of q-exponential polynomials and numbers was successfully established, which may be considered as the Hankel transform of a certain q-analogue of Bell polynomials and numbers.

There are many ways to define q-analogue of Stirling-type and Bell-type numbers (see [6, 8–10, 12, 14]). However, in the desire to establish the Hankel transform of q-analogue of generalized Bell numbers, Corcino et al. [11] were motivated to define a q-analogue of r-Whitney numbers of the second kind parallel to that in (1) as follows:

$$W_{m,r}[n,k]_q = q^{m(k-1)-r}W_{m,r}[n-1,k-1]_q + [mk-r]_q W_{m,r}[n-1,k]_q. \tag{2}$$

Two more forms of this q-analogue, denoted by $W^*_{m,r}[n,k]_q$ and $\widetilde{W}_{m,r}[n,k]_q$, were respectively defined by

$$\begin{aligned} W^*_{m,r}[n,k]_q &:= q^{-kr+m\binom{k}{2}}W_{m,r}[n,k]_q, \\ \widetilde{W}_{m,r}[n,k]_q &:= q^{-kr}W^*_{m,r}[n,k]_q = q^{-m\binom{k}{2}}W_{m,r}[n,k]. \end{aligned}$$

The corresponding q-analogues of generalized Bell numbers, also known as q-analogues of r-Dowling numbers, were also defined in three forms as (see [3, 11, 13, 15])

$$D_{m,r}[n]_q := \sum_{k=0}^{n} W_{m,r}[n,k]_q,$$

$$D^*_{m,r}[n]_q := \sum_{k=0}^{n} W^*_{m,r}[n,k]_q,$$

and

$$\widetilde{D}_{m,r}[n]_q := \sum_{k=0}^{n} \widetilde{W}_{m,r}[n,k]_q.$$

where $D_{m,r}[n]_q, D^*_{m,r}[n]_q$ and $\widetilde{D}_{m,r}[n]_q$ denote the first, second and third form of the q-analogues of r-Dowling numbers, respectively. The Hankel transforms of $D_{m,r}[n]_q, D^*_{m,r}[n]_q$ and $\widetilde{D}_{m,r}[n]_q$ were successfully established in [3, 11, 15].

To extend these research studies, a certain (p,q)-analogue of r-Whitney numbers of the second kind, denoted by $W_{m,r}[n,k]_{p,q}$, was defined in [16] as coefficients of the following generating function:

$$[mt+r]^n_{p,q} = \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q}[mt|m]^{\underline{k}}_{p,q} \tag{3}$$

where

$$[t|m]^{\underline{n}}_{p,q} = \prod_{j=0}^{n-1}[t-jm]_{p,q}. \tag{4}$$

The orthogonality and inverse relations, an explicit formula, and a kind of exponential generating function of $W_{m,r}[n,k]_{p,q}$ were already obtained. Unfortunately, its Hankel transform was not successfully established using the method applied in [3, 11, 15]. This motivated Corcino et al. [19] to define the type 2 (p,q)-analogue of r-Whitney numbers of the second kind, denoted by $W_{m,r}[n,k;t]_{p,q}$, as follows:

$$W_{m,r}[n+1,k;t]_{p,q} = q^{m(k-1)+r}W_{m,r}[n,k-1;t]_{p,q} + [mk+r]_{p,q}p^{mt-km}W_{m,r}[n,k;t]_{p,q}. \tag{5}$$

The second form was then defined as follows:

$$W^*_{m,r}[n,k;t]_{p,q} := q^{-kr-m\binom{k}{2}}W_{m,r}[n,k;t]_{p,q}. \tag{6}$$

Several properties of these (p,q)-analogues were established in [19] including their Hankel transforms, which are given by

$$\det\left(W_{m,r}[s+i+j,s+j;t]_{p,q}\right)_{0\le i,j\le n} = \prod_{k=0}^{n} q^{m\binom{s+k}{2}+(s+k)r}p^{nmt}[m(s+k)+r]^k_{p,q}$$

$$\det(W^*_{m,r}[s+i+j,s+j;t]_{p,q})_{0\le i,j\le n} = \prod_{k=0}^{n} p^{nmt}[m(s+k)+r]^k_{p,q}.$$

On the other hand, the first, second and third forms of type 2 (p,q)-analogue of the r-Dowling numbers, denoted by $D_{m,r}[n]_{p,q}$, $D^*_{m,r}[n]_{p,q}$ and $\widetilde{D}_{m,r}[n]_{p,q}$ were defined respectively in [19] as follows:

$$D_{m,r}[n]_{p,q} := \sum_{k=0}^{n} W_{m,r}[n,k;t]_{p,q},$$

$$D^*_{m,r}[n]_{p,q} := \sum_{k=0}^{n} W^*_{m,r}[n,k;t]_{p,q},$$
$$\widetilde{D}_{m,r}[n]_{p,q} := \sum_{k=0}^{n} \widetilde{W}_{m,r}[n,k;t]_{p,q},$$

where

$$\widetilde{W}_{m,r}[n,k;t]_{p,q} = q^{kr} W^*_{m,r}[n,k;t]_{p,q} \tag{7}$$

denotes the third form of the (p,q)-analogue of the r-Whitney numbers of the second kind. Among these three forms, only the second form was provided a Hankel transform, which is given by

$$H(D^*_{m,r}[n]_{p,q}) = \left(\frac{q}{p}\right)^{\frac{n(n^2+3n+8)}{6}+r-1\binom{n}{2}} ([m]_{\frac{q}{p}})^{\binom{n}{2}} \prod_{k=0}^{n-1} [k]_{\left(\frac{q}{p}\right)^m}!.$$

The main objective of this study is to establish additional property of the type 2 (p,q)-analogues of the r-Whitney numbers of the second kind. More precisely, the divisibility property of these type 2 (p,q)-analogues will be discussed thoroughly.

2. Preliminary Results

This section provides a brief discussion on some relations that are necessary in deriving the divisibility property of the type 2 (p,q)-analogue of the r-Whitney numbers of the second kind $W^*_{m,r}[n,k;t]_{p,q}$.

Multiplying both sides of the recurrence relation in (5) by $q^{-kr-m\binom{k}{2}}$ yields

$$\begin{aligned} q^{-kr-m\binom{k}{2}} W_{m,r}[n+1,k;t]_{p,q} &= q^{-kr-m\binom{k}{2}} q^{m(k-1)+r} W_{m,r}[n,k-1;t]_{p,q} \\ &\quad + q^{-kr-m\binom{k}{2}} [mk+r]_{p,q} p^{mt-km} W_{m,r}[n,k;t]_{p,q} \\ q^{-kr-m\binom{k}{2}} W_{m,r}[n+1,k;t]_{p,q} &= q^{-(k-1)r-m\binom{k-1}{2}} W_{m,r}[n,k-1;t]_{p,q} \\ &\quad + [mk+r]_{p,q} p^{mt-km} q^{-kr-m\binom{k}{2}} W_{m,r}[n,k;t]_{p,q}. \end{aligned}$$

Applying (6) consequently gives

$$W^*_{m,r}[n+1,k;t]_{p,q} = W^*_{m,r}[n,k-1;t]_{p,q} + [mk+r]_{p,q} p^{mt-km} W^*_{m,r}[n,k;t]_{p,q}. \tag{8}$$

This relation can be used to generate the following first few values of $W^*_{m,r}[n,k;t]_{p,q}$:

By repeated application of (8), we can easily derive the following vertical recurrence relation.

Theorem 2.1. *For nonnegative integers n and k, and real number r, the (p,q)-analogue of r-Whitney numbers of the second kind satisfies the following vertical recurrence relation*

$$W^*_{m,r}[n+1,k+1;t]_{p,q} = \sum_{j=k}^{n} [m(k+1)+r]_{p,q}^{n-j} p^{(n-j)[mt-(k+1)m]} W^*_{m,r}[j,k;t]_{p,q}. \tag{9}$$

n/k	0	1	2	3
0	1			
1	$[r]_{p,q}p^{mt}$	1		
2	$[r]_{p,q}^2p^{2mt}$	$[r]_{p,q}p^{mt}+[m+r]_{p,q}p^{m(t-1)}$	1	
3	$[r]_{p,q}^3p^{3mt}$	$[r]_{p,q}^2p^{2mt}+[r]_{p,q}[m+r]_{p,q}p^{m(2t-1)}$	$[r]_{p,q}p^{mt}+2[m+r]_{p,q}p^{m(t-1)}$	1
		$+[m+r]_{p,q}^2p^{2m(t-1)}$		

Table 1: The First Values of $W^*_{m,r}[n,k;t]_{p,q}$

One can easily verify relation (9) using the values of $W^*_{m,r}[n,k;t]_{p,q}$ in Table 1.

Now, let us derive the rational generating function for $W^*_{m,r}[n,k;t]_{p,q}$. Suppose that

$$\Psi^*_k(x)=\sum_{n=k}^{\infty}W^*_{m,r}[n,k;t]_{p,q}x^{n-k}.$$

When $k=0$, (8) reduces to

$$W^*_{m,r}[n+1,0;t]_{p,q}=[r]_{p,q}p^{mt}W^*_{m,r}[n,0;t]_{p,q}.$$

By repeated application of (8), this inductively gives

$$\begin{aligned}W^*_{m,r}[n+1,0;t]_{p,q}&=[r]_{p,q}p^{mt}W^*_{m,r}[n,0;t]_{p,q}=\left([r]_{p,q}p^{mt}\right)^2W^*_{m,r}[n-1,0;t]_{p,q}\\&\vdots\\&=\left([r]_{p,q}p^{mt}\right)^{n+1}W^*_{m,r}[0,0;t]_{p,q}=\left([r]_{p,q}p^{mt}\right)^{n+1}.\end{aligned}$$

Hence,

$$\Psi^*_0(x)=\sum_{n=0}^{\infty}W^*_{m,r}[n,0;t]_{p,q}x^n=\frac{1}{(1-xp^{mt}[r]_{p,q})}.$$

When $k>0$ and applying the triangular recurrence relation in (5), we have

$$\begin{aligned}\Psi^*_k(x)&=\sum_{n=k}^{\infty}W^*_{m,r}[n,k;t]_{p,q}x^{n-k}\\&=\sum_{n-1=k-1}^{\infty}W^*_{m,r}[n-1,k-1;t]_{p,q}x^{(n-1)(k-1)}\\&\quad+xp^{mt-km}[mk+r]_{p,q}\sum_{n-1=k}^{\infty}W^*_{m,r}[n-1,k;t]_{p,q}x^{n-1-k}\\&=\Psi^*_{k-1}(x)+xp^{m(t-k)}[mk+r]_{p,q}\Psi^*_k(x)\end{aligned}$$

Solving for $\Psi^*_k(t)$ yields

$$\Psi^*_k(x)=\frac{1}{1-xp^{m(t-k)}[mk+r]_{p,q}}\Psi^*_{k-1}(x).$$

Applying backward substitution gives the following rational generating function for $W_{m,r}[n,k;t]_{p,q}$.

Theorem 2.2. *For nonnegative integers n and k, and real number r, the (p,q)-analogue $W_{m,r}[n,k;t]_{p,q}$ satisfies the following rational generating function*

$$\Psi_k^*(x) = \sum_{n=k}^{\infty} W_{m,r}^*[n,k;t]_{p,q} x^{n-k} = \frac{1}{\prod_{j=0}^{k}(1 - xp^{m(t-j)}[mj+r]_{p,q})}. \tag{10}$$

Remark 2.3. *This rational generating function plays an important role in proving the main result of the paper.*

3. Divisibility Property

In this section, the congruence relation modulo pq for the type 2 (p,q)-analogue of the r-Whitney numbers of the second kind $W_{m,r}^*[n,k;t]_{p,q}$ will be established using the rational generating function in (10).

Using the values of $W_{m,r}^*[n,k;t]_{p,q}$ in Table 1, we observe that, with

$$[t]_{p,q} = p^{t-1} + p^{t-2}q + p^{t-3}q^2 + \ldots + pq^{t-2} + q^{t-1},$$

the polynomial expressions of $W_{m,r}^*[n,k]_q$ from row 0 to row 3, if they are divided by pq, the remainders form the following triangle of expressions in p:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & p^{mt+r-1} & & 1 & & \\ & p^{2(mt+r-1)} & & 2p^{mt+r-1} & & 1 & \\ p^{3(mt+r-1)} & & 3p^{2(mt+r-1)} & & 3p^{mt+r-1} & & 1. \end{array}$$

This can further be written as

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0}p^{mt+r-1} & & \binom{1}{1} & & \\ & \binom{2}{0}p^{2(mt+r-1)} & & \binom{2}{1}p^{mt+r-1} & & \binom{2}{2} & \\ \binom{3}{0}p^{3(mt+r-1)} & & \binom{3}{1}p^{2(mt+r-1)} & & \binom{3}{2}p^{mt+r-1} & & \binom{3}{3}, \end{array}$$

To generalize this observation, the next theorem contains the divisibility property of $W_{m,r}^*[n,k;t]_{p,q}$.

Theorem 3.1. *For nonnegative integers n and k, the type 2 (p,q)-analogue of the r-Whitney numbers of the second kind $W_{m,r}[n,k;t]_{p,q}$ satisfies the following congruence relation*

$$W_{m,r}^*[n,k;t]_{p,q} \equiv \binom{n}{k} p^{(n-k)(mt+r-1)} \mod pq. \tag{11}$$

Proof. The polynomial $[t]_{p,q}$ can be written as

$$[t]_{p,q} = p^{t-1} + q^{t-1} + pqy,$$

where y is a polynomial in p and q. Then, we have

$$\begin{aligned}\frac{1}{\prod_{j=0}^{k}(1-xp^{m(t-j)}[mj+r]_{p,q})} &= \sum_{n=0}^{\infty}\left(xp^{m(t-j)}[mj+r]_{p,q}\right)^n \\ &= \sum_{n=0}^{\infty}p^{nm(t-j)}(p^{mj+r-1}+q^{mj+r-1}+pqy)^n x^n \\ &= \sum_{n=0}^{\infty}p^{n(mt+r-j)}x^n + pq\sum_{n=0}^{\infty}\hat{z}_n x^n,\end{aligned}$$

where $\hat{z}_n$ is a polynomial in p and q. It follows that

$$\begin{aligned}\frac{1}{\prod_{j=0}^{k}(1-xp^{m(t-j)}[mj+r]_{p,q})} &= \sum_{n=0}^{\infty}p^{n(mt+r-j)}x^n \mod pq \\ &= \frac{1}{1-p^{mt+r-1}x} \mod pq.\end{aligned}$$

Thus, using (10), we have

$$\begin{aligned}\sum_{n=k}^{\infty}W^*_{m,r}[n,k;t]_{p,q}x^{n-k} &\equiv \frac{1}{(1-p^{mt+r-1}x)^{k+1}} \mod pq \\ &\equiv \sum_{n=0}^{\infty}\binom{n+(k+1)-1}{n}p^{n(mt+r-1)}x^n \mod pq \\ &\equiv \sum_{n=k}^{\infty}\binom{n}{k}p^{(n-k)(mt+r-1)}x^{n-k} \mod pq.\end{aligned}$$

Comparing the coefficients of x^{n-k} completes the proof of the theorem.

Remark 3.2. Using (6) and Theorem 3.1, the first form of the type 2 (p,q)-analogues of the r-Whitney numbers of the second kind satisfies the following congruence relation modulo pq:

$$\begin{aligned}W_{m,r}[n,k;t]_{p,q} &\equiv \binom{n}{k}p^{(n-k)(mt+r-1)}q^{kr+m\binom{k}{2}} \mod pq \\ &\equiv \begin{cases} q^{nr+m\binom{n}{2}} \mod pq, & \text{for } n=k \\ 0 \mod pq, & \text{otherwise.}\end{cases}\end{aligned} \tag{12}$$

Moreover, using (7) and Theorem 3.1, the third form of the type 2 (p,q)-analogues of the r-Whitney numbers of the second kind satisfies the following congruence relation modulo pq:

$$\widetilde{W}_{m,r}[n,k;t]_{p,q} \equiv \binom{n}{k}p^{(n-k)(mt+r-1)}q^{kr} \mod pq \tag{13}$$

$$\equiv \begin{cases} q^{nr} \mod pq, & \text{for } n = k \\ 0 \mod pq, & \text{otherwise.} \end{cases}$$

Remark 3.3. When $p = 1$, the congruence relation in (11) reduces to

$$W^*_{m,r}[n,k]_q = W^*_{m,r}[n,k;t]_{1,q} \equiv \binom{n}{k} \mod q,$$

which is exactly the congruence relation in [15, Theorem 2.1] for the second form of (q,r)-Whitney numbers of the second kind. Moreover, the congruence relations in (12) and (13) reduce to

$$W_{m,r}[n,k]_q = W_{m,r}[n,k;t]_{1,q} \equiv \binom{n}{k} q^{kr+m\binom{k}{2}} \equiv 0 \mod q$$
$$\widetilde{W}_{m,r}[n,k]_q = \widetilde{W}_{m,r}[n,k;t]_{1,q} \equiv \binom{n}{k} q^{kr} \equiv 0 \mod q,$$

which are the congruence relations for the first and third forms of (q,r)-Whitney numbers of the second kind. We recall that, for a prime p, the p-adic valuation $\nu_p(n)$ of n is defined to be the largest exponent k such that $p^k|n$. Moreover, the p-adic valuation of the rational number $\frac{n}{m}$ is defined by

$$\nu_p\left(\frac{n}{m}\right) = \nu_p(n) - \nu_p(m).$$

Furthermore, the p-adic absolute value $|n|_p$ of n is defined by

$$|n|_p = \frac{1}{p^{\nu_p(n)}}.$$

Clearly, when q is prime,

$$\nu_q\left(W_{m,r}[n,k]_q\right) = kr + m\binom{k}{2}$$
$$\nu_q\left(\widetilde{W}_{m,r}[n,k]_q\right) = kr.$$

Consequently,

$$\nu_q\left(\frac{W_{m,r}[n,k]_q}{\widetilde{W}_{m,r}[n,k]_q}\right) = \nu_q\left(W_{m,r}[n,k]_q\right) - \nu_q\left(\widetilde{W}_{m,r}[n,k]_q\right) = m\binom{k}{2}.$$

Also, one can easily see that

$$\nu_q\left(W^*_{m,r}[n,k]_q - \binom{n}{k}\right) = q^{\nu_p\left(W^*_{m,r}[n,k]_q - \binom{n}{k}\right)} \left|W^*_{m,r}[n,k]_q - \binom{n}{k}\right|_q = 1.$$

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