EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 899-918
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Dynamical Behaviors of Four-Dimensional Prey-Predator Model

A. G. Farhan[1,*], Nihad Sh. Khalaf[2], Talat J. Aldhlki [1]

[1] *Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq*
[2] *Department of Mathematics, College of Education for Women, Tikrit University, Tikrit, Iraq*

Abstract. In this paper, we study the dynamic behavior of a four-dimensional prey-predator suggested model of four species. The four species are two prey and two predator species, each of them grows logistically. The two prey live in diverse habitats and have the ability of group defense. In the mentioned model, one predator feeds on the two prey, the top predator feeds on other three species. The existents and, the boundedness of the positive solution, the existence and the local stability of all possible equilibrium points, of the model are investigated. The model has seven equilibrium points at most, four of them always exist and the others exist under certain conditions. Three equilibrium points are not stable while the others are locally asymptotically stable, under given conditions. For the coexistence point, a basin of attraction for it has been found. The steady-state bifurcation relative to the mortality rate of the predators in the neighborhood of three of the equilibrium points and the Hopf-bifurcation relative to the growth rate of the prey in the neighborhood of two of the equilibrium points has been found. Finally, two numerical example has been given to support the theoretical results.

2020 Mathematics Subject Classifications: 65P30, 34C23, 92D25

Key Words and Phrases: Basin of attraction, Bifurcation, Equilibrium point, Group defence, stability

---

1. Introduction

Predators feed in a habitat that is relatively rich in food for some time, which means to them that there are large numbers of prey, or that such prey is easy to catch. When food is scarce, or prey is minor in the habitat, the predators look for another habitat with enough food to live in for another period of time, this phenomenon is called switching, see [8, 9, 13]. Group defense means that all prey animals attack predators collectively and do

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4716

Email addresses: `a.g.farhan.edbs@uomustansiriyah.edu.iq` (A. G. Farhan),
`nihad.shreef16@tu.edu.iq` (N.Sh. Khalaf),
`dr.talatjassim.edbs@uomustansiriyah.edu.iq` (T.G. Aldhlki)

not give them an opportunity to attack and prey on them. In 1920, Volterra introduced a mathematical model that represents the interaction between prey and predator [1, 10]. Later many modified Volterra models were introduced by researchers interested in this field. Various models, including prey with one predator, two predators with one prey, two prey with two predators, and food chains of three or more species have been studied in [2–4, 15, 16]. In the wild, lions are at the top of the food chain because of their strength and ability to kill all animals and prey on them, but it is possible for a herd or a small group of hyenas to attack a young lion or an old lion outside its kingdom. In general, predators such as lions, hyenas, wolves, and others attack prey such as zebra, wild buffalo, and other prey with the aim of killing and devouring them. Prey by nature has the ability to live in groups that move from one habitat to another in search of food. Some prey animals, such as buffaloes and zebras, have the ability to collectively defend and attack predators in masse [3] and [8]. In [3], two prey-predatory models were studied, one of the two models was with group prey defense, while the other model was without group prey defense. The prey was supposed to live in two different habitats. Both predatory species tend to move from one habitat to another in search of food. The first model has been expanded into the following mathematical model that proposed and studied, in [5] and [7] with switching index n = 1,2, respectively. The expanded model deals with, two prey and two predators, all species grow logistically, the prey lives in diverse habitats with prey group defense, one predator tends to switch habitats and feeds on the prey, while the other predator feeds on only one prey.

$$
\begin{aligned}
\dot{x}_1 &= x_1\left(g_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1 x_2^n y_1}{x_1^n+x_2^n}-\beta y_2\right),\\
\dot{x}_2 &= x_2\left(g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\alpha_2 x_1^n y_1}{x_1^n+x_2^n}\right),\\
\dot{y}_1 &= y_1\left(-\mu_1+\frac{\delta_1 x_1 x_2^n}{x_1^n+x_2^n}+\frac{\delta_2 x_1^n x_2}{x_1^n+x_2^n}\right),\\
\dot{y}_2 &= y_2(-\mu_2+\gamma x_1)
\end{aligned}
\tag{1}
$$

The aim of this work is to study the local qualitative behaviors in the neighborhood of the equilibrium points of a proposed prey-predator mathematical model for four species with an incomplete food chain. The four species, consist of two prey and two predators, and each of them grows logistically. The two prey live in two different habitats and have the ability to group defense. One predator or the top feeds on the other predator, in addition to the two prey indicated in the model. We show that all positive solutions are bounded under some conditions. In the third section, it was found that the model has, at most seven equilibrium points. Local stability and local bifurcation of the equilibrium points have been analyzed in section4 and section5 respectively. Two sets of parameters form two systems of 4-dimensional differential equations, each of them is an example of the model (2) presented in the sixth section of this paper to simulate the theoretical results obtained in the previous sections. Finally a brave conclusion of this paper is given in the last section.

2. THE MATHEMATICAL MODEL

In this work, the following model was proposed, which is a modification of model (1)

$$
\begin{aligned}
\dot{x}_1 &= x_1\left(g_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1 x_2 y_1}{x_1+x_2}-\frac{\alpha_2 x_2 y_2}{x_1+x_2}\right),\\
\dot{x}_2 &= x_2\left(g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\beta_1 x_1 y_1}{x_1+x_2}-\frac{\beta_2 x_1 y_2}{x_1+x_2}\right),\\
\dot{y}_1 &= y_1\left(-\mu_1+\frac{\epsilon_1 x_1 x_2}{x_1+x_2}-\rho_2 y_2\right),\\
\dot{y}_2 &= y_2\left(-\mu_2+\frac{\epsilon_2 x_1 x_2}{x_1+x_2}+\rho_1 y_1\right)
\end{aligned}
\tag{2}
$$

where x_i denote the density of the prey $i=1,2$; y_1 denote the density of the predator that fed on the two prey; y_2 denote the density of the predator that fed on the other three species (top predator). $g_i, i=1,2$ is the growth rate of $x_i,\ i=1,2$; $k_i,\ i=1,2$, is the carrying environmental capacity to $x_i,\ i=1,2$; $\alpha_i, i=1,2$, is the rate of predation by the predator$y_i,\ i=1,2$ with the prey $x_1; \beta_i,\ i=1,2$ is the rate of predation by the predator $y_i,\ i=1,2$ on prey x_2; $\mu_i,\ i=1,2$, is the mortality rate of predators$y_i,\ i=1,2; \varepsilon_i, i=1,2$, is the corresponding conversion rates to $\mu_i,\ i=1,2$; ρ_1, is the rate of change of y_2 due to the presence of y_1,; and ρ_2, is the rate of change of y_1 due to the presence of y_2. The switching behavior of predators y_1 and y_2 is shown by, the functions $\alpha_1 x_2 y_1 (x_1+x_2)^{-1}$, $\beta_1 x_1 y_1 (x_1+x_2)^{-1}$ and, the functions $\alpha_2 x_2 y_2 (x_1+x_2)^{-1}$, $\beta_2 x_1 y_2 (x_1+x_2)^{-1}$ respectively. It is easy to show that all functions of the model (2) and their partial derivatives are continuous, so that these functions are Lipschitizion functions on

$$R^{+4}=\left\{(x_1,x_2,y_1,y_2)\in\mathcal{R}^4 : x_i(0)\geq 0, y_i(0)\geq 0, i=1,2, x_1+x_2>0\right\}.$$

Hence, the existence and the uniqueness of the solution of model (2) is guaranteed. Now, we will show that the trajectories of all the positive solutions of the system (2) whose initial conditions lie within the following region $\mathbb{D}$ are bounded

$$\mathbb{D}:=\left\{(x_1,x_2,y_1,y_2)\in\mathcal{R}^4, 0<x_i<k_i, y_i>0, i=1,2\right\}. \tag{3}$$

Theorem 1. *If $\varepsilon_i \leq \alpha_i+\beta_i,\ i=1,2,\ \rho_1\leq\rho_2$, then the trajectories of all the positive solutions of the system* (2) *whose initial conditions belong to $\mathbb{D}$ are bounded.*

Proof. The following real valued function:

$$z(t)=x_1(t)+x_2(t)+y_1(t)+y_2(t),$$

is a positive definitive on $\mathbb{D}$. Therefore, we get

$$\dot{z}=\sum_{i=1}^{2}\left(x_i g_i\left(1-\frac{x_i}{k_i}\right)-\mu_i y_i-\frac{(\rho_2-\rho_1)}{2}y_1 y_2-\frac{x_1 x_2 y_i}{x_1+x_2}\left(\alpha_i+\beta_i-\varepsilon_i\right)\right)$$

If $0 < \rho \leq max\{\mu_1, \mu_2\}$, then we obtain:

$$\dot{z} \ +\rho u = \sum_{i=1}^{2}\left(x_i g_i\left(1-\frac{x_i}{k_i}+\frac{\rho}{g_i}\right)+(\rho-\mu_i)y_i-\frac{(\rho_2-\rho_1)}{2}y_1y_2-\frac{x_1x_2y_i}{x_1+x_2}\left(\alpha_i+\beta_i-\varepsilon_i\right)\right).$$

It is obvious that

$$\begin{aligned}\dot{z}+\rho z &< \sum_{i=1}^{2} x_i g_i\left(1-\frac{x_i}{k_i}+\frac{\rho}{g_i}\right)\\ &= \sum_{i=1}^{2}\frac{x_i}{k_i}(k_ig_i-g_ix_i+k_i\rho) < \sum_{i=1}^{2}\frac{x_i}{k_i}(k_ig_i+k_i\rho) < \sum_{i=1}^{2}k_i(g_i+\rho).\end{aligned}$$

Put $\alpha =: \sum_{i=1}^{2} k_i\left(g_i+\rho\right)$. So it is clear that: $0 \leq z\left(t\right) \leq \frac{\alpha}{\rho} + z\left(0\right)e^{-\rho t}$, and when

$$t \to \infty, 0 \leq z(t) \leq \frac{\alpha}{\rho}. \tag{4}$$

So that, from (4), the trajectories of all the positive solutions of the system (2) with initial conditions lie within the region $\mathbb{D}$ defined above and satisfy $\varepsilon_i \leq \alpha_i + \beta_i,\ i = 1, 2,\ \ \rho_1 \leq \rho_2$ are bounded. and with this, we have completed the proof.

3. EXISTENCE OF EQUILIBRIUM POINTS

The system has four equilibrium points always exist, regardless of the values of the system parameters plus three others, whose existence depends on the change in the values of the system parameters. The equilibrium point are given as follows:

(i) The equilibrium points $P_0 = (0, 0, 0, 0), P_1 = (k_1, 0, 0, 0), P_2 = (0, k_2, 0, 0)$ and $P_3 = (k_1, k_2, 0, 0)$ always exist.

(ii) The equilibrium point $P_4 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$, where:

$$\tilde{x}_1 = \frac{\mu_1(1+\tilde{x})}{\epsilon_1},\ \tilde{x}_2 = \frac{\tilde{x}_1}{\check{x}},\ \tilde{y}_1 = g_1\left(1-\frac{\tilde{x}_1}{k_1}\right)\frac{(1+\tilde{x})}{\alpha_1} \tag{5}$$

and $\tilde{x}$ is a positive root of the equation (7) below, provided that

$$\tilde{x} < \frac{\varepsilon_1 k_1 - \mu_1}{\mu_1} \tag{6}$$

$$\mathbb{A}_1\tilde{x}^3 + \mathbb{A}_2\tilde{x}^2 + \mathbb{A}_3\tilde{x} + \mathbb{A}_4 = 0, \tag{7}$$

such that

$$\mathbb{A}_1 = g_1\beta_1k_2\mu_1 > 0, \qquad \mathbb{A}_2 = g_1\beta_1k_2(\mu_1 - k_1\varepsilon_1) < 0,$$

$$\mathbb{A}_3 = g_2\alpha_1 k_1(k_2\varepsilon_1 - \mu_1), \qquad \mathbb{A}_4 = -g_2\alpha_1 k_1\mu_1 < 0.$$

The number of sign changes between $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$, and $\mathbb{A}_4$ is one if $\mathbb{A}_3 < 0$, and three if $\mathbb{A}_3 > 0$. According to Descartes' rule of sign, equation (7) has at most one positive root if $\mathbb{A}_3 < 0$, and at most three positive root if $\mathbb{A}_3 > 0$. Therefore, the existence of P_4 depends on the existence of the positive root $\tilde{x}$ achieves the two equations (6) and (7).

(iii) The equilibrium point $P_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$, where:

$$\check{x}_1 = \frac{\mu_2(1+\check{x})}{\varepsilon_2},\ \check{x}_2 = \frac{\check{x}_1}{\check{x}},\ \check{y}_2 = g_1\left(1 - \frac{\mu_2(1+\check{x})}{\varepsilon_2 k_1}\right)\frac{(1+\check{x})}{\alpha_2} \tag{8}$$

and $\check{x}$ is a positive root of equation (10) below, provided that:

$$\check{x} < \frac{\varepsilon_2 k_1 - \mu_2}{\mu_2}, \tag{9}$$

$$\mathbb{B}_1\check{x}^3 + \mathbb{B}_2\check{x}^2 + \mathbb{B}_3\check{x} + \mathbb{B}_4 = 0, \tag{10}$$

such that:

$$\mathbb{B}_1 = g_1\beta_2 k_2\mu_2, \qquad \mathbb{B}_2 = g_1\beta_2 k_2(\mu_2 - k_1\varepsilon_2) < 0,$$
$$\mathbb{B}_3 = g_2\alpha_2 k_1(k_2\varepsilon_2 - \mu_2), \qquad \mathbb{B}_4 = -g_2\alpha_2 k_1\mu_2.$$

The number of sign changes between $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$, and $\mathbb{B}_4$ is one if $\mathbb{B}_3 < 0$, and, three if $\mathbb{B}_3 > 0$. Based on Descartes' rule of sign., eq.(10) has at most one positive root if $\mathbb{B}_3 < 0$, and at most three positive root if $\mathbb{B}_3 > 0$. Therefore, the existence of P_5 depends on the existence of the positive root $\check{x}$ achieves the two equations (9) and (10).

(iv) The interior equilibrium point or the coexistence point $P_6 = (\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2)$, where

$$\begin{aligned}\overline{x}_1 &= \frac{k_1k_2(1+h)\left((\alpha_1\beta_2 - \alpha_2\beta_1)h\mu_2 + \rho_1(1+h)(\alpha_2 g_2 h - \beta_2 g_1)\right)}{\rho_1(1+h)^2\left(k_1\ \alpha_2 g_2 h^2 - k_2\beta_2 g_1\right) + k_1k_2\varepsilon_2 h^2\left(\alpha_1\beta_2 - \ \alpha_2\beta_1\right)},\\ \overline{x}_2 &= h\overline{x}_1,\ \overline{y}_1 = \frac{(1+h)\mu_2 - h\varepsilon_2}{\rho_1(1+h)},\ \overline{y}_2 = \frac{h\varepsilon_1 - (1+h)\mu_1}{\rho_2(1+h)},\end{aligned} \tag{11}$$

and $h = \overline{x}_2\overline{x}_1^{-1}$, is a positive real root of the following equation:

$$\mathbb{A}h^4 + \mathbb{B}h^3 + \mathbb{C}h^2 + \mathbb{D}h + \mathbb{E} = 0, \tag{12}$$

where:

$$\mathbb{A} = k_1 g_2(\alpha_2\mu_1\rho_1 - \alpha_1\mu_2\rho_2 + \rho_1\rho_2 g_1),$$
$$\mathbb{B} = 2k_1 g_2(\alpha_2\mu_1\rho_1 - \alpha_1\mu_2\rho_2) + 3\rho_1\rho_2 g_1 g_2 k_1 - k_2\rho_1\rho_2 g_1 g_2$$

$$+ k_1k_2g_2(\alpha_1\epsilon_2\rho_2 - \alpha_2\epsilon_1\rho_1),$$
$$\mathbb{C} = k_1g_2(\alpha_2\mu_1\rho_1 - \alpha_1\mu_2\rho_2) + k_2g_1(\beta_1\mu_2\rho_2 - \beta_2\mu_1\rho_1) - k_1k_2g_2(\alpha_2\epsilon_1\rho_1 - \alpha_1\epsilon_2\rho_2)$$
$$+ k_1k_2g_1(\beta_2\epsilon_1\rho_1 - \beta_1\epsilon_2\rho_2) + k_1k_2(\alpha_1\beta_2 - \alpha_2\beta_1)(\mu_1\epsilon_2 - \mu_2\epsilon_1) + 3\rho_1\rho_2g_1g_2(k_1 - k_2),$$
$$\mathbb{D} = 2k_2g_1(\beta_1\mu_2\rho_2 - \beta_2\mu_1\rho_1) - 3k_2\rho_1\rho_2g_1g_2 + \rho_1\rho_2g_1g_2k_1 + k_1k_2g_1(\beta_2\epsilon_1\rho_1 - \beta_1\epsilon_2\rho_2),$$
$$\mathbb{E} = k_2g_1(\beta_1\mu_2\rho_2 - \beta_2\mu_1\rho_1 - \rho_1\rho_2g_2).$$

If the signs of the coefficients $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$, and $\mathbb{E}$ are positive, a positive root cannot be obtained, and this means that there is no interior equilibrium point. Otherwise, there is a possibility of finding one positive root, two positive roots, three positive roots, or four at most, according to Descartes' rule of signs.

4. LOCAL STABILITY OF EQUILIBRIUM POINTS

In this section, the local stability of the equilibrium points of the system is studied, by using the following Jacobian matrix $J(x_1, x_2, y_1, y_2)$ of the system (2) at each equilibrium point [6], [11], [12].

$$J(x_1,x_2,y_1,y_2) = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \frac{\partial \dot{x}_1}{\partial y_1} & \frac{\partial \dot{x}_1}{\partial y_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \frac{\partial \dot{x}_2}{\partial y_1} & \frac{\partial \dot{x}_2}{\partial y_2} \\ \frac{\partial \dot{y}_1}{\partial x_1} & \frac{\partial \dot{y}_1}{\partial x_2} & \frac{\partial \dot{y}_1}{\partial y_1} & \frac{\partial \dot{y}_1}{\partial y_2} \\ \frac{\partial \dot{y}_2}{\partial x_1} & \frac{\partial \dot{y}_2}{\partial x_2} & \frac{\partial \dot{y}_2}{\partial y_1} & \frac{\partial \dot{y}_2}{\partial y_2} \end{bmatrix}$$

where,

$$\frac{\partial \dot{x}_1}{\partial x_1} = g_1\left(1 - \frac{2x_1}{k_1}\right) - \frac{(\alpha_1y_1+\alpha_2y_2)x_2^2}{(x_1+x_2)^2}, \qquad \frac{\partial \dot{x}_1}{\partial x_2} = -\frac{(\alpha_1y_1+\alpha_2y_2)x_1^2}{(x_1+x_2)^2},$$
$$\frac{\partial \dot{x}_1}{\partial y_1} = -\frac{\alpha_1x_1x_2}{x_1+x_2}, \qquad \frac{\partial \dot{x}_1}{\partial y_2} = -\frac{\alpha_2x_1x_2}{x_1+x_2},$$
$$\frac{\partial \dot{x}_2}{\partial x_1} = -\frac{(\beta_1y_1+\beta_2y_2)x_2^2}{(x_1+x_2)^2}, \qquad \frac{\partial \dot{x}_2}{\partial x_2} = g_2\left(1 - \frac{2x_2}{k_2}\right) - \frac{(\beta_1y_1+\beta_2y_2)x_1^2}{(x_1+x_2)^2},$$
$$\frac{\partial \dot{x}_2}{\partial y_1} = -\frac{\beta_1x_1x_2}{x_1+x_2}, \qquad \frac{\partial \dot{x}_2}{\partial y_2} = -\frac{\beta_2x_1x_2}{x_1+x_2},$$
$$\frac{\partial \dot{y}_1}{\partial x_1} = \frac{\varepsilon_1x_2^2y_1}{(x_1+x_2)^2}, \qquad \frac{\partial \dot{y}_1}{\partial x_2} = \frac{\varepsilon_1x_1^2y_1}{(x_1+x_2)^2},$$
$$\frac{\partial \dot{y}_1}{\partial y_1} = -\mu_1 + \frac{\varepsilon_1x_1x_2}{x_1+x_2} - \rho_2y_2, \qquad \frac{\partial \dot{y}_1}{\partial y_2} = -\rho_2y_1,$$
$$\frac{\partial \dot{y}_2}{\partial x_1} = \frac{\varepsilon_2x_2^2y_2}{(x_1+x_2)^2}, \qquad \frac{\partial \dot{y}_2}{\partial x_2} = \frac{\varepsilon_2x_1^2y_2}{(x_1+x_2)^2},$$
$$\frac{\partial \dot{y}_2}{\partial y_1} = \rho_1y_2, \qquad \frac{\partial \dot{y}_2}{\partial y_2} = -\mu_2 + \frac{\varepsilon_1x_1x_2}{x_1+x_2} + \rho_1y_1,$$

In Theorems 2, 3 and 4, we will prove that the points P_0, P_1, and P_2 are unstable point while the remaining points are locally asymptoticaly stable points if they meet certain conditions.

Theorem 2. *Consider the system* (2)*, then the equilibrium points*

(i) $P_0 = (0,0,0,0), P_1 = (k_1,0,0,0)$ *and* $P_2 = (0,k_2,0,0)$ *are unstable equilibrium points.*

(ii) $P_3 = (k_1,k_2,0,0)$ *locally asymptotically stable point if* $k_1k_2\varepsilon_i < (k_1+k_2)\,\mu_i,\ \ i = 1,2.$

Proof.

(i) Suppose that $P_0 = (0,0,0,0)$ is locally asymptotically stable. So that all the trajectories (x_1,x_2,y_1,y_2) of the system (2) converge to $(0,0,0,0)$ as t $\to \infty$. Then since $x_1 > 0$, we have that $\frac{d}{dt}(\ln x_1) \to g_1$, as t $\to \infty$.

It is possible to find a small ball with center P_0 and radius g_1, such that inside it we have $\frac{d}{dt}(\ln x_1) \geq \frac{g_1}{2}$. if (x_1,x_2,y_1,y_2) converges to $(0,0,0,0)$, when t converges to ∞, then there exists $t_0 > 0$, such that;

$x_1(t_0) > 0, x_1(t) \geq x_1(t_0)exp\left(\frac{g_1(t-t_0)}{2}\right) \to \infty$, as $t \to \infty$. So $x_1 \to \infty$. Similarly, if $x_2(0) > 0, x_1 \to \infty$, there is no a trajectory to the system (2) converges to $(0,0,0,0)$. Hence $P_0 = (0,0,0,0)$ is unstable point.

The Jacobian matrices of the system (2) at the equilibrium points P_1 and P_2 are

$$J(P_1) = \begin{bmatrix} -g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & -\mu_1 & 0 \\ 0 & 0 & 0 & -\mu_2 \end{bmatrix} \text{ and } J(P_2) = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & -g_2 & 0 & 0 \\ 0 & 0 & -\mu_1 & 0 \\ 0 & 0 & 0 & -\mu_2 \end{bmatrix},$$

respectively. The eigenvalues of the two matrices $J(P_1)$ and $J(P_2)$ are:

$$\lambda_1 = -g_1 < 0,\ \lambda_2 = g_2 > 0,\ \lambda_3 = -\mu_1 < 0, \lambda_4 = -\mu_2 < 0$$

and

$$\lambda_1 = g_1 > 0,\ \ \lambda_2 = -g_2 < 0,\ \lambda_3 = -\mu_1 < 0,\ \lambda_4 = -\mu_2 < 0,$$

respectively. It is clear that, we have one positive eigenvalue in both cases. Therefore, the two points P_1 and P_2 are saddle points and this means that they are not stable.

(ii) The Jacobian matrix of the system (2) at the equilibrium point $P_3 = (k_1,k_2,0,0)$ is

$$J(P_3) = \begin{bmatrix} -g_1 & 0 & -\alpha_1k_1k_2(k_1+k_2)^{-1} & -\alpha_2k_1k_2(k_1+k_2)^{-1} \\ 0 & -g_2 & -\beta_2k_1k_2(k_1+k_2)^{-1} & -\beta_2k_1k_2(k_1+k_2)^{-1} \\ 0 & 0 & \varepsilon_1k_1k_2(k_1+k_2)^{-1} - \mu_1 & 0 \\ 0 & 0 & 0 & \varepsilon_2k_1k_2(k_1+k_2)^{-1} - \mu_2 \end{bmatrix}$$

The solutions of the following characteristic equations $|J(P_3) - \lambda I_{4\times 4}| = 0$, where $I_{4\times 4}$ is the identity matrix, are:

$$\lambda_1 = -g_1 < 0,\ \lambda_2 = -g_2 < 0,\ \lambda_3 = \frac{\varepsilon_1 k_1 k_2}{k_1 + k_2} - \mu_1,\ \lambda_4 = \frac{\varepsilon_2 k_1 k_2}{k_1 + k_2} - \mu_2.$$

So that, $P_3 = (k_1, k_2, 0, 0)$ is locally asymptotically stable if the following conditions are met

$$k_1 k_2 \varepsilon_i < (k_1 + k_2)\,\mu_i,\ i = 1, 2. \tag{13}$$

Theorem 3. *Consider the system* (2)*, then*

(i) The equilibrium point $P_4 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$ is locally asymptotically stable if

$$\widetilde{\Gamma}\varepsilon_2\tilde{x}_1 + \rho_1\tilde{y}_1 - \mu_2 < 0, \mathrm{tr}\widetilde{\mathcal{M}} < 0, \det\widetilde{\mathcal{M}} < 0,\ \textit{and}\ \sum_{i=1}^{3}\mathrm{tr}\widetilde{\mathcal{M}}\widetilde{\mathcal{M}}_{ii} < \det\widetilde{\mathcal{M}}. \tag{14}$$

$$\textit{where}\ \widetilde{\mathcal{M}} = \begin{bmatrix} \widetilde{\Gamma}^2\alpha_1\tilde{x}\tilde{y}_1 - k_1^{-1}\tilde{x}_1 g_1 & -\widetilde{\Gamma}^2\alpha_1\tilde{x}^2\tilde{y}_1 & -\widetilde{\Gamma}\alpha_1\tilde{x}_1 \\ -\widetilde{\Gamma}^2\beta_1\tilde{y}_1 & \widetilde{\Gamma}^2\beta_1\tilde{x}\tilde{y}_1 - k_2^{-1}\tilde{x}_2 g_2 & -\widetilde{\Gamma}\beta_2\tilde{x}_1 \\ \widetilde{\Gamma}\varepsilon_1\tilde{y}_1 & \widetilde{\Gamma}\varepsilon_1\tilde{x}^2\tilde{y}_1 & \widetilde{\Gamma}\varepsilon_1\tilde{x}_1 - \mu_1 \end{bmatrix}\ \textit{and}\ \widetilde{\Gamma} = \tfrac{1}{1+\tilde{x}}.$$

(ii) The equilibrium point $P_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$ is locally asymptotically stable point if

$$\check{\Gamma}\varepsilon_1\check{x}_1 - \rho_2\check{y}_2 - \mu_1 < 0,\ \mathrm{tr}\check{\mathcal{M}} < 0,\ \det\check{\mathcal{M}} < 0,\ \ \textit{and}\ \sum_{i=1}^{3}\mathrm{tr}\check{\mathcal{M}}\check{\mathcal{M}}_{ii} < \det\check{\mathcal{M}}. \tag{15}$$

$$\textit{where}\ \check{\mathcal{M}} = \begin{bmatrix} \check{\Gamma}^2\alpha_1\check{x}\check{y}_2 - k_1^{-1}\check{x}_1 g_1 & -\check{\Gamma}^2\alpha_1\check{x}^2\check{y}_2 & -\ \check{\Gamma}\alpha_1\check{x}_1 \\ -\check{\Gamma}^2\beta_2\check{y}_2 & \check{\Gamma}^2\beta_2\check{x}\check{y}_2 - k_2^{-1}\check{x}_2 g_2 & \check{\Gamma}\beta_2\check{x}_1 \\ \check{\Gamma}^2\varepsilon_2\check{y}_2 & \check{\Gamma}^2\varepsilon_2\check{x}^2\check{y}_2 & \check{\Gamma}\varepsilon_1\check{x}_1 - \mu_2 \end{bmatrix}\ \textit{and}\ \check{\Gamma} = \tfrac{1}{1+\check{x}}$$

Proof.

(i) The Jacobian matrix of the system (2) at equilibrium points $P_4 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$, is

$$J(P_4) = \begin{bmatrix} \widetilde{\Gamma}^2\alpha_1\tilde{x}\tilde{y}_1 - k_1^{-1}\tilde{x}_1 g_1 & -\widetilde{\Gamma}^2\alpha_1\tilde{x}^2\tilde{y}_1 & -\widetilde{\Gamma}\alpha_1\tilde{x}_1 & -\widetilde{\Gamma}\alpha_2\tilde{x}_1 \\ -\widetilde{\Gamma}^2\beta_1\tilde{y}_1 & \widetilde{\Gamma}^2\beta_1\tilde{x}\tilde{y}_1 - k_2^{-1}\tilde{x}_2 g_2 & -\widetilde{\Gamma}\beta_1\tilde{x}_1 & -\widetilde{\Gamma}\beta_2\tilde{x}_1 \\ \widetilde{\Gamma}^2\varepsilon_1\tilde{y}_1 & \widetilde{\Gamma}^2\varepsilon_1\tilde{x}^2\tilde{y}_1 & \widetilde{\Gamma}\varepsilon_1\tilde{x}_1 - \mu_1 & -\rho_2\tilde{y}_1 \\ 0 & 0 & 0 & \widetilde{\Gamma}\varepsilon_2\tilde{x}_1 + \rho_1\tilde{y}_1 - \mu_2 \end{bmatrix}$$

such that $\widetilde{\Gamma} = \frac{1}{1+\tilde{x}}$.

The characteristic equation for the matrix $J(P_4)$ is as follows:

$$\left(\lambda - \widetilde{\Gamma}\varepsilon_2\tilde{x}_1 - \rho_1\tilde{y}_1 + \mu_2\right)\left(\lambda^3 - \mathrm{tr}\widetilde{\mathcal{M}}\ \lambda^2 + \sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii}\lambda - \det\widetilde{\mathcal{M}}\right) = 0,$$

where, $\widetilde{\mathcal{M}} = \begin{bmatrix} \widetilde{\Gamma}^2\alpha_1\tilde{x}\tilde{y}_1 - k_1^{-1}\tilde{x}_1 g_1 & -\widetilde{\Gamma}^2\alpha_1\tilde{x}^2\tilde{y}_1 & -\widetilde{\Gamma}\alpha_1\tilde{x}_1 \\ -\widetilde{\Gamma}^2\beta_1\tilde{y}_1 & \widetilde{\Gamma}^2\beta_1\tilde{x}\tilde{y}_1 - k_2^{-1}\tilde{x}_2 g_2 & -\widetilde{\Gamma}\beta_2\tilde{x}_1 \\ \widetilde{\Gamma}^2\varepsilon_1\tilde{y}_1 & \widetilde{\Gamma}^2\varepsilon_1\tilde{x}^2\tilde{y}_1 & \widetilde{\Gamma}\varepsilon_1\tilde{x}_1 - \mu_1 \end{bmatrix}$

and $\widetilde{\mathcal{M}}_{ii},\ i = 1, 2, 3$ is the diagonal minor's of the matrix $\widetilde{\mathcal{M}}$.

According to the criteria of Routh-Hurwitz, $P_4 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$, is a locally asymptotically stable point provided that (14) are satisfied.

(ii) The Jacobi's matrix of the system (2) at $P_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$ is

$$J(P_5) = \begin{bmatrix} \check{\Gamma}^2\alpha_1\check{x}\check{y}_2 - k_1^{-1}\check{x}_1 g_1 & -\check{\Gamma}^2\alpha_2\check{x}^2\check{y}_2 & -\check{\Gamma}\alpha_1\check{x}_1 & -\check{\Gamma}\alpha_2\check{x}_1 \\ -\check{\Gamma}^2\beta_2 y_2 & \check{\Gamma}^2\beta_2\check{x}\check{y}_2 - k_2^{-1}\check{x}_2 g_2 & -\check{\Gamma}\beta_1\check{x}_1 & -\check{\Gamma}\beta_2\check{x}_1 \\ 0 & 0 & \check{\Gamma}\varepsilon_1\check{x}_1 - \rho_2\check{y}_2 - \mu_1 & 0 \\ \check{\Gamma}^2\varepsilon_2\check{y}_2 & \check{\Gamma}^2\varepsilon_2\check{x}^2\check{y}_2 & \rho_1\check{y}_2 & \check{\Gamma}\varepsilon_1\check{x}_1 - \mu_2 \end{bmatrix}$$

such that $\check{\Gamma} = \frac{1}{1+\check{x}}$.

Simple calculations yield that the characteristic equation of the matrix$J(P_5)$ is:

$$\left(\check{\Gamma}\varepsilon_1\check{x}_1 - \rho_2 y_2 - \mu_1 - \lambda\right)\left(\lambda^3 - \mathrm{tr}\check{\mathcal{M}}\ \lambda^2 + \sum_{i=1}^{3}\check{\mathcal{M}}_{ii}\ \lambda - \det\check{\mathcal{M}}\right) = 0,$$

where

$$\check{\mathcal{M}} = \begin{bmatrix} \check{\Gamma}^2\alpha_1\check{x}\check{y}_2 - k_1^{-1}\check{x}_1 g_1 & -\check{\Gamma}^2\alpha_1\check{x}^2\check{y}_2 & -\check{\Gamma}\alpha_1\check{x}_1 \\ -\check{\Gamma}^2\beta_2\check{y}_2 & \check{\Gamma}^2\beta_2\check{x}\check{y}_2 - k_2^{-1}\check{x}_2 g_2 & \check{\Gamma}\beta_2\check{x}_1 \\ \check{\Gamma}^2\varepsilon_2\check{y}_2 & \check{\Gamma}^2\varepsilon_2\check{x}^2\check{y}_2 & \check{\Gamma}\varepsilon_1\check{x}_1 - \mu_2 \end{bmatrix}$$

and $\check{\mathcal{M}}_{ii},\ i = 1, 2, 3$ is the diagonal minor's matrix $\check{\mathcal{M}}$.

According to the criteria of Routh-Hurwitz, any of the equilibrium points $P_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$, is locally asymptotically provided that (15) are satisfied. And with this, we have completed the proof.

Theorem 4. *Consider the system* (2)*, then* $P_6 = (\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2)$ *is a locally asymptotically stable point if*

$$\begin{cases} \mathrm{tr}\overline{\mathcal{M}} > 0, \\ \mathrm{tr}\overline{\mathcal{M}}\ \overline{\Delta} - \sum_{i=1}^{4}\overline{\mathcal{M}}_{ii} > 0, \\ \sum_{i=1}^{4}\overline{\mathcal{M}}_{ii}\left(\mathrm{tr}\overline{\mathcal{M}}\ \overline{\Delta} - \sum_{i=1}^{4}\overline{\mathcal{M}}_{ii}\right) - \det\overline{\mathcal{M}}\ \mathrm{tr}\overline{\mathcal{M}}^2 > 0, \end{cases} \tag{16}$$

where, for $i = 1, 2, 3, 4,$ $\overline{\mathcal{M}}_{ii}$ *is the diagonal minor's of the matrix* $\overline{\mathcal{M}} = J(P_6)$

Proof. The Jacobian matrix of the system (2) at equilibrium points $P_6 = (\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2)$ is as follows

$$J\left(P_6\right) = \begin{bmatrix} Q_1 - \frac{g_1 \overline{x}_1}{k_1} & -\frac{Q_1}{h} & \alpha_1 E & \alpha_2 E \\ -h^2 Q_2 & hQ_2 - \frac{g_2 \overline{x}_2}{k_2} & \beta_1 E & \beta_2 E \\ h^2 F_1 & F_1 & 0 & -\rho_2 \overline{y}_1 \\ h^2 F_2 & F_2 & \rho_1 \overline{y}_2 & 0 \end{bmatrix} =: \overline{\mathcal{M}},$$

such that, for $i = 1, 2$, $Q_i = g_i \left(1 - \frac{\overline{x}_i}{k_i}\right) \frac{1}{1+h}$, $E = \frac{-\overline{x}_2}{1+h}$, and $F_i = \frac{\varepsilon_i \overline{y}_i}{(1+h)^2}$, further, the characteristic equation of the matrix $\overline{\mathcal{M}}$ is

$$\lambda^4 - \mathrm{tr}\overline{\mathcal{M}}\ \lambda^3 + \overline{\Delta}\ \lambda^2 - \sum_{i=1}^{4} \overline{\mathcal{M}}_{ii}\ \lambda + \det\overline{\mathcal{M}} = 0,$$

where, $\overline{\mathcal{M}}_{ii},\ i = 1, 2, 3, 4$ is the diagonal minor's of the matrix $\overline{\mathcal{M}}$, and

$$\overline{\Delta} = \frac{g_1 \overline{x}_1 g_2 \overline{x}_2}{k_1 k_2} + \rho_1 \rho_2 \overline{y}_1 \overline{y}_2 - \sum_{i=1}^{2} \left(EF_i \left(h^2 \alpha_i + \beta_i\right) + \overline{x}_i \frac{g_i Q_{\left(i-(-1)^i\right)}}{k_i} \right).$$

According to the criteria of Routh-Hurwitz, $P_6 = (\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2)$ is locally asymptotically stable point if the (16) is satisfied. And with this, we have completed the proof.

In the following theorem, we give conditions through which the stability region of the coexistence point be asymptotically stable.

Theorem 5. *Assume that $P_6 = (\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2)$ is locally asymptotically stable point, and $\overline{x}_i \geq k_i,\ i = 1, 2$. Then, the set $\overline{B}$, which is defined below represents an attraction basin for P_6.*

$$\overline{B} = \left\{ (x_1, x_2, y_1, y_2) : x_i \geq \overline{x}_i, y_1 \leq \overline{y}_1, y_2 = \overline{y}_2 \right\}.$$

Proof. The function

$$V\left(x_1, x_2, y_1, y_2\right) = \sum_{i=1}^{2} \left(x_i - \overline{x}_i - \overline{x}_i \ln \frac{x_i}{\overline{x}_i} + y_i - \overline{y}_i - \overline{y}_i \ln \frac{y_i}{\overline{y}_i} \right)$$

is positive definite .

$$\begin{aligned} \dot{V}\left(x_1, x_2, y_1, y_2\right) &= \sum_{i=1}^{2} \left(\dot{x}_i \left(1 - \frac{k_1}{x_i}\right) + \dot{y}_i \left(1 - \frac{y_i}{\overline{y}_i}\right) \right) \\ &= \sum_{i=1}^{2} \left(x_i - \overline{x}_i\right) \mathrm{G}_i\left(x_1, x_2\right) + \left(y_i - \overline{y}_i\right) \mathrm{G}_3\left(x_1, x_2\right), \end{aligned}$$

such that

$$\mathrm{G}_1\left(x_1, x_2\right) = \left(g_1 \left(1 - \frac{x_1}{k_1}\right) - \frac{\alpha_1 x_2 y_1}{x_1 + x_2} - \frac{\alpha_2 x_2 y_2}{x_1 + x_2} \right),$$

$$\mathrm{G}_2\left(x_1, x_2\right)=\left(g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\alpha_2 x_1 y_1}{x_1+x_2}\right),$$
$$\mathrm{G}_3\left(x_1, x_2\right)=\left(-\mu_1+\frac{\varepsilon_1 x_1 x_2}{x_1+x_2}-\rho_2 \overline{y}_2\right).$$

It is obvious that $\left(x_1-\overline{x}_1\right)>0,\left(x_2-2\right)>0, \mathrm{G}_1\left(x_1, x_2\right)<0$ and $\mathrm{G}_2\left(x_1, x_2\right)<0$, so that,

$$\sum_{i=1}^2\left(x_i-\overline{x}_i\right) \mathrm{G}_i\left(x_1, x_2\right)<0.$$

The function $\mathrm{G}_3\left(x_1, x_2\right)$ is increasing with respect to $x_i,\ i=1,2$, because of the positivity of it derivative with respect to x_i for $x_i \geq \overline{x}_i, i=1,2$ as shown below:

$$\frac{\partial \mathrm{G}_3\left(x_1, x_2\right)}{\partial x_1}=\frac{\varepsilon_1 x_2^2}{\left(x_1+x_2\right)^2}>0, \frac{\partial \mathrm{G}_3\left(x_1, x_2\right)}{\partial x_2}=\frac{\varepsilon_1 x_1^2}{\left(x_1+x_2\right)^2}>0, \text{ for } x_i \geq \overline{x}_i, i=1,2.$$

Now, since $\mathrm{G}_3\left(\overline{x}_1, \overline{x}_2\right)=0$, then it is obtained that $\mathrm{G}_3\left(x_1, x_2\right)>0$, for $x_i \geq \overline{x}_i, i=1,2$. Hence, $\left(y_1-\overline{y}_1\right) \mathrm{G}_3\left(x_1, x_2\right)<0$. So that,

$$\sum_{i=1}^2\left(x_i-\overline{x}_i\right) \mathrm{G}_i\left(x_1, x_2\right)+\left(y_i-\overline{y}_i\right) \mathrm{G}_3\left(x_1, x_2\right)<0,$$

$$\dot{V}\left(x_1, x_2, y_1, y_2\right)<0,\ \forall\left(x_1, x_2, y_1, y_2\right) \in \overline{B}-\left\{\left(\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2\right)\right\},$$

and $\dot{V}\left(\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2\right)=0$. So that, $\overline{B}$ is an attraction basin for $P_6=\left(\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2\right)$ and with this, we have completed the proof.

5. LOCAL BIFURCATION

This section is dedicated to the study of the local bifurcation of the three equilibrium points. Local bifurcation occurs when a small change in the parameter value changes the behavior of the equilibrium. If the corresponding real part of one eigenvalue of the equilibrium passes through zero, a steady-state bifurcation occurs, and in the case that the eigenvalues is not zeros but is purely imaginary, then this is a Hopf-bifurcation [12, 14].

(I) If the equilibrium point $P(\vartheta)$ is locally asymptotically stable and for $\vartheta=\vartheta^*$ we have one eigenvalue $\lambda\left(\vartheta^*\right)=0$, the bifurcation that occurs is a steady state bifurcation [6]. The following theorems show that steady-state bifurcation occurs at points $P_3=\left(k_1, k_2 .0,0\right), P_4=\left(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0\right)$, and $P_5=\left(\check{x}_1, \check{x}_2, 0, \check{y}_2\right)$ for the parameters $\mu_i,\ i=1$ or 2, μ_1, and μ_2, respectively.

Theorem 6. *Suppose that, the equilibrium point $P_3=\left(k_1, k_2 .0,0\right)$ of the system (2) is locally asymptotically stable point, then*

(i) A steady state bifurcation occurs as μ_1 pass through $\mu_1^*=\frac{k_1 k_2 \varepsilon_1}{k_1+k_2}$,

(ii) A steady state bifurcation occurs as μ_1 pass through $\mu_2^*=\frac{k_1 k_2 \varepsilon_2}{k_1+k_2}$.

Proof. (i) From Theorem 2 and the assumption or the theorem that we have that the eigenvalues of the characteristic equation of $J(P_3)$ are

$$\lambda_1 = -g_1, \quad \lambda_2 = -g_2, \lambda_3 = \frac{\varepsilon_1 k_1 k_2}{k_1 + k_2} - \mu_1 < 0 \ , \quad \lambda_4 = \frac{\varepsilon_2 k_1 k_2}{k_1 + k_2} - \mu_2 < 0.$$

It obvious that $P_3 = (k_1, k_2.0, 0)$ do not depends on the parameter μ_1, which means that P_3 do not change with the change of the value of μ_1. Hence, if $\mu_1 > \mu_1^*$ then P_3 of the system (2) is locally asymptotically stable point (nod point) and if $\mu_1 < \mu_1^*$ then P_3 of the system (2) is unstable point (saddle point).
(ii) The proof is the same as the proof of the first part of this theorem, and with this, we have completed the proof.

Theorem 7. *(i) Suppose that, the equilibrium point $P_4 = (\breve{x}_1, \breve{x}_2, 0, \breve{y}_2)$ of the system* (2) *exists and is locally asymptotically stable point then a steady state bifurcation occurs as μ_2 pass through $\mu_2^* = \frac{\varepsilon_2 \breve{x}_1}{1+\breve{x}} + \rho_1 \breve{y}_1$.*

(ii) Suppose that, the equilibrium point P_5 of the system (2) *exist and is locally asymptotically stable point then a steady state bifurcation occurs as μ_1 pass through $\mu_1^* = \frac{\varepsilon_1 \tilde{x}_1}{1+\tilde{x}} - \rho_2 \tilde{y}_2$.*

Proof. (i) From Theorem 3, the characteristic equation of $J(P_4)$ is

$$\left(\lambda - \widetilde{\Gamma}\varepsilon_2 \tilde{x}_1 - \rho_1 \tilde{y}_1 + \mu_2\right)\left(\lambda^3 - \mathrm{tr}\widetilde{\mathcal{M}}\ \lambda^2 + \sum_{i=1}^{3} \widetilde{\mathcal{M}}_{ii}\ \lambda - \mathrm{det}\widetilde{\mathcal{M}}\right) = 0.$$

As it is clear from the equation that second algebraic expression $\left(\lambda^3 - \mathrm{tr}\widetilde{\mathcal{M}}\lambda^2 + \sum_{i=1}^{3} \widetilde{\mathcal{M}}_{ii}\lambda - \mathrm{det}\widetilde{\mathcal{M}}\right)$ of the left side the characteristic equation does not depend on the parameter μ_2. The equations (5), (6) and (8), show that the existence of P_4 does not depends on μ_2. Hence, if $\mu_2 > \mu_2^*$ the P_3 of the system (2) is locally asymptotically stable point (nod point) and if $\mu_2 < \mu_2^*$ the P_4 of the system (2) is unstable point (saddle point).

(ii) The proof is the same as the proof of first part of this theorem.
And with this, we have completed the proof.

(II) The local birth or death of a periodic solution from equilibrium as a parameter ϑ passes through a critical value ϑ^* is called *Hopf bifurcation.*
For the parameter $g = g_1 = g_2$, let P_4 be a locally asymptotically stable equilibrium point and

$$\mathrm{tr}\tilde{M} \sum_{i=1}^{3} \tilde{M}_{ii} = \mathrm{det}\tilde{M}, \tag{17}$$

$g^* = \frac{S_2 \mp \sqrt{S_2^2 - 4S_1S_3}}{2S_1}$, is the positive root of the equation (17) where,

$$S_1 = \tilde{A}\tilde{N}_{22},\ S_2 = \mathrm{det}\tilde{N} - \tilde{A}\left(\tilde{N}_{11} + \tilde{N}_{22}\right) - \tilde{B}\tilde{N}_{33},\ S_3 = \tilde{B}\left(\tilde{N}_{11} + \tilde{N}_{22}\right),$$

$$\tilde{A} = \sum_{i=1}^{2} \left(k_i^{-1}\tilde{x}_i\right) - \widetilde{\Gamma}^2\tilde{x}\tilde{y}\left(\alpha_1+\beta_1\right), \tilde{B} = (\mu_1 - \widetilde{\Gamma}\varepsilon_1\tilde{x}_1),$$

$$\tilde{N} = \begin{bmatrix} k_1^{-1}\tilde{x}_1 - \widetilde{\Gamma}^2\alpha_1\tilde{x}\tilde{y} & \widetilde{\Gamma}^2\alpha_1\tilde{x}^2\tilde{y} & \widetilde{\Gamma}\alpha_1\tilde{x}_1 \\ \widetilde{\Gamma}^2\beta_1\tilde{y} & k_2^{-1}\tilde{x}_2 - \widetilde{\Gamma}^2\beta_1\tilde{x}\tilde{y} & \widetilde{\Gamma}\beta_2\tilde{x}_1 \\ -\widetilde{\Gamma}\varepsilon_1\tilde{x}_2^{-1}\tilde{y} & -\widetilde{\Gamma}\varepsilon_1\tilde{x}\tilde{x}_2^{-1}\tilde{y} & \mu_1 - \widetilde{\Gamma}\varepsilon_1\tilde{x}_1 \end{bmatrix},$$

$$\tilde{y} = \left(1 - \frac{\tilde{x}_1}{k_1}\right)\frac{(1+\tilde{x})}{\alpha_1}.$$

From (17), eigenvalues of $(P_4(g^*))$ are

$$\lambda_{1,2}\left(g^*\right) = \pm i\sqrt{\sum_{i=1}^{3}\check{\mathcal{M}}_{ii}},\ \lambda_3\left(g^*\right) = \mathrm{tr}\widetilde{\mathcal{M}}(g),\ \lambda_4(g^*) = \widetilde{\Gamma}\varepsilon_2\tilde{x}_1 + \rho_1\tilde{y}_1 - \mu_2.$$

Theorem 8. *Suppose that the equilibrium point $P_4 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$, of the system* (2) *exists and is locally asymptotically stable for the parameter $g = g_1 = g_2$, and g^* be the positive root of the equation* (17)*, then a simple Hopf bifurcation occurs as g passes through g^* provided that*

$$\left(\widetilde{\mathcal{M}}_{11} + \widetilde{\mathcal{M}}_{22}\right)\left(\mu_1 - \widetilde{\Gamma}\varepsilon_1\tilde{x}_1\right)g^{-1} - \widetilde{\mathcal{M}}_{33}\left(k_1^{-1}\tilde{x}_1 - \widetilde{\Gamma}^2\alpha_1\tilde{x}\tilde{y} + k_2^{-1}\tilde{x}_2 - \widetilde{\Gamma}^2\beta_1\tilde{x}\tilde{y}\right) \neq 0.$$

Proof. Since $\frac{\partial\tilde{x}_1}{\partial g} = \frac{\partial\tilde{x}_2}{\partial g} = \frac{\partial\tilde{y}_2}{\partial g} = 0, \frac{\partial\tilde{y}_1}{\partial g} = \frac{\tilde{y}_1}{g} = \tilde{y}$, then the point $P_4 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$, depends smoothly on the parameter g. If there is a simple pair of complex eigenvalues $\lambda_{1,2}\left(g\right) = u \pm iv$ of the Jacobian matrix $\left(J\left(P_4(g)\right)\right)$ at the equilibrium point $P_4\left(g\right)$, such that, it becomes a purely imaginary at $g = g^*$, while all other eigenvalues are real and negative; and $\frac{du(g^*)}{dg} \neq 0$, then at g^*we have a simple Hopf bifurcation, [5]. It is easy to deduce,

$$\frac{du\left(g^*\right)}{dg} = \frac{\left(\frac{d(\det\widetilde{\mathcal{M}})}{dg} - \sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii}\frac{d\mathrm{tr}\widetilde{\mathcal{M}}}{dg} - \mathrm{tr}\widetilde{\mathcal{M}}\frac{d\bar{b}}{dg}\right)}{2\sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii} + 2\left(\mathrm{tr}\widetilde{\mathcal{M}}\right)^2}.$$

Because $2\sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii} + 2\left(\mathrm{tr}\widetilde{\mathcal{M}}\right)^2$ is positive, so it is sufficient to prove that

$$W := \frac{d(\det\widetilde{\mathcal{M}})}{dg} - \mathrm{tr}\widetilde{\mathcal{M}}\ \frac{d\sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii}}{dg} - \sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii}\frac{d\ \mathrm{tr}\widetilde{\mathcal{M}}}{dg} \neq 0.$$

Simple calculation show that

$$\begin{aligned} W &= \frac{d(\det\widetilde{\mathcal{M}})}{dg} - \sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii}\frac{d\ \mathrm{tr}\widetilde{\mathcal{M}}}{dg} - \mathrm{tr}\widetilde{\mathcal{M}}\frac{d\left(\sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii}\right)}{dg} \\ &= (2g)\ \det\ \tilde{N} - \mathrm{tr}\widetilde{\mathcal{M}}\left(\tilde{N}_{11} + \tilde{N}_{22} + 2g\tilde{N}_{33}\right) - \sum_{i=1}^{3}\widetilde{\mathcal{M}}_{ii}\left(k_1^{-1}\tilde{x}_1 - \widetilde{\Gamma}^2\alpha_1\tilde{x}\tilde{y} + k_2^{-1}\tilde{x}_2 - \widetilde{\Gamma}^2\beta_1\tilde{x}\tilde{y}\right) \end{aligned}$$

$$= \mathrm{tr}\widetilde{\mathcal{M}}\left(\tilde{N}_{11}+\tilde{N}_{22}\right)-\left(g\tilde{N}_{11}+g\tilde{N}_{22}+g^{2}\tilde{N}_{33}\right)\left(k_1^{-1}\tilde{x}_1-\widetilde{\Gamma}^2\alpha_1\tilde{x}\tilde{y}+k_2^{-1}\tilde{x}_2-\widetilde{\Gamma}^2\beta_1\tilde{x}\tilde{y}\right)$$
$$= \left(\widetilde{\mathcal{M}}_{11}+\widetilde{\mathcal{M}}_{22}\right)(\mu_1-\widetilde{\Gamma}\varepsilon_1\tilde{x}_1)g^{-1}-\widetilde{\mathcal{M}}_{33}\left(k_1^{-1}\tilde{x}_1-\widetilde{\Gamma}^2\alpha_1\tilde{x}\tilde{y}+k_2^{-1}\tilde{x}_2-\widetilde{\Gamma}^2\beta_1\tilde{x}\tilde{y}.\right)$$

And with this, we have completed the proof.

Now consider the following equation

$$\mathrm{tr}\check{\mathcal{M}}\sum_{i=1}^{3}\check{\mathcal{M}}_{ii}=\det\check{\mathcal{M}} \tag{18}$$

$g^{**}=\frac{\mathrm{r}_2\mp\sqrt{r_2^2-4r_1r_3}}{2r_1}$ is the positive root of the (18) where

$$r_1=\check{A}\check{N}_{22},\ r_2=\det\check{N}-\check{A}\left(\check{N}_{11}+\check{N}_{22}\right)-\check{B}\check{N}_{33}, r_3=\check{B}\left(\check{N}_{11}+\check{N}_{22}\right),$$

$$\check{A}=\sum_{i=1}^{2}\left(k_i^{-1}\check{x}_i\right)-\widetilde{\Gamma}^2\check{x}\check{y}\left(\alpha_1+\beta_1\right),\ \check{B}=(\mu_1-\widetilde{\Gamma}\varepsilon_1\check{x}_1),$$

$$\check{N}=\begin{bmatrix}\check{\Gamma}^2\alpha_1\check{x}\check{y}-k_1^{-1} & -\check{\Gamma}^2\alpha_1\check{x}^2\check{y} & -\check{\Gamma}\alpha_1\check{x}_1\\ -\check{\Gamma}^2\beta_2\check{y} & \check{\Gamma}^2\beta_2\check{x}\check{y}-k_2^{-1}\check{x}_2 & \check{\Gamma}\beta_2\check{x}_1\\ \check{\Gamma}^2\varepsilon_2\check{y} & \check{\Gamma}^2\varepsilon_2\check{x}^2\check{y} & \check{\Gamma}\varepsilon_2\check{x}_1-\mu_2\end{bmatrix}\ \text{and}\ \check{\Gamma}=\frac{1}{1+\check{x}}$$

Theorem 9. *Suppose that the equilibrium point $P_5=(\check{x}_1,\check{x}_2,0,\check{y}_2)$ of the system (2) exists and is locally asymptotically stable for the parameter $g=g_1=g_2$, and g^{**} be the positive root of the equation (17), then a simple Hopf bifurcation occurs as g passes through g^{**} provided that $\left(\check{M}_{11}+\check{M}_{22}\right)(\mu_2-\check{\Gamma}\varepsilon_1\check{x}_1)g^{-1}-\check{M}_{33}\left(k_1^{-1}\check{x}_1-\check{\Gamma}^2\alpha_1\check{x}\check{y}+k_2^{-1}\check{x}_2-\check{\Gamma}^2\beta_2\check{x}\check{y}.\right)\neq 0$, where $\check{y}=g^{-1}\check{y}_2$, and g^{**} is the positive root of the (17) given above.*

Proof. The proof is the same as the proof of Theorem 8.

6. NUMERICAL EXAMPLE

In this section, two numerical examples are given to confirm the obtained theoretical results in the above sections. Consider is the set of parameters in the following table

Table 1: model parameters1

I	g_i	k_i	α_i	β_i	μ_i	ϵ_i	ρ_i
1	0.5	1.9	0.2	0.49	0.12	0.19	0.1
2	0.51	1.8	0.21	0.5	0.19	0.2	0.12

The point $P_3=(k_1,k_1,0,0)$ is unstable, but with $\mu_1>\mu_1^*=0.1756$ is locally asymptotically stable as shown in Figure 1, this means P_3 has a steady state bifurcation when μ_1 pass throw $\mu_1^*=0.1756$. The point $P_4=(1.6927,1.0075,0.731,0)$, is unstable, see Figure

2, whether the equilibrium point $P_5 = (1.6934, 0.7095, 0, 0.87686)$, exist when $\mu_2 = 0.1$, and is locally asymptotically stable as shown in Figure 3. The point $P_6 = (1.7098,\ 1.1715,\ 0.5096,\ 0.1007)$, exist and is locally asymptotically stable as shown in Figure 4.
Now consider the table bellow

Table 2: model parameters 2

I	g_i	k_i	α_i	β_i	μ_i	ϵ_i	ρ_i
1	0.3	2	0.1	0.4	0.12	0.1	0.1
2	0.4	3.2	0.2	0.5	0.13	0.2	0.11

The point $P_3 = (k_1, k_1, 0, 0)$ is unstable, but with $\mu_1 > \mu_1^* = 0.12308$ and $\mu_2 > \mu_2^* = 0.24615$ is locally asymptotically stable as shown in Figure 6 This means P_3 has a steady bifurcation when μ_1 pass throw $\mu_1^* = 0.12308$, and μ_2 pass throw $\mu_2^* = 0.24615$.
The point $P_4 = (1.9630, 3.0872, 0.0907, 0)$, is unstable, see Figure 7, whether the equilibrium point $P_5 = (1.4549, 1.1749, 0, 0.9151)$, exist and is locally asymptotically stable as shown in Figure 8. The point $P_6 =$ does not exist.

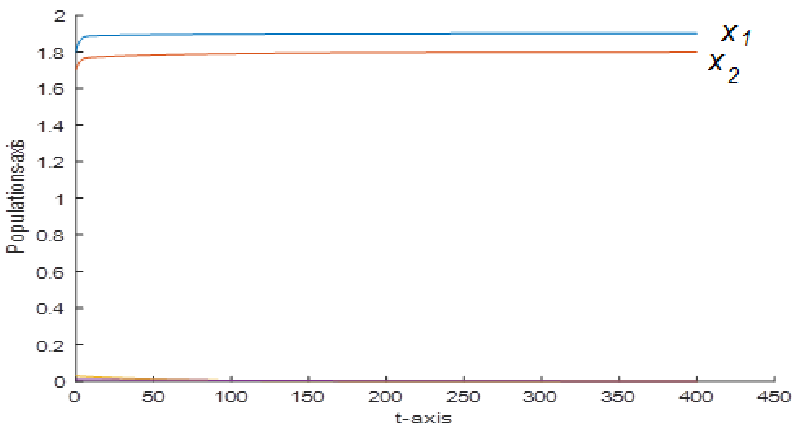

Figure 1: The trajectory of system (2), according to the parameters listed in Table 1 and $\mu_1 > \mu_1^* = 0.1756$, starts in (1.7,1.6,0.05,0.08), which located close to P_3 tend to P_3.

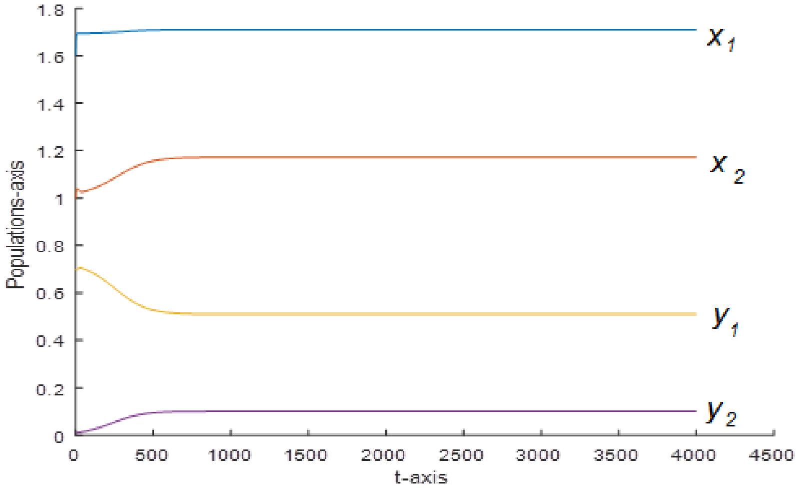

Figure 2: The trajectory of system (2), according to the parameters listed in Table 1, starts in the initial point $(1.6, , 1, 0.7, 0.01)$, which located close to P_4 and it is moving away from P_4 and approaching P_6.

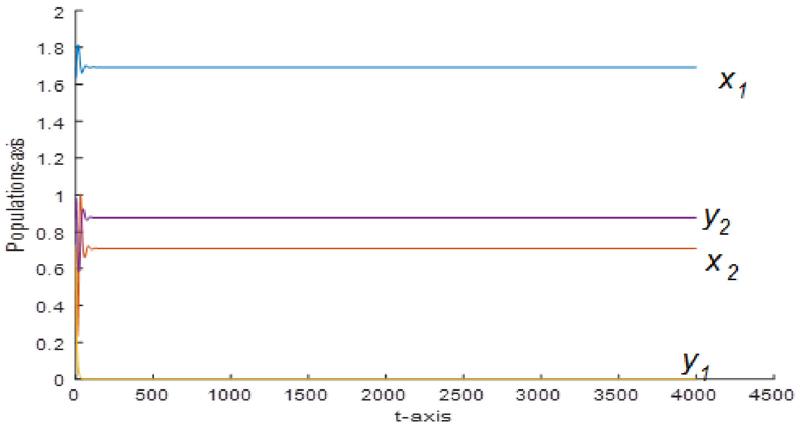

Figure 3: The trajectory of system (2), according to the parameters listed in table 1, starts in the initial point $(1.6, 0.7, 0.02, 0.8)$, which located close to P_5, is approaching P_5.

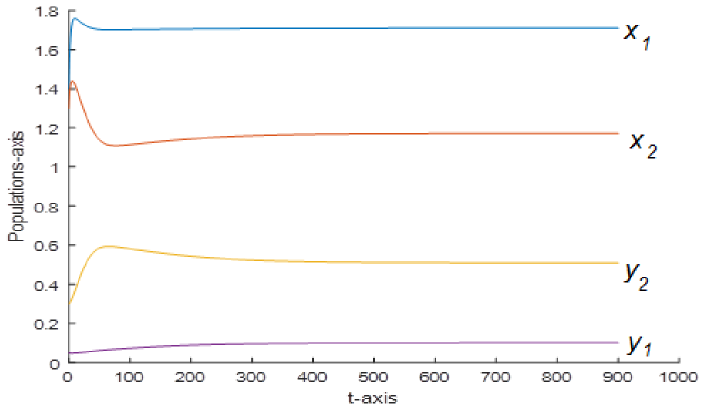

Figure 4: The trajectory of system (2), according to the parameters listed in Table 1, starts in the initial point $(1.4, 1.3, 0.3, 0.05)$, which located close to P_6 is approaching P_6.

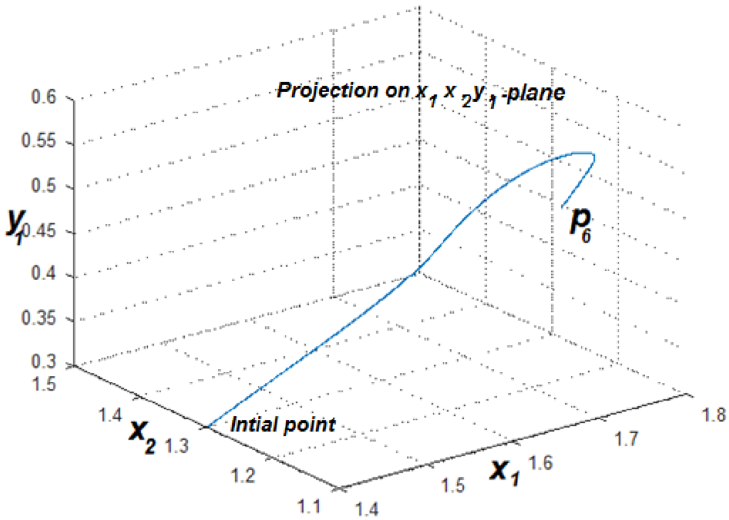

Figure 5: The trajectory of system (2), according to the parameters listed in Table 1, starts in the initial point $(1.4, 1.3, 0.3, 0.05)$, which located close to P_6 is approaching P_6.

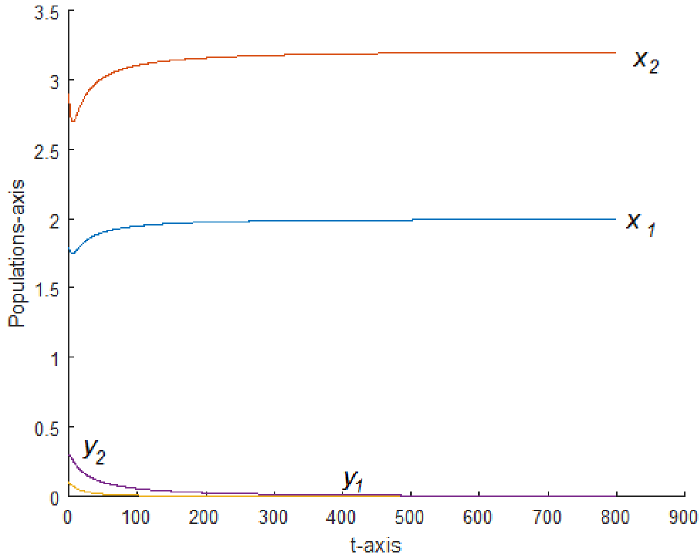

Figure 6: The trajectory of system (2), according to the parameters listed in Table 2 and $\mu_1 > \mu_1^* = 0.12308$, and $\mu_2 > \mu_2^* = 0.24615$ starts in the initial point (1.8,2.9,0.3,0.1) that located close to P_3 tends to P_3.

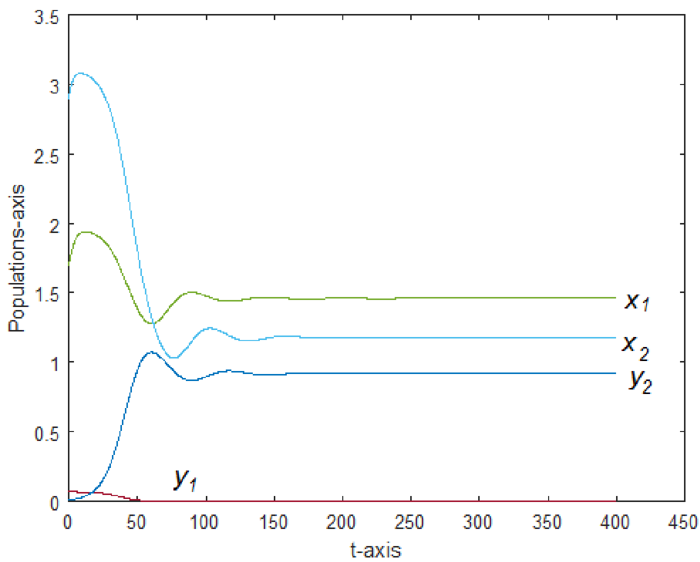

Figure 7: The trajectory of system (2), according to the parameters listed in Table 2 starts in the initial point $(1.7, 2.9, 0.07, 0.03)$ that located close to P_4 tends to P_5

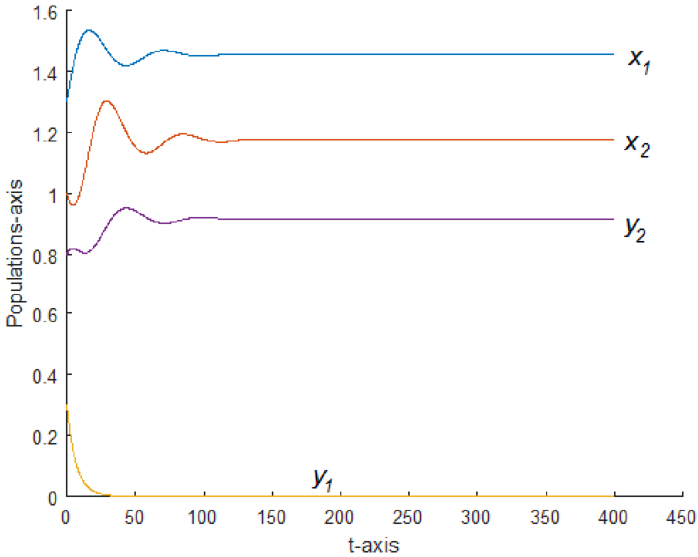

Figure 8: The trajectory of system (2), according to the parameters listed in Table 2 starts in the initial point $(1.3, 1, 0.3, 0.8)$ is located close to P_5 tends to P_5

7. CONCLUSION

In this paper, the dynamical behavior of a four-dimensional prey-predator model for four species was presented. The model has seven equilibrium points, four of them always exist, whatever the values of the model parameters. The positive trajectory of the model are bounded under some certain conditions. The system has three, unstable equilibrium points and four equilibrium points that are asymptotically locally stable under some conditions. The appearance of steady-state and, Hopf bifurcation near the equilibrium points has been discussed. In the numerical example represented by Table 1, all the possible equilibrium points exist, and in the example represented by Table 2, the coexistence point disappeared. In both examples, the steady-state bifurcation was present in the neighborhood of P_3; P_4 is not stable while P_5 is locally asymptotically stable.

References

[1] A H Adamu. Mathematical analysis of predator-prey model with two preys and one predator. *International Journal of Engineering and Applied Sciences*, 5:17–23, 2018.

[2] S Antipova. Mathematical modeling of competition two ideologies with internal conflicts. *Bulletin of Sib GUTI*, 4:27–42, 2022.

[3] R Bhattacharyya and B Mukhopadhyay. Spatial dynamics of nonlinear prey-predator models with pry migration and predator switching. *Ecological complexity*, 3:160–169, 2006.

[4] A Chatterjee and S Pal. Switching effects driven by predation on diffusive predator prey system. *Applications and Applied Mathematics: An International journal*, 16:682–704, 2021.

[5] A G Farhan. Study of a nonlinear model for two prey and two predators species. *Journal of the College of Basic Education, Mustansiriyah University*, 100:73–86, 2018.

[6] A G Farhan. Lotka-volterra model with prey-predators food chain. *Iraqi Journal of Science*, 0:56–63, 2020.

[7] A G Farhan. On the mathematical model of two-prey and two-predator species. *Iraqi Journal of Science*, 3:608–619, 2020.

[8] A G Farhan and M A Rasheed. On dynamics of nonlinear prey–predator model involving group defense. *J. of College of Education, Mustansiriyah University*, 5:249–264, 2014.

[9] P. Fisher. Sur legilibre de faunas: Interactions des moules, des poupreset des cripedaed. *C.R. Soc. Biologeogr*, 92:47–48, 1934.

[10] V Hadžiabdić, M Mehuljić, and J Bektešević. Lotka-volterra model with two predators and their prey. *TEM Journal*, 6:132–136, 2017.

[11] C S Holling. Principles of insect predation. *Annual Review of Entomology*, 6:163–182, 1961.

[12] S L Jianhua and W H Ni. Steady-state bifurcation and hopf bifurcation for a diffusive leslie–gower predator–prey model. *Computers and Mathematics with Applications*, 70:3043–3056, 2015.

[13] J H Lawton, J R Beddington, and R Bonser. *Switching in invertebrate predators.* Springer US, Boston, MA, 1974.

[14] J L Moiola and G Chen. *Hopf bifurcation analysis: a frequency domain approach.* World Scientific, 1996.

[15] S Rahman, S Islam, and S Sarwardi. Effects of prey refuge with holling type iv functional response dependent prey predator model. *International Journal of Modelling and Simulation*, 0:1–19, 2023.

[16] M Tansky. Switching effects in prey–predator system. *J. Theor. Biol*, 70:263–271, 1978.