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# Upper and lower $s\beta(\star)$-continuous multifunctions

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**Abstract.** Our main purpose is to introduce the concepts of upper and lower $s\beta(\star)$-continuous multifunctions. In particular, some characterizations of upper and lower $s\beta(\star)$-continuous multifunctions are investigated. Moreover, the relationships between $s\beta(\star)$-continuous multifunctions and almost $s\beta(\star)$-continuous multifunctions are discussed.

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**Key Words and Phrases**: Upper $s\beta(\star)$-continuous multifunction, lower $s\beta(\star)$-continuous multifunction

## 1. Introduction

The concept of semi-continuity was first introduced by Levine [13]. In 1982, Mashhour et al. [15] introduced and investigated the notion of precontinuous functions. Abd El-Monsef et al. [7] introduced the notion of $\beta$-continuous functions as a generalization of semi-continuous functions [13] and precontinuous functions [15]. Borsík and Doboš [4] introduced the notion of almost quasi-continuity which is weaker than that of quasi-continuity [14] and investigated a decomposition theorem of quasi-continuity. Popa and Noiri [17] investigated some characterizations of $\beta$-continuity and showed that almost quasi-continuity is equivalent to $\beta$-continuity. In 1993, Popa and Noiri [18] extended the concept of $\beta$-continuous functions to multifunctions and introduced the notions of upper and lower $\beta$-continuous multifunctions. Moreover, the relationships between $\beta$-continuous mulfunctions and quasi-continuous multifunctions were established in [17]. Noiri and Popa [16] introduced and studied the concepts of upper and lower almost $\beta$-continuous mulfunctions. In 2003, Hatir et al. [8] introduced and investigated the notions of strong $\beta$-$\mathscr{I}$-open sets and strongly $\beta$-$\mathscr{I}$-continuous functions in ideal topological spaces. Hatir et al. [9] investigated further properties of strong $\beta$-$\mathscr{I}$-open sets and strongly $\beta$-$\mathscr{I}$-continuous functions. In 2019, Boonpok [2] introduced and studied the concepts of upper and lower $\star$-continuous multifunctions in ideal topological spaces. In [3], the present

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author introduced and investigated the notions of upper and lower $\beta(\star)$-continuous multifunctions. The purpose of the present paper is to introduce the notions of upper and lower $s\beta(\star)$-continuous multifunctions. Furthermore, several characterizations of upper and lower $s\beta(\star)$-continuous multifunctions are investigated. Moreover, the relationships between $s\beta(\star)$-continuous multifunctions and almost $s\beta(\star)$-continuous multifunctions are discussed.

## 2. Preliminaries

Throughout the present paper, spaces $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of a topological space $(X, \tau)$. The closure of $A$ and the interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space $(X, \tau)$ with an ideal $\mathscr{I}$ on $X$ is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset $A$ of $X$, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I}$ for every open neighbourhood $U$ of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [12], $A^\star$ is called the local function of $A$ with respect to $\mathscr{I}$ and $\tau$ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than $\tau$. A subset $A$ is said to be $\star$-closed [11] if $A^\star \subseteq A$. The interior of a subset $A$ in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from $X$ into $Y$, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set $B$ of $Y$ by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

**Lemma 1.** *For a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

*(1) If $V \in \tau$, then $V \cap Cl^\star(A) \subseteq Cl^\star(V \cap A)$* [9].

*(2) If $F$ is closed in $X$, then $Int^\star(A \cup F) \subseteq Int^\star(A) \cup F$.*

A subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$ is called *semi-$\mathscr{I}$-open* [10] (resp. *pre$^\star_{\mathscr{I}}$-open* [5], *strong $\beta$-$\mathscr{I}$-open* [8]) if $A \subseteq \text{Cl}^\star(\text{Int}(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}(A))$, $A \subseteq \text{Cl}^\star(\text{Int}(\text{Cl}^\star(A)))$). The complement of a semi-$\mathscr{I}$-open (resp. pre$^\star_{\mathscr{I}}$-open, strong $\beta$-$\mathscr{I}$-open) set is called *semi-$\mathscr{I}$-closed* [10] (resp. *pre$^\star_{\mathscr{I}}$-closed* [5], *strong $\beta$-$\mathscr{I}$-closed* [8]). The *strong $\beta$-$\mathscr{I}$-closure* (resp. *semi-$\mathscr{I}$-closure*) [6] of a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$, denoted by $s\beta\text{Cl}_{\mathscr{I}}(A)$ (resp. $s\text{Cl}_{\mathscr{I}}(A)$), is defined by the intersection of all strong $\beta$-$\mathscr{I}$-closed (resp. semi-$\mathscr{I}$-closed) sets of $X$ containing $A$. Let $A$ be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. The union of all strong $\beta$-$\mathscr{I}$-open sets of $X$ contained in $A$ is called the *strong $\beta$-$\mathscr{I}$-interior* of $A$ and is denoted by $s\beta\mathrm{Int}_{\mathscr{I}}(A)$.

**Lemma 2.** *For a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

*(1)* $s\mathit{Cl}_{\mathscr{I}}(A) = A \cup \mathit{Int}^{\star}(\mathit{Cl}(A))$ [6].

*(2)* $s\beta \mathit{Cl}_{\mathscr{I}}(A) = A \cup \mathit{Int}^{\star}(\mathit{Cl}(\mathit{Int}^{\star}(A)))$ [6].

*(3)* $s\beta \mathit{Int}_{\mathscr{I}}(A) = A \cap \mathit{Cl}^{\star}(\mathit{Int}(\mathit{Cl}^{\star}(A)))$.

*Proof.* (3) We observe that

$$\begin{aligned} A \cap \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A))) &\subseteq \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A))) \\ &= \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A) \cap \mathrm{Int}(\mathrm{Cl}^{\star}(A)))) \\ &\subseteq \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A \cap \mathrm{Int}(\mathrm{Cl}^{\star}(A))))) \\ &\subseteq \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A \cap \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A)))))). \end{aligned}$$

Thus, $A \cap \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A)))$ is strong $\beta$-$\mathscr{I}$-open and so $A \cap \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A))) \subseteq s\beta\mathrm{Int}_{\mathscr{I}}(A)$. On the other hand, since $s\beta\mathrm{Int}_{\mathscr{I}}(A)$ is strong-$\beta$-$\mathscr{I}$-open, we have

$$s\beta\mathrm{Int}_{\mathscr{I}}(A) \subseteq \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(s\beta\mathrm{Int}_{\mathscr{I}}(A)))) \subseteq \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A)))$$

and hence $s\beta\mathrm{Int}_{\mathscr{I}}(A) \subseteq A \cap \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A)))$. Thus, $s\beta\mathrm{Int}_{\mathscr{I}}(A) = A \cap \mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(A)))$.

**Lemma 3.** *Let $V$ be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. If $V$ is $\star$-open, then $s\mathit{Cl}_{\mathscr{I}}(V) = \mathit{Int}^{\star}(\mathit{Cl}(V))$.*

*Proof.* Suppose that $V$ is $\star$-open. Then, we have $V \subseteq \mathrm{Int}^{\star}(\mathrm{Cl}(V))$, by Lemma 2, $s\mathrm{Cl}_{\mathscr{I}}(V) = V \cup \mathrm{Int}^{\star}(\mathrm{Cl}(V)) = \mathrm{Int}^{\star}(\mathrm{Cl}(V))$.

**Lemma 4.** *Let $A$ be a subset of an ideal topological space $(X, \tau, \mathscr{I})$ and $x \in X$. Then, $x \in s\beta \mathit{Cl}_{\mathscr{I}}(A)$ if and only if $U \cap A \neq \emptyset$ for every strong $\beta$-$\mathscr{I}$-open set $U$ containing $x$.*

*Proof.* Let $x \in s\beta\mathrm{Cl}_{\mathscr{I}}(A)$. Suppose that $U \cap A = \emptyset$ for some strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$. Then, $A \subseteq X - U$ and $X - U$ is strong $\beta$-$\mathscr{I}$-closed. Since $x \in s\beta\mathrm{Cl}_{\mathscr{I}}(A)$, we have $x \in s\beta\mathrm{Cl}_{\mathscr{I}}(X - U) = X - U$; hence $x \notin U$, which is a contradiction that $x \in U$. Thus, $U \cap A \neq \emptyset$ for every strong $\beta$-$\mathscr{I}$-open set $U$ containing $x$.

Conversely, assume that $U \cap A \neq \emptyset$ for every strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$. We shall show that $x \in s\beta\mathrm{Cl}_{\mathscr{I}}(A)$. Suppose that $x \notin s\beta\mathrm{Cl}_{\mathscr{I}}(A)$. Then, there exists a strong $\beta$-$\mathscr{I}$-closed set $F$ such that $A \subseteq F$ and $x \notin F$. Thus, $X - F$ is a strong $\beta$-$\mathscr{I}$-open set containing $x$ such that $(X - F) \cap A = \emptyset$. This a contradiction to $U \cap A \neq \emptyset$; hence $x \in s\beta\mathrm{Cl}_{\mathscr{I}}(A)$.

**Lemma 5.** *For a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are hold:*

*(1)* $X - s\beta Cl_{\mathscr{I}}(A) = s\beta Int_{\mathscr{I}}(X - A)$.

*(2)* $X - s\beta Int_{\mathscr{I}}(A) = s\beta Cl_{\mathscr{I}}(X - A)$.

*Proof.* (1) Let $x \in X - s\beta\text{Cl}_{\mathscr{I}}(A)$. Then, $x \notin s\beta\text{Cl}_{\mathscr{I}}(A)$ and there exists a strong $\beta$-$\mathscr{I}$-open set $V$ of $X$ containing $x$ such that $V \cap A = \emptyset$. Thus, $V \subseteq X - A$ and hence $x \in s\beta\text{Int}_{\mathscr{I}}(X-A)$. This shows that $X - s\beta\text{Cl}_{\mathscr{I}}(A) \subseteq s\beta\text{Int}_{\mathscr{I}}(X-A)$. On the other hand, let $x \in s\beta\text{Int}_{\mathscr{I}}(X-A)$. Then, there exists a strong $\beta$-$\mathscr{I}$-open set $V$ of $X$ containing $x$ such that $V \subseteq X - A$ and so $V \cap A = \emptyset$. By Lemma 4, $x \notin s\beta\text{Cl}_{\mathscr{I}}(A)$; hence $x \in X - s\beta\text{Cl}_{\mathscr{I}}(A)$. Thus, $s\beta\text{Int}_{\mathscr{I}}(X - A) \subseteq X - s\beta\text{Cl}_{\mathscr{I}}(A)$ and so $X - s\beta\text{Cl}_{\mathscr{I}}(A) = s\beta\text{Int}_{\mathscr{I}}(X - A)$.

(2) This follows from (1).

## 3. Upper and lower $s\beta(\star)$-continuous multifunctions

In this section, we introduce the notions of upper and lower $s\beta(\star)$-continuous multifunctions. Moreover, several characterizations of upper and lower $s\beta(\star)$-continuous multifunctions are discussed.

**Definition 1.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

*(1) upper $s\beta(\star)$-continuous at a point $x \in X$ if, for each $\star$-open set $V$ of $Y$ containing $F(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$;*

*(2) lower $s\beta(\star)$-continuous at a point $x \in X$ if, for each $\star$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(z) \cap V \neq \emptyset$ for every $z \in U$;*

*(3) upper (resp. lower) $s\beta(\star)$-continuous if $F$ has this property at each point of $X$.*

**Theorem 1.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is upper $s\beta(\star)$-continuous at $x \in X$ if and only if $x \in s\beta Int_{\mathscr{I}}(F^+(V))$ for every $\star$-open set $V$ of $Y$ containing $F(x)$.*

*Proof.* Let $V$ be any $\star$-open set of $Y$ containing $F(x)$. Then, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$. Then, $U \subseteq F^+(V))$. Since $U$ is strong $\beta$-$\mathscr{I}$-open, we have $x \in U \subseteq \text{Cl}^\star(\text{Int}(\text{Cl}^\star(U))) \subseteq \text{Cl}^\star(\text{Int}(\text{Cl}^\star(F^+(V))))$. Since $x \in F^+(V)$ and by Lemma 2, $x \in F^+(V) \cap \text{Cl}^\star(\text{Int}(\text{Cl}^\star(F^+(V)))) = s\beta\text{Int}_{\mathscr{I}}(F^+(V))$.

Conversely, let $V$ be any $\star$-open set of $Y$ containing $F(x)$. By (2), $x \in s\beta\text{Int}_{\mathscr{I}}(F^+(V))$ and so there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that $F$ is upper $s\beta(\star)$-continuous at $x$.

**Theorem 2.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower $s\beta(\star)$-continuous at $x \in X$ if and only if $x \in s\beta Int_{\mathscr{I}}(F^-(V))$ for every $\star$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$.*

*Proof.* The proof is similar to that of Theorem 1.

**Definition 2.** *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called $s\beta(\star)$-continuous at a point $x \in X$ if, for each $\star$-open set $V$ of $Y$ containing $f(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called $s\beta(\star)$-continuous if $f$ has this property at each point of $X$.*

**Corollary 1.** *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $s\beta(\star)$-continuous at $x \in X$ if and only if $x \in s\beta Int_{\mathscr{I}}(f^{-1}(V))$ for every $\star$-open set $V$ of $Y$ containing $f(x)$.*

**Theorem 3.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is upper $s\beta(\star)$-continuous;*

*(2) $F^+(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$ for every $\star$-open set $V$ of $Y$;*

*(3) $F^-(K)$ is strong $\beta$-$\mathscr{I}$-closed in $X$ for every $\star$-closed set $K$ of $Y$;*

*(4) $s\beta Cl_{\mathscr{I}}(F^-(B)) \subseteq F^-(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(5) $Int^\star(Cl(Int^\star(F^-(B)))) \subseteq F^-(Cl^\star(B))$ for every subset $B$ of $Y$.*

*Proof.* (1) ⇒ (2): Let $V$ be any $\star$-open set of $Y$ and $x \in F^+(V)$. There exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$. Thus,

$$x \in U \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(U))) \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(V))))$$

and hence $F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(V))))$. This shows that $F^+(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$.

(2) ⇒ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset $B$ of $Y$.

(3) ⇒ (4): For any subset $B$ of $Y$, $\mathrm{Cl}^\star(B)$ is $\star$-closed in $Y$ and by (3), we have $F^-(\mathrm{Cl}^\star(B))$ is strong $\beta$-$\mathscr{I}$-closed in $X$. Thus, $s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(B)) \subseteq F^-(\mathrm{Cl}^\star(B))$.

(4) ⇒ (5): Let $B$ be any subset of $Y$. By (4) and Lemma 2,

$$\mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^-(B)))) \subseteq s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(B)) \subseteq F^-(\mathrm{Cl}^\star(B)).$$

(5) ⇒ (2): Let $V$ be any $\star$-open set of $Y$. Then, $Y - V$ is $\star$-closed in $Y$ and by (5),

$$\begin{aligned} X - F^+(V) &= F^-(Y - V) \\ &\supseteq \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^-(Y - V)))) \\ &= \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(X - F^+(V)))) \\ &= X - \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(V)))). \end{aligned}$$

Thus, $F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(V))))$ and so $F^+(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$.

(2) ⇒ (1): Let $x \in X$ and $V$ be any $\star$-open set of $Y$ containing $F(x)$. By (2), we have $F^+(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$. Put $U = F^+(V)$. Then, $U$ is a strong $\beta$-$\mathscr{I}$-open set of $X$ containing $x$ such that $F(U) \subseteq V$. This shows that $F$ is upper $s\beta(\star)$-continuous.

**Theorem 4.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is lower $s\beta(\star)$-continuous;*

*(2) $F^-(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$ for every $\star$-open set $V$ of $Y$;*

*(3) $F^+(K)$ is strong $\beta$-$\mathscr{I}$-closed in $X$ for every $\star$-closed set $K$ of $Y$;*

*(4) $s\beta Cl_{\mathscr{I}}(F^+(B)) \subseteq F^+(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(5) $Int^\star(Cl(Int^\star(F^+(B)))) \subseteq F^+(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(6) $F(Int^\star(Cl(Int^\star(A)))) \subseteq Cl^\star(F(A))$ for every subset $A$ of $X$;*

*(7) $F(s\beta Cl_{\mathscr{I}}(A)) \subseteq Cl^\star(F(A))$ for every subset $A$ of $X$.*

*Proof.* It is shown similarly to the proof of Theorem 3 that the statements (1), (2), (3), (4) and (5) are equivalent. We shall prove only the following implications.

(5) ⇒ (6): Let $A$ be any subset of $X$. By (5), we have

$$\text{Int}^\star(\text{Cl}(\text{Int}^\star(F^+(F(A))))) \subseteq F^+(\text{Cl}^\star(F(A)))$$

and hence $F(\text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))) \subseteq \text{Cl}^\star(F(A))$.

(6) ⇒ (7): Let $A$ be any subset of $X$. By (6) and Lemma 2, we have

$$\begin{aligned} F(s\beta\text{Cl}_{\mathscr{I}}(A)) &= F(A \cup \text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))) \\ &= F(A) \cup F(\text{Int}^\star(\text{Cl}(\text{Int}^\star(A)))) \\ &\subseteq \text{Cl}^\star(F(A)). \end{aligned}$$

(7) ⇒ (3): Let $K$ be any $\star$-closed set of $Y$. By (7),

$$F(s\beta\text{Cl}_{\mathscr{I}}(F^+(K))) \subseteq \text{Cl}^\star(F(F^+(K))) \subseteq \text{Cl}^\star(K) = K.$$

Thus, $s\beta\text{Cl}_{\mathscr{I}}(F^+(K)) \subseteq F^+(K)$ and hence $F^+(K)$ is strong $\beta$-$\mathscr{I}$-closed in $X$.

**Corollary 2.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $f$ is $s\beta(\star)$-continuous;*

*(2) $f^{-1}(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$ for every $\star$-open set $V$ of $Y$;*

*(3) $f^{-1}(K)$ is strong $\beta$-$\mathscr{I}$-closed in $X$ for every $\star$-closed set $K$ of $Y$;*

*(4) $s\beta Cl_{\mathscr{I}}(f^{-1}(B)) \subseteq f^{-1}(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(5) $Int^\star(Cl(Int^\star(f^{-1}(B)))) \subseteq f^{-1}(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(6) $f(Int^\star(Cl(Int^\star(A)))) \subseteq Cl^\star(f(A))$ for every subset $A$ of $X$;*

*(7) $f(s\beta Cl_{\mathscr{I}}(A)) \subseteq Cl^\star(f(A))$ for every subset $A$ of $X$.*

## 4. Upper and lower almost $s\beta(\star)$-continuous multifunctions

We begin this section by introducing the notions of upper and lower almost $s\beta(\star)$-continuous multifunctions.

**Definition 3.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

*(1) upper almost $s\beta(\star)$-continuous at a point $x \in X$ if, for each $\star$-open set $V$ of $Y$ containing $F(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq Int^\star(Cl(V))$;*

*(2) lower almost $s\beta(\star)$-continuous at a point $x \in X$ if, for each $\star$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(z) \cap Int^\star(Cl(V)) \neq \emptyset$ for every $z \in U$;*

*(3) upper (resp. lower) almost $\beta(\star)$-continuous if $F$ has this property at each point of $X$.*

**Remark 1.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication holds:*

*upper $s\beta(\star)$-continuity $\Rightarrow$ upper almost $s\beta(\star)$-continuity.*

The converse of the implication is not true in general. We give an example for the implication as follows.

**Example 1.** *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, X\}$ and an ideal $\mathscr{I} = \{\emptyset\}$. Let $Y = \{a, b, c\}$ with a topology $\sigma = \{\emptyset, \{b\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset, \{b\}\}$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $F(1) = \{b\}$ and $F(2) = F(3) = \{a, c\}$. Then, $F$ is upper almost $s\beta(\star)$-continuous but $F$ is not upper $s\beta(\star)$-continuous, since $\{a, c\}$ is $\star$-open in $Y$ but $F^+(\{a, c\})$ is not strong $\beta$-$\mathscr{I}$-open in $X$.*

**Theorem 5.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is upper almost $s\beta(\star)$-continuous at $x \in X$ if and only if $x \in s\beta Int_{\mathscr{I}}(F^+(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set $V$ of $Y$ containing $F(x)$.*

*Proof.* Let $V$ be any $\star$-open set of $Y$ containing $F(x)$. Then, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \mathrm{Int}^\star(\mathrm{Cl}(V)) = s\mathrm{Cl}_{\mathscr{J}}(V)$; hence $U \subseteq F^+(s\mathrm{Cl}_{\mathscr{J}}(V))$. Since $U$ is strong $\beta$-$\mathscr{I}$-open, we have

$$x \in U \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(U))) \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}}(V))))).$$

Since $x \in F^+(V) \subseteq F^+(s\mathrm{Cl}_{\mathscr{J}}(V))$ and by Lemma 2,

$$x \in F^+(s\mathrm{Cl}_{\mathscr{J}}(V)) \cap \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(s\mathrm{Cl}_{\mathscr{J}}(V)))) = s\beta\mathrm{Int}_{\mathscr{I}}(F^+(s\mathrm{Cl}_{\mathscr{J}}(V))).$$

Conversely, let $V$ be any $\star$-open set of $Y$ containing $F(x)$. Then, we have

$$x \in s\beta\mathrm{Int}_{\mathscr{I}}(F^+(s\mathrm{Cl}_{\mathscr{J}}(V)))$$

and so there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^+(s\mathrm{Cl}_{\mathscr{J}}(V))$; hence $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}}(V) = \mathrm{Int}^\star(\mathrm{Cl}(V))$. This shows that $F$ is upper almost $\beta(\star)$-continuous at $x$.

**Theorem 6.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower almost $s\beta(\star)$-continuous at $x \in X$ if and only if $x \in s\beta Int_{\mathscr{I}}(F^-(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$.*

*Proof.* The proof is similar to that of Theorem 5.

**Definition 4.** *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called almost $s\beta(\star)$-continuous at a point $x \in X$ if, for each $\star$-open set $V$ of $Y$ containing $f(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq Int^\star(Cl(V))$. A function*

$$f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$$

*is called almost $\beta(\star)$-continuous if $f$ has this property at each point of $X$.*

**Corollary 3.** *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is almost $s\beta(\star)$-continuous at $x \in X$ if and only if $x \in s\beta Int_{\mathscr{I}}(f^{-1}(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set $V$ of $Y$ containing $f(x)$.*

Recall that a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$R^\star$-$\mathscr{I}$-open* [2] if $A = \mathrm{Int}^\star(\mathrm{Cl}(A))$. The complement of a $R^\star$-$\mathscr{I}$-open set is said to be *$R^\star$-$\mathscr{I}$-closed.*

**Theorem 7.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is upper almost $s\beta(\star)$-continuous;*

*(2) for each $x \in X$ and each $\star$-open set $V$ of $Y$ containing $F(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq sCl_{\mathscr{J}}(V)$;*

*(3) for each $x \in X$ and each $R^\star$-$\mathscr{J}$-open set $V$ of $Y$ containing $F(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$;*

*(4) $F^+(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$ for every $R^\star$-$\mathscr{J}$-open set $V$ of $Y$;*

*(5) $F^-(K)$ is strong $\beta$-$\mathscr{I}$-closed in $X$ for every $R^\star$-$\mathscr{J}$-closed set $K$ of $Y$;*

*(6) $F^+(V) \subseteq s\beta Int_{\mathscr{I}}(F^+(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set $V$ of $Y$;*

*(7) $s\beta Cl_{\mathscr{I}}(F^-(sInt_{\mathscr{J}}(K))) \subseteq F^-(K)$ for every $\star$-closed set $K$ of $Y$;*

*(8) $s\beta Cl_{\mathscr{I}}(F^-(Cl^\star(Int(K)))) \subseteq F^-(K)$ for every $\star$-closed set $K$ of $Y$;*

*(9) $s\beta Cl_{\mathscr{I}}(F^-(Cl^\star(Int(Cl^\star(B))))) \subseteq F^-(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(10) $Int^\star(Cl(\ Int^\star(F^-(Cl^\star(Int(K)))))) \subseteq F^-(K)$ for every $\star$-closed set $K$ of $Y$;*

*(11)* $Int^{\star}(Cl(Int^{\star}(F^{-}(sInt_{\mathscr{I}}(K))))) \subseteq F^{-}(K)$ *for every* $\star$*-closed set* $K$ *of* $Y$*;*

*(12)* $F^{+}(V) \subseteq Cl^{\star}(Int(Cl^{\star}(F^{+}(sCl_{\mathscr{I}}(V)))))$ *for every* $\star$*-open set* $V$ *of* $Y$*.*

*Proof.* (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3): The proofs are obvious.

(3) $\Rightarrow$ (4): Let $V$ be any $\star$-open set of $Y$ and $x \in F^{+}(V)$. Then, $F(x) \subseteq V$ and so there exists a strong $\beta$-$\mathscr{I}$-open set $U_x$ of $X$ containing $x$ such that $F(U_x) \subseteq V$. Thus, $x \in U_x \subseteq F^{+}(V)$ and hence $F^{+}(V) = \cup_{x \in F^{+}(V)} U_x$. This shows that $F^{+}(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$.

(4) $\Rightarrow$ (5): This follows from the fact that $F^{+}(Y - B) = Y - F^{-}(B)$ for every subset $B$ of $Y$.

(5) $\Rightarrow$ (6): Let $V$ be any $\star$-open set of $Y$ and $x \in F^{+}(V)$. Then, $F(x) \subseteq V \subseteq s\mathrm{Cl}_{\mathscr{I}}(V)$ and hence $x \in F^{+}(s\mathrm{Cl}_{\mathscr{I}}(V)) = X - F^{-}(Y - s\mathrm{Cl}_{\mathscr{I}}(V))$. Since $Y - s\mathrm{Cl}_{\mathscr{I}}(V)$ is $R^{\star}$-$\mathscr{I}$-closed, we have $F^{-}(Y - s\mathrm{Cl}_{\mathscr{I}}(V))$ is strong $\beta$-$\mathscr{I}$-closed in $X$. Thus, $F^{+}(s\mathrm{Cl}_{\mathscr{I}}(V))$ is a strong $\beta$-$\mathscr{I}$-open set of $X$ containing $x$ and so $x \in s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{I}}(V)))$. This shows that $F^{+}(V) \subseteq s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{I}}(V)))$.

(6) $\Rightarrow$ (7): Let $K$ be any $\star$-closed set of $Y$. Then, since $Y - K$ is $\star$-open and by (6),

$$\begin{aligned} X - F^{-}(K) &= F^{+}(Y - K) \\ &\subseteq s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{I}}(Y - K))) \\ &= s\beta\mathrm{Int}_{\mathscr{I}}(F^{+}(Y - s\mathrm{Int}_{\mathscr{I}}(K))) \\ &= s\beta\mathrm{Int}_{\mathscr{I}}(X - F^{-}(s\mathrm{Int}_{\mathscr{I}}(K))) \\ &= X - s\beta\mathrm{Cl}_{\mathscr{I}}(F^{-}(s\mathrm{Int}_{\mathscr{I}}(K))). \end{aligned}$$

Thus, $s\beta\mathrm{Cl}_{\mathscr{I}}(F^{-}(s\mathrm{Int}_{\mathscr{I}}(K))) \subseteq F^{-}(K)$.

(7) $\Rightarrow$ (8): The proof is obvious since $s\mathrm{Int}_{\mathscr{I}}(K) = \mathrm{Cl}^{\star}(\mathrm{Int}(K))$ for every $\star$-closed set $K$ of $Y$.

(8) $\Rightarrow$ (9): The proof is obvious.

(9) $\Rightarrow$ (10): By (9) and Lemma 2,

$$\begin{aligned} \mathrm{Int}^{\star}(\mathrm{Cl}(\mathrm{Int}^{\star}(F^{-}(\mathrm{Cl}^{\star}(\mathrm{Int}^{\star}(K)))))) &\subseteq s\beta\mathrm{Cl}_{\mathscr{I}}(F^{-}(\mathrm{Cl}^{\star}(\mathrm{Int}(K)))) \\ &\subseteq s\beta\mathrm{Cl}_{\mathscr{I}}(F^{-}(\mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(K))))) \\ &\subseteq F^{-}(\mathrm{Cl}^{\star}(K)) = F^{-}(K). \end{aligned}$$

(10) $\Rightarrow$ (11): The proof is obvious since $s\mathrm{Int}_{\mathscr{I}}(K) = \mathrm{Cl}^{\star}(\mathrm{Int}(K))$ for every $\star$-closed set $K$ of $Y$.

(11) $\Rightarrow$ (12): Let $V$ be any $\star$-open set of $Y$. Then, $Y - V$ is $\star$-closed in $Y$ and by (11),

$$\mathrm{Int}^{\star}(\mathrm{Cl}(\mathrm{Int}^{\star}(F^{-}(s\mathrm{Int}_{\mathscr{I}}(Y - V))))) \subseteq F^{-}(Y - V) = X - F^{+}(V).$$

Moreover, we have

$$\begin{aligned} \mathrm{Int}^{\star}(\mathrm{Cl}(\mathrm{Int}^{\star}(F^{-}(s\mathrm{Int}_{\mathscr{I}}(Y - V))))) &= \mathrm{Int}^{\star}(\mathrm{Cl}(\mathrm{Int}^{\star}(F^{-}(Y - s\mathrm{Cl}_{\mathscr{I}}(V))))) \\ &= \mathrm{Int}^{\star}(\mathrm{Cl}(\mathrm{Int}^{\star}(X - F^{+}(s\mathrm{Cl}_{\mathscr{I}}(V))))) \end{aligned}$$

$$= X - \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}}(V))))).$$

Thus, $F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}}(V)))))$.

(12) $\Rightarrow$ (1): Let $x$ be any point of $X$ and $V$ be any $\star$-open set of $Y$ containing $F(x)$. Then, we have $x \in F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}}(V)))))$ and hence

$$x \in s\beta\mathrm{Int}_{\mathscr{I}}(F^+(s\mathrm{Cl}_{\mathscr{J}}(V))).$$

Thus, $F$ is upper almost $s\beta(\star)$-continuous at $x$ by Theorem 5.

**Theorem 8.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is lower almost $s\beta(\star)$-continuous;*

*(2) for each $x \in X$ and each $\star$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^-(sCl_{\mathscr{J}}(V))$;*

*(3) for each $x \in X$ and each $R^\star$-$\mathscr{J}$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^-(V)$;*

*(4) $F^-(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$ for every $R^\star$-$\mathscr{J}$-open set $V$ of $Y$;*

*(5) $F^+(K)$ is strong $\beta$-$\mathscr{I}$-closed in $X$ for every $R^\star$-$\mathscr{J}$-closed set $K$ of $Y$;*

*(6) $F^-(V) \subseteq s\beta Int_{\mathscr{I}}(F^-(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set $V$ of $Y$;*

*(7) $s\beta Cl_{\mathscr{I}}(F^+(sInt_{\mathscr{J}}(K))) \subseteq F^+(K)$ for every $\star$-closed set $K$ of $Y$;*

*(8) $s\beta Cl_{\mathscr{I}}(F^+(Cl^\star(Int(K)))) \subseteq F^+(K)$ for every $\star$-closed set $K$ of $Y$;*

*(9) $s\beta Cl_{\mathscr{I}}(F^+(Cl^\star(Int(Cl^\star(B))))) \subseteq F^+(Cl^\star(B))$ for every subset $B$ of $Y$;*

*(10) $Int^\star(Cl(Int^\star(F^+(Cl^\star(Int(K)))))) \subseteq F^+(K)$ for every $\star$-closed set $K$ of $Y$;*

*(11) $Int^\star(Cl(Int^\star(F^+(sInt_{\mathscr{J}}(K))))) \subseteq F^+(K)$ for every $\star$-closed set $K$ of $Y$;*

*(12) $F^-(V) \subseteq Cl^\star(Int(Cl^\star(F^-(sCl_{\mathscr{J}}(V)))))$ for every $\star$-open set $V$ of $Y$.*

*Proof.* The proof is similar to that of Theorem 7.

**Corollary 4.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $f$ is almost $s\beta(\star)$-continuous;*

*(2) for each $x \in X$ and each $\star$-open set $V$ of $Y$ containing $f(x)$, there exists a strong $\beta$-$\mathscr{I}$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq sCl_{\mathscr{J}}(V)$;*

(3) *for each* $x \in X$ *and each* $R^\star$-$\mathscr{J}$*-open set* $V$ *of* $Y$ *containing* $f(x)$*, there exists a strong* $\beta$-$\mathscr{I}$*-open set* $U$ *of* $X$ *containing* $x$ *such that* $f(U) \subseteq V$;

(4) $f^{-1}(V)$ *is strong* $\beta$-$\mathscr{I}$*-open in* $X$ *for every* $R^\star$-$\mathscr{J}$*-open set* $V$ *of* $Y$;

(5) $f^{-1}(K)$ *is strong* $\beta$-$\mathscr{I}$*-closed in* $X$ *for every* $R^\star$-$\mathscr{J}$*-closed set* $K$ *of* $Y$;

(6) $f^{-1}(V) \subseteq s\beta Int_{\mathscr{I}}(f^{-1}(sCl_{\mathscr{J}}(V)))$ *for every* $\star$*-open set* $V$ *of* $Y$;

(7) $s\beta Cl_{\mathscr{I}}(f^{-1}(sInt_{\mathscr{J}}(K))) \subseteq f^{-1}(K)$ *for every* $\star$*-closed set* $K$ *of* $Y$;

(8) $s\beta Cl_{\mathscr{I}}(f^{-1}(Cl^\star(Int(K)))) \subseteq f^{-1}(K)$ *for every* $\star$*-closed set* $K$ *of* $Y$;

(9) $s\beta Cl_{\mathscr{I}}(f^{-1}(Cl^\star(Int(Cl^\star(B))))) \subseteq f^{-1}(Cl^\star(B))$ *for every subset* $B$ *of* $Y$;

(10) $Int^\star(Cl(\, Int^\star(f^{-1}(Cl^\star(Int(K)))))) \subseteq f^{-1}(K)$ *for every* $\star$*-closed set* $K$ *of* $Y$;

(11) $Int^\star(Cl(\, Int^\star(f^{-1}(sInt_{\mathscr{J}}(K))))) \subseteq f^{-1}(K)$ *for every* $\star$*-closed set* $K$ *of* $Y$;

(12) $f^{-1}(V) \subseteq Cl^\star(\, Int(Cl^\star(f^{-1}(sCl_{\mathscr{J}}(V)))))$ *for every* $\star$*-open set* $V$ *of* $Y$.

**Theorem 9.** *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

(1) $F$ *is upper almost* $s\beta(\star)$*-continuous;*

(2) $s\beta Cl_{\mathscr{I}}(F^-(V)) \subseteq F^-(Cl^\star(V))$ *for every strong* $\beta$-$\mathscr{J}$*-open set* $V$ *of* $Y$;

(3) $s\beta Cl_{\mathscr{I}}(F^-(V)) \subseteq F^-(Cl^\star(V))$ *for every semi-*$\mathscr{J}$*-open set* $V$ *of* $Y$;

(4) $F^+(V) \subseteq s\beta Int_{\mathscr{I}}(F^+(Int^\star(Cl(V))))$ *for every* $pre^\star_{\mathscr{J}}$*-open set* $V$ *of* $Y$.

*Proof.* (1) $\Rightarrow$ (2): Let $V$ be any strong $\beta$-$\mathscr{J}$-open set of $Y$. Since $\mathrm{Cl}^\star(V)$ is $R^\star$-$\mathscr{J}$-closed, by Theorem 7, $F^-(\mathrm{Cl}^\star(V))$ is strong $\beta$-$\mathscr{I}$-closed in $X$ and hence

$$s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(V)) \subseteq F^-(\mathrm{Cl}^\star(V)).$$

(2) $\Rightarrow$ (3): This is obvious since every semi-$\mathscr{J}$-open set is strong $\beta$-$\mathscr{J}$-open.

(3) $\Rightarrow$ (4): Let $V$ be any $\mathrm{pre}^\star_{\mathscr{J}}$-open set of $Y$. Then, we have $V \subseteq \mathrm{Int}^\star(\mathrm{Cl}(V))$ and $Y - V \supseteq \mathrm{Cl}^\star(\mathrm{Int}(Y - V))$. Since $\mathrm{Cl}^\star(\mathrm{Int}(Y - V))$ is semi-$\mathscr{J}$-open in $Y$ and by (3),

$$\begin{aligned}
X - F^+(V) &= F^-(Y - V) \\
&\supseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}(Y - V))) \\
&\supseteq s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Cl}^\star(\mathrm{Int}(Y - V)))) \\
&= s\beta\mathrm{Cl}_{\mathscr{I}}(F^-(Y - \mathrm{Int}^\star(\mathrm{Cl}(V)))) \\
&= s\beta\mathrm{Cl}_{\mathscr{I}}(X - F^+(\mathrm{Int}^\star(\mathrm{Cl}(V)))) \\
&= X - s\beta\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Int}^\star(\mathrm{Cl}(V)))).
\end{aligned}$$

Thus, $F^+(V) \subseteq s\beta\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Int}^\star(\mathrm{Cl}(V))))$.

(4) $\Rightarrow$ (1): Let $V$ be any $R^\star$-$\mathscr{J}$-open set of $Y$. Then, $V$ is $\mathrm{pre}^\star_{\mathscr{J}}$-open in $Y$ and by (4), $F^+(V) \subseteq s\beta\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Int}^\star(\mathrm{Cl}(V)))) = s\beta\mathrm{Int}_{\mathscr{I}}(F^+(V))$ and hence $F^+(V)$ is strong $\beta$-$\mathscr{I}$-open in $X$. It follows from Theorem 7 that $F$ is upper almost $s\beta(\star)$-continuous.

**Theorem 10.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $F$ is lower almost $s\beta(\star)$-continuous;*

*(2) $s\beta Cl_{\mathscr{I}}(F^-(V)) \subseteq F^-(Cl^\star(V))$ for every strong $\beta$-$\mathscr{J}$-open set $V$ of $Y$;*

*(3) $s\beta Cl_{\mathscr{I}}(F^-(V)) \subseteq F^-(Cl^\star(V))$ for every semi-$\mathscr{J}$-open set $V$ of $Y$;*

*(4) $F^+(V) \subseteq s\beta Int_{\mathscr{I}}(F^+(Int^\star(Cl(V))))$ for every $pre^\star_{\mathscr{J}}$-open set $V$ of $Y$.*

*Proof.* The proof is similar to that of Theorem 9.

**Corollary 5.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

*(1) $f$ is almost $s\beta(\star)$-continuous;*

*(2) $s\beta Cl_{\mathscr{I}}(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every strong $\beta$-$\mathscr{J}$-open set $V$ of $Y$;*

*(3) $s\beta Cl_{\mathscr{I}}(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every semi-$\mathscr{J}$-open set $V$ of $Y$;*

*(4) $f^{-1}(V) \subseteq s\beta Int_{\mathscr{I}}(f^{-1}(Int^\star(Cl(V))))$ for every $pre^\star_{\mathscr{J}}$-open set $V$ of $Y$.*

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