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Solving geometry problems by alternative methods in mathematics education

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Abstract. Solving geometry problems is both difficult and interesting. Difficult because there is no general algorithm to solve more or less non-trivial problems as every single problem requires individual and creative approach. At the same time, this is a very interesting activity, because for almost every problem there are plenty of ways to solve it.
In this work, we present the method of auxiliary circle divided into equal parts. This method allows finding solution algorithm for some geometry problems which are hard to solve by the method of additional constructions.

2020 Mathematics Subject Classifications: 97G10, 97G30, 97G40

Key Words and Phrases: Geometry problem, Ceva's theorem, regular polygon, solution algorithm.

1. Introduction

In geometry, there is a class of problems for which the traditional methods (such as the method of equal triangles, the method of geometric transformations, the vector method, etc.) are either inapplicable or provide complicated and cumber-some solutions. In many cases, it becomes possible to solve such kind of problems by adding some lines (so called additional constructions) to the drawing. Sometimes these constructions suggest themselves, but in other cases situation may depend on your experience, expertise and geometric intuition.

The method of solving problems by adding some constructions to the drawing is called the method of additional constructions. The essence of the method of additional constructions lies in the fact that you add some new (auxiliary) elements to the drawing of

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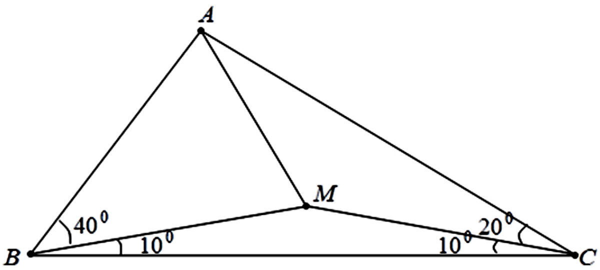

Figure 1: (Triangle ABC with the given angles)

your problem. As a result, the relationships between the problem data and the unknown quantities, which have been hard to see before, become more tangible, or even obvious.

There are problems where additional constructions are the only way of solution. Solving geometry problems by the method of additional constructions is a very hard task, because there is no general algorithm to solve your problem by this method. Even the schoolchildren with a good knowledge of geometry are not able to solve such kind of problems.

In this work, we overcome the above mentioned difficulties by using the method of auxiliary circle divided into equal parts. Namely, we present a solution algorithm for geometry problems which are hard to solve by the method of additional constructions. In conclusion, let's mention the works [1–3, 5, 7] and [6], which deal with the method of additional constructions. We solve some of problems considered in these works by using our new method.

2. The Method

Let's illustrate our new method on the problem solved by using the method of additional constructions in [4].

Problem 1. The point M is given inside the triangle ABC so that $\angle ABM = 40°$, $\angle CBM = 10°$, $\angle ACM = 20°$, $\angle BCM = 10°$ (Fig.1). Prove that $\angle CAM = 30°$.

Before proceeding to solve this problem by our method, let's recall how it was solved by the method of additional constructions in [4].

Let the circle centered at M with a radius $BM = CM$ intersect the continuation of AC at the point D . Then, as $DM = CM$, the triangle CMD is equilateral and $\angle MDC = \angle MCA = 20°$ (Fig.2).

If we join the points D and B, then the triangle BDM becomes equilateral, because $\angle BMC = 160°$ and $\angle CMD = 140°$. In the triangle ABD, we find $\angle BAD = 180° - (20° + 80°) = 80°$. Hence, $BD = AD$. On the other hand, as $BD = BM$, we have $AB = BM$, i.e. the triangle ABM is isosceles. Since the angle at the vertex of this triangle is equal to $40°$, the adjacent angles at the base are equal to $70°$. Consequently, $\angle BAM = 70°$, and therefore, $\angle CAM = 30°$.

Now let's solve this problem by our method.

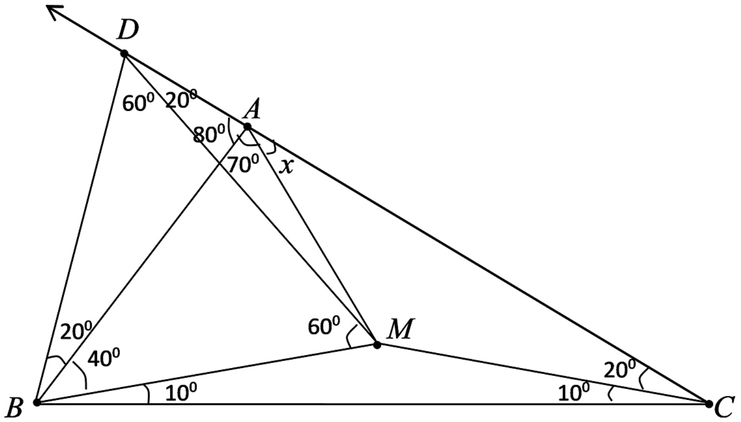

Figure 2: (Triangle BDM is equilateral, while triangles ABD and ABM are isosceles)

Divide the circle into 18 equal parts and name the division points. Degree measure of the arc between two consecutive points is equal to $\frac{360°}{18} = 20°$. If we place the triangle ABC inside the circle as shown on Fig.3 ($A \equiv C_{18}$, $B \equiv C_{15}$, $C \equiv C_5$), then the conditions of the problem are satisfied: $\angle C_{18}C_{14}C_4 = 40°$, $\angle C_5C_{15}C_4 = 10°$, $\angle C_{18}C_5C_{16} = 20°$, $\angle C_{15}C_5C_{16} = 10°$.

From Fig.3 we can see that the sought angle $C_5C_{18}C_8$ is equal to $30°$.

Consequently, the considered problem is equivalent to the following statement: In the regular octadecagon, the diagonals C_4C_{15}, C_5C_{16} and C_8C_{18} intersect each other at one point. To verify the intersection of these three diagonals at one point, it is convenient to use the following trigonometric (angular) theorem of Giovanni Ceva [6].

Theorem. Let the points A_1, B_1, C_1 be given on the sides BC, AC, AB of the triangle ABC, respectively. Then the segments AA_1, BB_1 and CC_1 intersect each other at one point if and only if (Fig.4)

$$\frac{\sin \angle \mathrm{BAA}_1}{\sin \angle C\mathrm{AA}_1} \cdot \frac{\sin \angle CBB_1}{\sin \angle ABB_1} \cdot \frac{\sin \angle ACC_1}{\sin \angle BCC_1} = 1.$$

So, the verification of whether three diagonals C_4C_{15}, C_5C_{16} and C_8C_{18} on Fig.3 intersect each other at one point is reduced to the verification of the following identity:

$$\frac{sin 10°}{sin 40°} \cdot \frac{sin 20°}{sin 10°} \cdot \frac{sin 70°}{sin 30°} = 1$$

And the validity of this identity follows from the equalities

$$sin 70° = cos 20°, \ sin 40° = 2 sin 20° cos 20°.$$

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If we compare the above two methods used to solve Problem 1, then we can easily see that the second one is simpler and more effective. Besides, the first method requires additional constructions, while the second one only needs the accurate drawing.

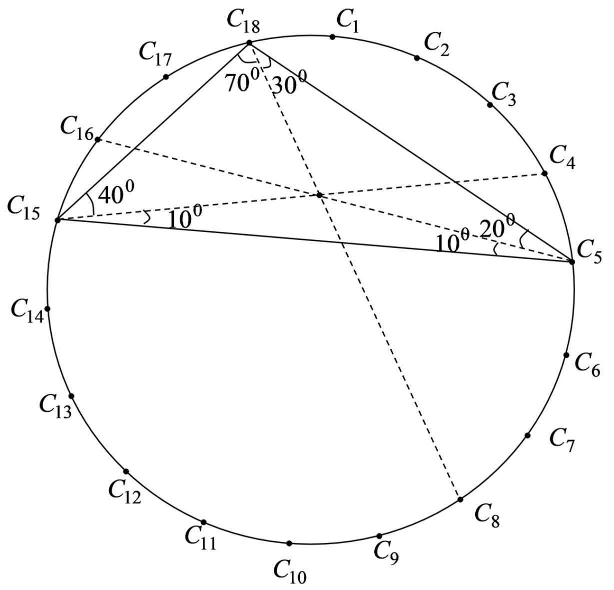

Figure 3: (Triangle $C_{18}C_{15}C_5$, equal to triangle ABC, placed inside the circle)

This example shows that many problems, even very difficult ones, can be solved using one property of regular polygon diagonals: property of intersection at one point.

When solving Problem 1 by our method, we divided the circle into 18 equal parts. This is because the triangle in this problem has the angles which are the multiples of $10°$. Depending on the sizes of the angles, the problem can be solved by dividing the circle into less than 18 equal parts. For example, to solve the problem below, we will divide the circle into 12 equal parts.

Problem 2. The point M is given inside the triangle ABC so that $\angle ABM = \angle CBM = \angle BCM = 15°$ and $\angle ACM = 30°$ (Fig.5). Prove that $\angle CAM = 75°$.

As the angles given in this problem are the multiples of $15°$, we divide the circle into 12, not 18 equal parts. Degree measure of the arc between two consecutive points is $\frac{360°}{12} = 30°$. If we place the triangle ABC inside the circle as shown on Fig.6 ($A \equiv P_1, B \equiv P_{10}, C \equiv P_3$), then the conditions of the problem are satisfied: $\angle P_1P_{10}P_2 = \angle P_3P_{10}P_2 = \angle P_{10}P_3P_{11} = 15°, \angle P_1P_3P_{11} = 30°$.

From Fig.6 we can see that the sought angle $P_8P_1P_3$ is equal to $75°$.

Consequently, the considered problem is equivalent to the following statement: In the regular dodecagon, the diagonals P_1P_8, P_2P_{10} and P_3P_{11} intersect each other at one point. Let's verify this by the above Ceva's theorem. Then it is reduced to the verification of the identity

$$\frac{sin15°}{sin15°} \cdot \frac{sin30°}{sin15°} \cdot \frac{sin30°}{sin75°} = 1$$

And the validity of this identity follows from the equalities

$$sin75° = cos15°,\ sin30° = 2sin15°cos15°.$$

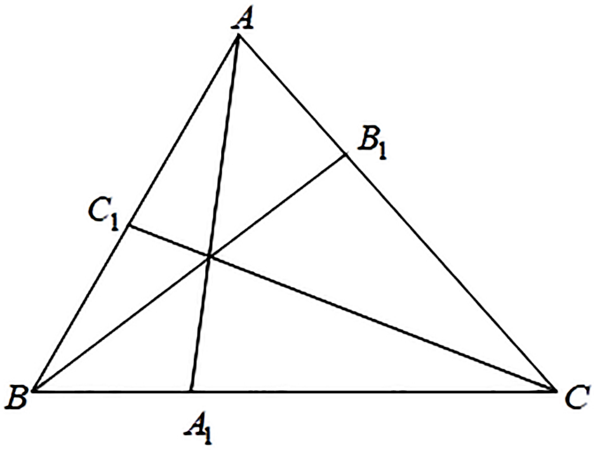

Figure 4: (Segments AA_1, BB_1 and CC_1 in the triangle ABC intersect each other at one point)

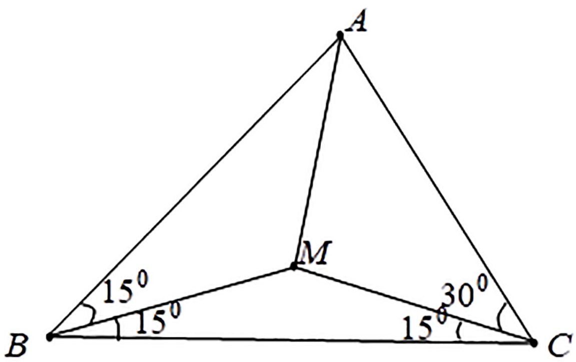

Figure 5: (Triangle ABC with the given angles)

Note that the presented method can be used to solve different kinds of problems, not only the ones mentioned above. Let's consider one of such problems.

Problem 3. The points D and E are given on the sides AB and AC of the triangle ABC, respectively, so that $\angle ABE = \angle CBE = 20°, \angle ACD = 30°, \angle BCD = 70°$ (Fig.7). Prove that $\angle BED = 60°$.

As the angles given in this problem are the multiples of $10°$, we divide the circle into 18 equal parts. If we place the triangle ABC inside the circle as shown on Fig.8 $(B \equiv C_{11}, C \equiv C_5, E \equiv C_3)$, then the conditions of the problem are satisfied: $\angle AC_{11}C_3 = \angle C_3C_{11}C_5 = 20°, vAC_5C_{18} = 30°, \angle C_{18}C_5C_{11} = 70°$.

From Fig.8 we can see that the sought angle $C_{17}C_3C_{11}$ is equal to $60°$.

As for Problem 3, it is equivalent to the following statement: In the regular octadecagon, the diagonals C_1C_{11}, C_3C_{17} and C_5C_{18} intersect each other at one point. Let's

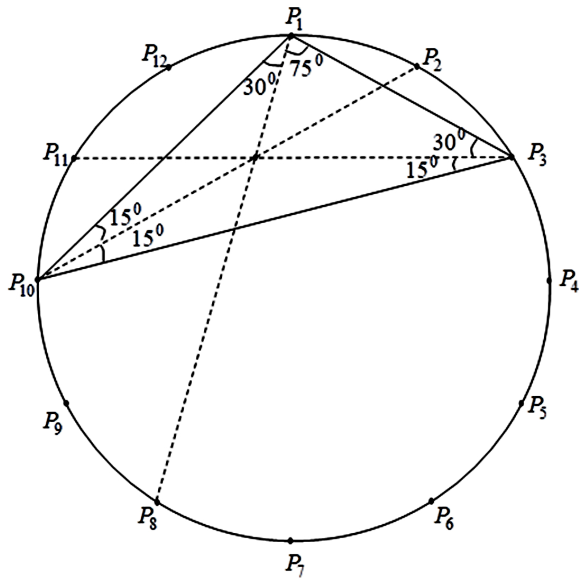

Figure 6: (Triangle $P_1P_{10}P_3$, equal to triangle ABC, placed inside the circle)

prove this statement as follows. It suffices to show that

$$\angle C_{17}DC_{18} + \angle C_{18}DC_1 + \angle C_1DC_3 = 180°.$$

Indeed,

$$\angle C_{17}DC_{18} = \frac{1}{2}\left(\cup C_{17}C_{18} + \cup C_3C_5\right) = 10° + 20° = 30°,$$

$$\angle C_{18}DC_1 = \frac{1}{2}\left(\cup C_1C_{18} + \cup C_5C_{11}\right) = 10° + 60° = 70°,$$

$$\angle C_1DC_3 = \frac{1}{2}\left(\cup C_1C_3 + \cup C_{11}C_{17}\right) = 20° + 60° = 80°,$$

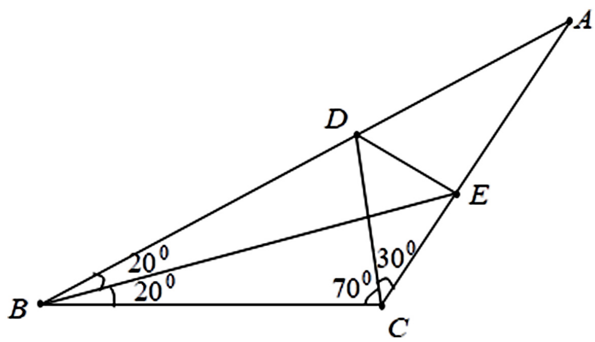

Figure 7: (Triangle ABC with the given angles)

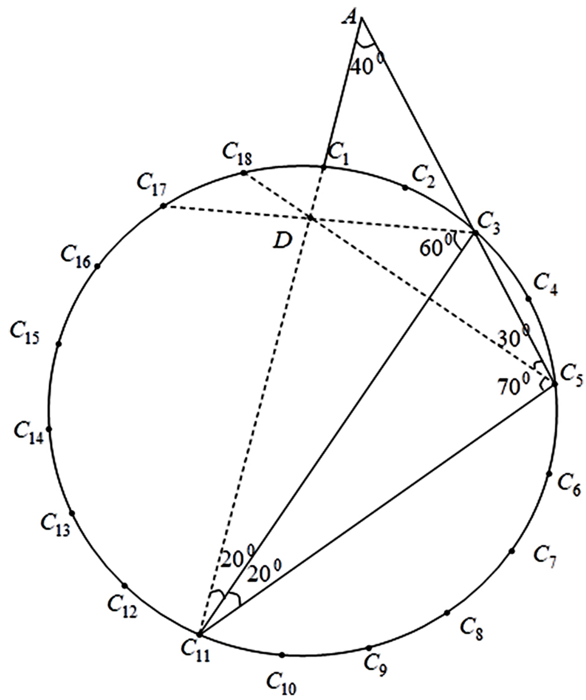

Figure 8: (Triangle $AC_{11}C_3$, equal to triangle ABC, placed inside the circle)

3. Concluding Remarks

This work deals with the solution of integer angled triangle problems in geometry. Usually, these problems are solved by the method of additional constructions, which requires a lot of experience and geometric intuition as there is no general algorithm for solving problems by this method. In this work, we present the method of auxiliary circle divided into equal parts, justification of which is based on one property of diagonals of regular polygons: three diagonals in regular polygons intersect each other at one point. From the problems solved above it can be seen that our method is simple and easy.

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