EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 751-762
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Hybrid Cubic B-spline Method for Solving A Class of Singular Boundary Value Problems

Ahmed Salem Heilat[1,*], Belal Batiha[1], Tariq Qawasmeh[1], Raed Hatamleh[1]

[1] *Department of Mathematics, Jadara University, P.O. Box(733), 21111 Irbid, Jordan*

Abstract. In this paper, a class of singular two-point boundary value problems are solved using hybrid cubic B-spline method. In this algorithm, a free parameter γ plays an important role to give accurate converge results for the solution. This parameter are chosen via optimization. Numerical examples are displayed to prove that our suggested method is flexible and effective, and the numerical results are compared with other numerical methods from the literature.

2020 Mathematics Subject Classifications: 34B05, 65L10

Key Words and Phrases: Singular two-point boundary value problems, Cubic B-spline, Trigonometric cubic B-spline, Hybrid cubic B-spline

1. Introduction

Consider a class of singular two-point boundary value problems:

$$
\begin{aligned}
x^{-\alpha}(x^{\alpha}y')' &= f(x,y), \quad 0 < x \leq 1 \\
y'(0) &= 0, \quad y(1) = \beta.
\end{aligned} \tag{1}
$$

where β is a finite constant and $\alpha \geq 1$. It is clear that (1) has a unique solution, if $f(x,y)$ is continuous, $\frac{\partial f}{\partial y}$ exists and is continuous and $\frac{\partial f}{\partial y} \geq 0$ [1]. Second-order singular boundary value problems arise frequently in several real life applications in chemical reaction, gas dynamics, thermal explosions, electrohydrodynamics, nuclear physics and atomic calculations can be modeled by linear and nonlinear singular differential equations. One important class of these equations is linear singular two-point boundary value problems which appeared clearly in many applications and has been examined in [2–11, 18, 19].

A lot of research has been carried out with the linear and non-linear singular of second-order boundary value problems. The finite difference method had been used to solve a class of singular two-point boundary value problems. Kumar suggested a difference method

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4725

Email addresses: ahmed_heilat@yahoo.com (A. S. Heilat),
b.bateha@jadara.edu.jo (B. Batiha), ta.qawasmeh@jadara.edu.jo (T. Qawasmeh),
raed@jadara.edu.jo (R. Hatamleh)

based on uniform mesh for a class of singular two-point boundary value problems [2]. a class of singular two-point boundary value problems using fourth-order finite difference method had been studied by Kanth and Reddy [3]. On the other hand, a class of singular two-point boundary value problems had been widely treated by using the spline method. Kanth and Reddy considered cubic spline method for solving a class of singular two-point boundary value problems [4] then he treated non-linear singular two-point boundary value problems by using Cubic spline polynomial [5]. Kadalbajoo and Aggarwal presented B-spline method for numerically solving singular two-point boundary value problems also they used Chebyshev polynomial to remove the singularity [6]. Singular two-point boundary value problems by using B-spline had been studied by N. Caglar and H. Caglar [7]. J. Rashidinia et al studied parametric spline method for a class of singular two-point boundary value problems [8]. B-spline of non-linear singular boundary value problems arising in physiology had been proposed by N. Caglar and H. Caglar [9]. Khuri and Sayfy suggested a new approach implementing a modified decomposition method in combination with the cubic B-spline collocation technique is introduced for the numerical solution of a class of singular boundary value problems arising in physiology [10]. Extended cubic uniform B-spline for a class of singular two-point boundary value problems had been suggested by Goh et al [11].
In this paper, we study a hybrid cubic B-spline method (HCBSM) for solving second-order singular boundary value problems. The approach used is basically that of [17]. The present method is examined by taking several problems to prove its accuracy.

2. Hybrid Cubic B-spline (HCBS)

In this paper, hybrid cubic B-spline function are utilized to solve singular two-point boundary value problems. Consider a partition π of $[a, b]$ is equally-spaced knots x_i into n segments $[x_i, x_{i+1}]$, $i = 0, 1, ..., n$, where $a = x_0 < x_1 < ... < x_n = b$, such that $h = \frac{b-a}{n}, x_0 = a, x_i = x_0 + ih$. Then, the hybrid cubic B-spline functions can be defined as the following relation [17]:

$$H_{4,i}(x) = \gamma B_{4,i}(x) + (1-\gamma)T_{4,i}(x) \tag{2}$$

where $\gamma \in \mathbb{R}$, $B_{4,i}(x)$ is cubic B-spline basis function [7].

$$B_{4,i}(x) = \frac{1}{24h^4}\begin{cases} (x-x_i)^3, & x \in [x_i, x_{i+1}], \\ h^3 + 3h^2(x - x_{i+1}) + 3h(x - x_{i+1})^2 - 3(x - x_{i+1})^3, & x \in [x_{i+1}, x_{i+2}], \\ h^3 + 3h^2(x_{i+3} - x) + 3h(x_{i+3} - x)^2 - 3(x_{i+3} - x)^3, & x \in [x_{i+2}, x_{i+3}], \\ (x_{i+4} - x)^3, & x \in [x_{i+3}, x_{i+4}], \end{cases} \tag{3}$$

and $T_{4,i}(x)$ is cubic trigonometric B-spline basis function [13] given as

$$T_{4,i}(x) = \frac{1}{\kappa}\begin{cases} p^3(x_i), & x \in [x_i, x_{i+1}], \\ p(x_i)[p(x_i)q(x_{i+2}) + p(x_{i+1})q(x_{i+3})] + p^2(x_{i+1})q(x_{i+4}), & x \in [x_{i+1}, x_{i+2}], \\ q(x_{i+4})[p(x_{i+1})q(x_{i+3}) + p(x_{i+2})q(x_{i+4})] + p(x_i)q^2(x_{i+3}), & x \in [x_{i+2}, x_{i+3}], \\ q^3(x_{i+4}), & x \in [x_{i+3}, x_{i+4}], \end{cases} \tag{4}$$

where $p(x_i) = \sin(\frac{x-x_i}{2})$, $q(x_i) = \sin(\frac{x_i-x}{2})$, $\kappa = \sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})$. The values of γ have essential role and influential in the hybrid cubic basis function. If $\gamma = 0$, the basis function is equal to cubic trigonometric B-spline basis function and if $\gamma = 1$, the basis function is equal to B-spline basis function.

3. Numerical method for singular boundary value problems

In linear case, the equation (1), can be taken as [14]

$$y''(x) + \frac{k}{x}y'(x) + b(x)y(x) = c(x), \quad 0 < x \le 1 \tag{5}$$

with boundary conditions

$$y'(0) = 0, \quad y(1) = \beta. \tag{6}$$

where $k \ge 1$.
By using L'Hôpital rule, we can modify Eq.(1) at the singular point $x = 0$, and then the boundary value problem is transform into [3, 7]

$$\begin{cases} (k+1)y''(x) + b(0)y(x) = c(0), & \text{for } x = 0, \\ y''(x) + \frac{k}{x}y'(x) + b(x)y(x) = c(x), & \text{for } x \neq 0, \end{cases} \tag{7}$$

In this section, hybrid cubic B-spline is used to solve a class of singular two-boundary value problems, the approach used is basically that of [15]. The approximate solution S(x), to the exact solution, y(x), is considered as

$$S(x) = \sum_{i=-1}^{n+1} C_i H_{4,i}(x) \tag{8}$$

where C_i are unknown real coefficients and $H_{4,i}(x)$ is hybrid cubic B-spline basis functions. There are only three nonzero basis functions; as follows, $H_{4,i-1}(x_i)$, $H_{4,i}(x_i)$, and $H_{4,i+1}(x_i)$ on sub interval $[x_i, x_{i+1}]$, as a result of local support properties of B-spline basis function. So, the approximate solution and its derivatives with respect to x are

$$S(x_i) = A_1C_{i-1} + A_2C_i + A_1C_{i+1}, \tag{9}$$
$$S^{'}(x_i) = A_3C_{i-1} - A_3C_{i+1}, \tag{10}$$
$$S^{''}(x_i) = A_4C_{i-1} + A_5C_i + A_4C_{i+1}, \tag{11}$$

where

$$A_i = \gamma\sigma_i + (1-\gamma)\eta_i, \quad \text{for } i = 1, 2, ..., 5, \tag{12}$$

with

$$\sigma_1 = \frac{1}{6}, \qquad \sigma_2 = \frac{4}{6}, \qquad \sigma_3 = \frac{-1}{2h}, \qquad \sigma_4 = \frac{1}{h^2}, \qquad \sigma_5 = \frac{-2}{h^2},$$
$$\eta_1 = \frac{\kappa_1^2}{\kappa_2\kappa_3}, \quad \eta_2 = \frac{2\kappa_1}{\kappa_3}, \quad \eta_3 = \frac{-3}{4\kappa_3}, \quad \eta_4 = \frac{3(\kappa_1 - 2\kappa_1^3 + \kappa_3)}{8\kappa_1\kappa_2\kappa_3}, \quad \eta_5 = \frac{-3(\kappa_4 + 2\kappa_1^2\kappa_2)}{4\kappa_1\kappa_2\kappa_3} \tag{13}$$

where

$$\kappa_1 = \sin(\frac{h}{2}), \qquad \kappa_2 = \sin(h), \qquad \kappa_3 = \sin(\frac{3h}{2}), \qquad \kappa_4 = \sin(2h), \tag{14}$$

Since $H_{4,i}(x_i) = 0$,

$$S_H(x_i) = C_{i-1}H_{4,i-1}(x_i) + C_iH_{4,i}(x_i) + C_{i+1}H_{4,i+1}(x_i)$$
$$= C_{i-1}[\gamma\sigma_1 + (1-\gamma)\eta_1] + C_i[\gamma\sigma_2 + (1-\gamma)\eta_2] + C_{i+1}[\gamma\sigma_1 + (1-\gamma)\eta_1] \approx y(x_i), \tag{15}$$

By finding the first and second derivative of (8) and calculating at x_i, to get (16) and (17).

$$S_H^{'}(x_i) = C_{i-1}[-\gamma\sigma_3 - (1-\gamma)\eta_3] + C_{i+1}[\gamma\sigma_3 + (1-\gamma)\eta_3] \approx y^{'}(x_i), \tag{16}$$

$$S_H^{''}(x_i) = C_{i-1}[\gamma\sigma_4 + (1-\gamma)\eta_4] + C_i[\gamma\sigma_5 + (1-\gamma)\eta_5] + C_{i+1}[\gamma\sigma_4 + (1-\gamma)\eta_4] \approx y^{''}(x_i), \tag{17}$$

the simplifications of $S_H(x)$, $S_H^{'}(x)$, and $S_H^{''}(x)$ at x_i are very beneficial for solving the problems.

In order to obtain the approximations of equations (5)-(6) at the point $x = x_i$, we put equations (15)-(17) into equations (7) and (6); this leads to

$$\begin{cases} (k+1)S''(0) + b(0)S(0) = c(0), \quad i = 0, \\ S''(x_i) + \dfrac{k}{x_i}S'(x_i) + b(x_i)S(x_i) = c(x_i), \quad i = 1, 2, ..., n \end{cases} \tag{18}$$

$S'(x_i) = 0 \quad$ for $i = 0, \quad$ and $\quad S(x_i) = \beta \quad$ for $i = n$.
To solve the singular two-point boundary value problems (5), suppose that $y(x) = S_H(x)$ and substituting (15), (16) and (17) into (18) and collecting the terms that only contain C_{i-1}, C_i, C_{i+1}, to get the following

$$\{C_{i-1}[\gamma\sigma_4 + (1-\gamma)\eta_4] + C_i[\gamma\sigma_5 + (1-\gamma)\eta_5] + C_{i+1}[\gamma\sigma_4 + (1-\gamma)\eta_4]\}$$
$$+\frac{k}{x_i}\{C_{i-1}[-\gamma\sigma_3 - (1-\gamma)\eta_3] + C_{i+1}[\gamma\sigma_3 + (1-\gamma)\eta_3]\}$$
$$+b(x_i)\{C_{i-1}[\gamma\sigma_1 + (1-\gamma)\eta_1] + C_i[\gamma\sigma_2 + (1-\gamma)\eta_2] + C_{i+1}[\gamma\sigma_1 + (1-\gamma)\eta_1]\} = c(x_i). \quad (19)$$

A system of $(n+3)$ equations with $(n+3)$ unknowns $C_{-1}, C_0, C_1, ..., C_{n+1}$ is obtained. This system can be written in the matrix-vector

$$[A]_{(n+3)\times(n+3)}[Q]_{(n+3)\times 1} = [R]_{1\times(n+3)}, \quad (20)$$

Where
$Q = [C_{-1}, C_0, C_1, ..., C_{n+1}]^T$,

$R = [0, c(x_0), c(x_1), ..., c(x_n), \beta]^T$
and A is an $(n+3)\times(n+3)$ -dimensional tri-diagonal matrix given by

$$A = \begin{pmatrix} -\gamma\sigma_3 - (1-\gamma)\eta_3 & 0 & \gamma\sigma_3 + (1-\gamma)\eta_3 & 0 & \cdots & 0 & 0 \\ a_0(x_0) & b_0(x_0) & c_0(x_0) & 0 & \cdots & 0 & 0 \\ 0 & a_1(x_1) & b_1(x_1) & c_1(x_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & a_n(x_n) & b_n(x_n) & c_n(x_n) \\ . & . & . & . & \gamma\sigma_1 + (1-\gamma)\eta_1 & \gamma\sigma_2 + (1-\gamma)\eta_2 & \gamma\sigma_1 + (1-\gamma)\eta_1 \end{pmatrix}$$

as well the coefficient in the matrix A as the following

$a_0(x_0) = (k+1)[\gamma\sigma_4 + (1-\gamma)\eta_4] + b(x_i)[\gamma\sigma_1 + (1-\gamma)\eta_1], \quad i = 0$
$b_0(x_0) = (k+1)[\gamma\sigma_5 + (1-\gamma)\eta_5] + b(x_i)[\gamma\sigma_2 + (1-\gamma)\eta_2], \quad i = 0$
$c_0(x_0) = (k+1)[\gamma\sigma_4 + (1-\gamma)\eta_4] + b(x_i)[\gamma\sigma_1 + (1-\gamma)\eta_1], \quad i = 0$

$a_i(x_i) = [\gamma\sigma_4 + (1-\gamma)\eta_4] - \frac{k}{x_i}[\gamma\sigma_3 + (1-\gamma)\eta_3] + b(x_i)[\gamma\sigma_1 + (1-\gamma)\eta_1], \quad i = 1, 2, ..., n$
$b_i(x_i) = [\gamma\sigma_5 + (1-\gamma)\eta_5] + b(x_i)[\gamma\sigma_2 + (1-\gamma)\eta_2], \quad i = 1, 2, ..., n$
$c_i(x_i) = [\gamma\sigma_4 + (1-\gamma)\eta_4] + \frac{k}{x_i}[\gamma\sigma_3 + (1-\gamma)\eta_3] + b(x_i)[\gamma\sigma_1 + (1-\gamma)\eta_1], \quad i = 1, 2, ..., n$

where Q is the unknown vector. Therefor, Q can be solved by taking

$$Q = A^{-1}R. \quad (21)$$

Finally, substitute the values of C_i , for $i = -1, 0, ..., n+1$ in equation (8) to get the approximated analytical solution to (18).

4. Optimization of the free parameter γ

The optimization for the values of γ has already been calculated as in [17].

5. Numerical Examples and Discussions

Hybrid cubic B-spline method was applied on problems (1), (2), (3), and (4) with different value of n to demonstrate the effectiveness of our proposed hybrid cubic B-spline method . The results obtained by hybrid cubic B-spline method are compared with the analytical solution. The discrete L_∞-norm and L_2-norm are defined as follows

$$L_\infty = \max_{i=1}^{n-1} \mid S(x_i) - u(x_i)) \mid$$

$$L_2 = \sqrt{\sum_{i=1}^{n-1} [S(x_i) - u(x_i)]^2}$$

We found that our method in comparison with the cubic spline method [4] and cubic B-spline method [7] and reproducing kernel spaces method [16] is much better with a view to accuracy and efficiency.

Example 1.Consider the following boundary value problem [4]

$$\begin{cases} y''(x) + \dfrac{2}{x} y'(x) - 4y(x) = -2, \quad 0 < x \leq 1, \\ y'(0) = 0, \qquad y(1) = 5.5 \end{cases} \tag{22}$$

The analytic solution is $y(x) = 0.5 + \frac{5sinh2x}{xsinh2}$. The numerical results are tabulated in Table 1 with $h = 0.05$ and compared with those obtained in [4, 7] and the exact solutions. The computational errors, L_∞ norm, and L_2 norm are tabulated in Table 2.

Table 1: Comparison of proposed method with cubic spline method [4] and cubic B-spline method [7] in the solution of example 1 with $h = 0.05$ and $\gamma = 1.85127150$ for $S(x)$

x	Exact solution	Cubic spline[4]	Cubic B-spline[7]	HCBSM
0.1	3.275624	$9.0800E-04$	$2.9500E-04$	$1.4257E-11$
0.2	3.331322	$2.8900E-04$	$2.9200E-04$	$3.2166E-06$
0.3	3.425641	$2.8100E-04$	$2.8500E-04$	$4.9187E-06$
0.4	3.560864	$2.7200E-04$	$2.7500E-04$	$6.4545E-06$
0.5	3.740271	$2.5500E-04$	$2.5800E-04$	$7.8650E-06$
0.6	3.968246	$2.3300E-04$	$2.3500E-04$	$8.9406E-06$
0.7	4.250393	$2.0000E-04$	$2.0200E-04$	$9.3166E-06$
0.8	4.593706	$1.5500E-04$	$1.5500E-04$	$8.4697E-06$
0.9	5.006766	$8.9000E-05$	$8.9000E-05$	$5.6839E-06$
1.0	5.500000	$0.0000E-00$	$0.0000E-00$	$0.0000E-00$

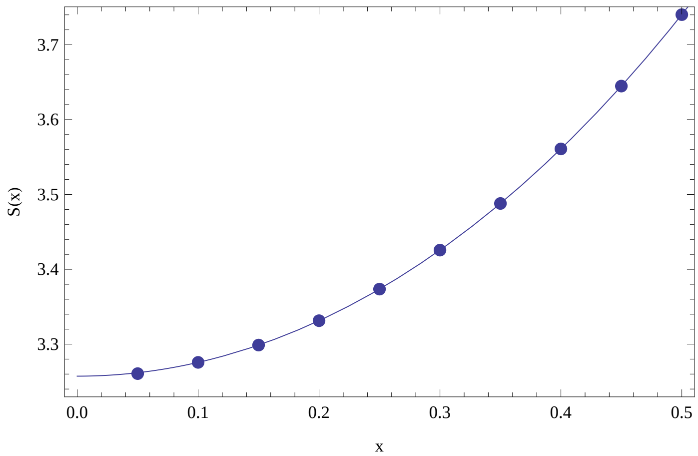

Figure 1: Numerical solution $S(x)$ and exact solution $y(x)$ for Example 1 with $h = \frac{1}{20}$ and $\gamma = 1.85127150$

Table 2: Comparison of error norms with cubic spline[4], cubic B-spline[7], and HCBSM for Example 1 with $h = 0.05$ and $\gamma = 1.85127150$

	Cubic spline[4]		Cubic B-spline[7]		HCBSM	
h	L_∞	L_2	L_∞	L_2	L_∞	L_2
0.05	$9.0800E-04$	$1.1190E-03$	$2.9500E-04$	$7.2366E-04$	$9.3166E-06$	$2.0219E-05$

Example 2. Consider the following boundary value problem [4]

$$\begin{cases} y''(x) + \dfrac{1}{x}y'(x) = (\dfrac{8}{8-x^2})^2, \\ y'(0) = 0, \qquad y(1) = 0 \end{cases} \tag{23}$$

The analytic solution is $y(x) = 2log(\frac{7}{8-x^2})$.

We can obtain absolute errors and the errors at different knots by using suggested hybrid cubic B-spline method with $h = 0.05$ and $\gamma = 1.4125$ are tabulated in Table 3 and compared with existing methods[4, 7]. The computational errors, L_∞ norm, and L_2 norm are tabulated in Table 4. It is clear that the current hybrid cubic B-spline method (HCBSM) is acceptable and accurate than cubic spline method [4] and cubic B-spline method [7]. The numerical results obtained by hybrid cubic B-spline method are shown in Figs. 1.

Table 3: Comparison of proposed method with cubic spline method [4] and cubic B-spline method [7] in the solution of example 2 with $h = 0.05$ and $\gamma = 1.4125$ for $S(x)$

x	Exact solution	Cubic spline[4]	Cubic B-spline[7]	HCBSM
0.1	−0.264561	$2.7000E-05$	$2.7000E-05$	$1.2355E-07$
0.2	−0.257038	$2.5000E-05$	$2.6000E-05$	$3.6696E-10$
0.3	−0.244435	$2.5000E-05$	$2.5000E-05$	$1.8901E-07$
0.4	−0.226657	$2.4000E-05$	$2.4000E-05$	$4.2654E-07$
0.5	−0.203565	$2.2000E-05$	$2.2000E-05$	$6.8326E-07$
0.6	−0.174975	$1.9000E-05$	$1.9000E-05$	$9.1680E-07$
0.7	−0.140651	$1.5000E-05$	$1.5000E-05$	$1.0672E-06$
0.8	−0.100300	$1.1000E-05$	$1.1000E-05$	$1.0503E-06$
0.9	−0.053562	$6.0000E-06$	$6.0000E-06$	$7.4911E-07$
1.0	0.000000	$0.0000E-00$	$0.0000E-00$	$0.0000E-00$

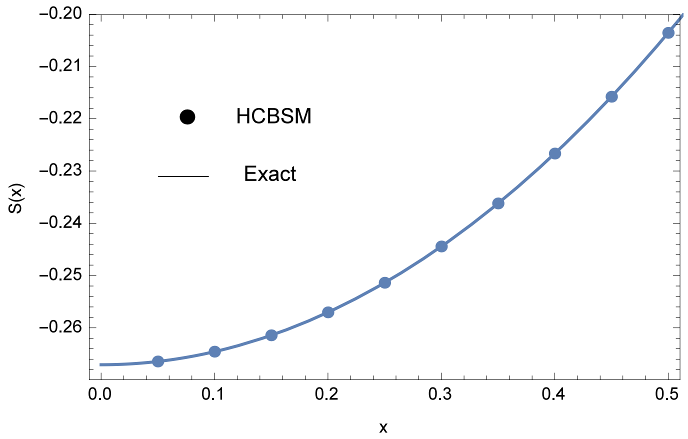

Figure 2: Numerical solution $S(x)$ and exact solution $y(x)$ for Example 2 with $h = 0.05$ and $\gamma = 1.4125$

Table 4: Comparison of error norms with cubic spline[4], cubic B-spline[7], and HCBSM for Example 2 with $h = 0.05$ and $\gamma = 1.4125$

	Cubic spline[4]		Cubic B-spline[7]		HCBSM	
h	L_∞	L_2	L_∞	L_2	L_∞	L_2
0.05	$2.7000E-05$	$6.1498E-05$	$2.7000E-05$	$6.1911E-05$	$1.0672E-06$	$2.0841E-06$

Example 3.Consider the following boundary value problem [7]

$$\begin{cases} -y''(x) - \dfrac{2}{x}y'(x) + (1-x^2)y(x) = x^4 - 2x^2 + 7, \\ y'(0) = 0, \qquad y(1) = 0 \end{cases} \tag{24}$$

The analytic solution is $y(x) = 1 - x^2$. The numerical results with $h = 0.05$ and $\gamma = 0.999999$ are given in Table 5 and Figs. 2. The computational errors, L_∞ norm, and L_2 norm are tabulated in Table 6.

Table 5: Comparison of proposed method with cubic spline method [4] and cubic B-spline method [7] in the solution of example 3 with $h = 0.05$ and $\gamma = 0.999999$ for $S(x)$

x	Exact solution	Cubic spline[4]	Cubic B-spline[7]	HCBSM
0.1	0.990000	$2.0000E-06$	$1.1405E-07$	$1.1405E-07$
0.2	0.960000	$2.0000E-06$	$4.5295E-08$	$4.5295E-08$
0.3	0.910000	$2.0000E-06$	$2.5596E-08$	$2.5596E-08$
0.4	0.840000	$2.0000E-06$	$1.6158E-08$	$1.6158E-08$
0.5	0.750000	$1.0000E-06$	$1.0650E-08$	$1.0650E-08$
0.6	0.640000	$1.0000E-06$	$7.0547E-09$	$7.0547E-09$
0.7	0.510000	$1.0000E-06$	$4.5235E-09$	$4.5235E-09$
0.8	0.360000	$1.0000E-06$	$2.6397E-09$	$2.6397E-09$
0.9	0.190000	$1.0520E-07$	$1.1767E-09$	$1.1767E-09$
1.0	0.000000	$0.0000E-00$	$0.0000E-00$	$0.0000E-00$

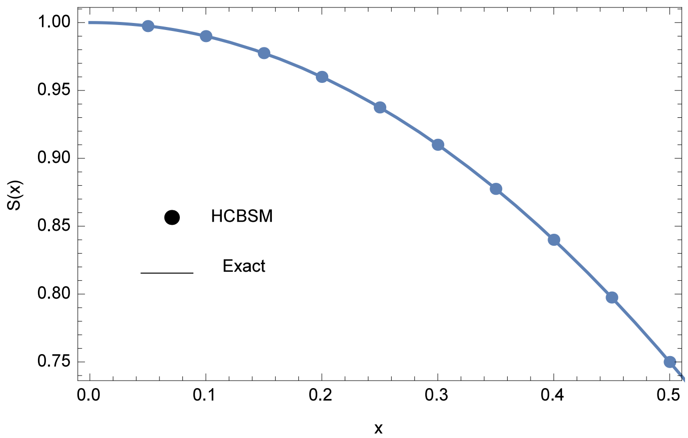

Figure 3: Numerical solution $S(x)$ and exact solution $y(x)$ for Example 3 with $h = 0.05$ and $\gamma = 0.999999$

Table 6: Comparison of error norms with cubic spline[4], cubic B-spline[7], and HCBSM for Example 3 with $h = 0.05$ and $\gamma = 0.999999$

	Cubic spline[4]		Cubic B-spline[7]		HCBSM	
h	L_∞	L_2	L_∞	L_2	L_∞	L_2
0.05	$2.0000E-06$	$4.4734E-06$	$1.1405E-07$	$1.2715E-07$	$1.1405E-07$	$1.2715E-07$

Example 4.Consider the following boundary value problem [16]

$$\begin{cases} y''(x) + \dfrac{1}{x}y'(x) + y(x) = 4 - 9x + x^2 - x^3, \\ y(0) = 0, \qquad y(1) = 0 \end{cases} \tag{25}$$

The analytic solution is $y(x) = x^2 - x^3$.
Hybrid cubic B-spline method is used to solve this problem numerically with $h = \frac{1}{26}$. Table 7 list approximate solutions. The numerical results obtained by hybrid cubic B-spline method are shown in Figs. 3. The computational errors, L_∞ norm, and L_2 norm are tabulated in Table 8. The results of this problem is compared with Minggen Cui [16]

which is listed in Table 7. The table denotes that HCBSM with $\gamma = 0.999999$ gives better results to this problems.

Table 7: Comparison of proposed method with reproducing kernel spaces method [16] in the solution of example 4 with $h = \frac{1}{26}$ and $\gamma = 0.999999$ for $S(x)$

x	Exact Solution	Reproducing kernel spaces [16]	HCBSM
0.001	$9.99E-07$	$2.0000E-11$	$2.6190E-09$
0.08	0.005888	$2.3000E-06$	$2.1606E-08$
0.16	0.021504	$1.1600E-05$	$1.7831E-08$
0.32	0.069632	$5.5800E-05$	$4.6929E-08$
0.48	0.119808	$1.1900E-04$	$5.4917E-08$
0.64	0.147456	$1.6800E-04$	$4.9914E-08$
0.80	0.128000	$1.5700E-04$	$3.2371E-08$
0.96	0.036864	$4.5000E-05$	$6.9600E-09$
1.00	0.000000	$0.0000E-00$	$0.0000E-00$

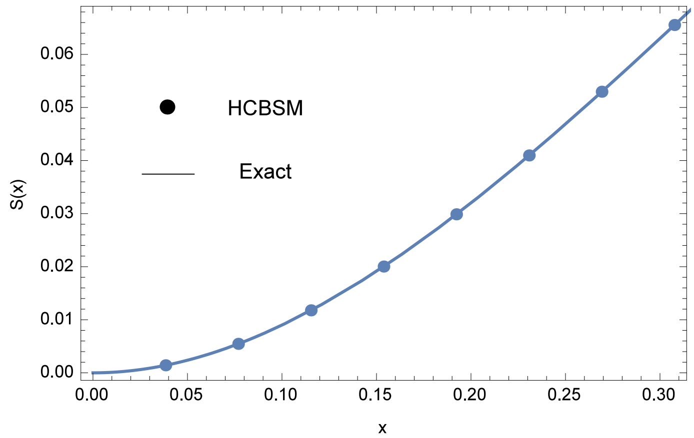

Figure 4: Numerical solution $S(x)$ and exact solution $y(x)$ for Example 4 with $h = \frac{1}{26}$ and $\gamma = 0.999999$

Table 8: Comparison of error norms with reproducing kernel spaces method [16], and HCBSM for Example 4 with $h = \frac{1}{26}$ and $\gamma = 0.999999$

	reproducing kernel spaces method [16]		HCBSM	
h	L_∞	L_2	L_∞	L_2
$\frac{1}{26}$	$1.6800E-04$	$2.6891E-04$	$5.4917E-08$	$9.7968E-08$

6. Conclusions

In this paper, a class of singular two point boundary value problems has been successfully solved using HCBSM. The hybrid cubic B-spline method for finding a numerical solution is easy and acceptable and flexible to apply. The numerical results presented indicate the proposed method is an accurate and reliable approach for the solution of a class of singular two point boundary value problems. Further, minimizing the one-norm term is sufficient to obtain the optimized values of γ. [17]

References

[1] Russell, R. D. and Shampine, L. F. (1975). Numerical methods for singular boundary value problems. SIAM Journal on Numerical Analysis, 12(1), 13-36.

[2] Kumar, M. (2003). A difference method for singular two-point boundary value problems. Applied mathematics and computation, 146(2), 879-884.

[3] Kanth, A. R. and Reddy, Y. N. (2004). Higher order finite difference method for a class of singular boundary value problems. Applied mathematics and computation, 155(1), 249-258.

[4] Kanth, A. R. and Reddy, Y. N. (2005). Cubic spline for a class of singular two-point boundary value problems. Applied mathematics and coputation, 170(2), 733-740.

[5] Kanth, A. R. (2007). Cubic spline polynomial for non-linear singular two-point boundary value problems. Applied mathematics and computation, 189(2), 2017-2022.

[6] Kadalbajoo, M. K. and Aggarwal, V. K. (2005). Numerical solution of singular boundary value problems via Chebyshev polynomial and B-spline. Applied mathematics and computation, 160(3), 851-863.

[7] Caglar, N. and Caglar, H. (2006). B-spline solution of singular boundary value problems. Applied mathematics and computation, 182(2), 1509-1513.

[8] Rashidinia, J., Mahmoodi, Z. and Ghasemi, M. (2007). Parametric spline method for a class of singular two-point boundary value problems. Applied mathematics and computation, 188(1), 58-63.

[9] Çağlar, H., Çağlar, N. and Özer, M. (2009). B-spline solution of non-linear singular boundary value problems arising in physiology. Chaos, Solitons and Fractals, 39(3), 1232-1237.

[10] Khuri, S. A. and Sayfy, A. (2010). A novel approach for the solution of a class of singular boundary value problems arising in physiology. Mathematical and Computer Modelling, 52(3), 626-636.

[11] Goh, J., Majid, A. A., Ismail, A. I. M. (2011). Extended cubic uniform B-spline for a class of singular boundary value problems. nuclear physics, 2, 4.

[12] Mat Zin, S., Abd Majid, A., Ismail, A. I. M., and Abbas, M. (2014). Application of Hybrid Cubic B-Spline Collocation Approach for Solving a Generalized Nonlinear Klien-Gordon Equation. Mathematical Problems in Engineering, 2014.

[13] Abbas, M., Majid, A. A., Ismail, A. I. M., and Rashid, A. (2014). The application of cubic trigonometric B-spline to the numerical solution of the hyperbolic problems. Applied Mathematics and Computation, 239, 74-88.

[14] Kadalbajoo, M. K. and Kumar, V. (2007). B-spline method for a class of singular two-point boundary value problems using optimal grid. Applied mathematics and computation, 188(2), 1856-1869.

[15] Prenter, P. M. (1989). Splines and variational methods. John Wiley and Sons.

[16] Cui, M. and Geng, F. (2007). Solving singular two-point boundary value problem in reproducing kernel space. Journal of Computational and Applied Mathematics, 205(1), 6-15.

[17] Heilat, A. S., Zureigat, H., and Batiha, B. (2021). New Spline Method for Solving Linear Two-Point Boundary Value Problems. European Journal of Pure and Applied Mathematics, 14(4), 1283-1294.

[18] Batiha, B., Ghanim, F., Alayed, O., Hatamleh, R. E., Heilat, A. S., Zureigat, H., and Bazighifan, O. (2022). Solving Multispecies Lotka–Volterra Equations by the Daftardar-Gejji and Jafari Method. International Journal of Mathematics and Mathematical Sciences.

[19] Visuvasam, J., Meena, A., and Rajendran, L. (2020). New analytical method for solving nonlinear equation in rotating disk electrodes for second-order ECE reactions. Journal of Electroanalytical Chemistry, 869, 106-114.