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# $\delta p(\Lambda, p)$-open sets in topological spaces

Chawalit Boonpok[1], Montri Thongmoon[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

**Abstract.** This paper deals with the notion of $\delta p(\Lambda, p)$-open sets. Some properties of $\delta p(\Lambda, p)$-open sets and $\delta p(\Lambda, p)$-closed sets are investigated. Moreover, several characterizations of $\delta p(\Lambda, p)$-$\mathscr{D}_1$ spaces and $\delta p(\Lambda, p)$-$R_0$ spaces are established.

**2020 Mathematics Subject Classifications**: 54A05, 54D10

**Key Words and Phrases**: $\delta p(\Lambda, p)$-open set, $\delta p(\Lambda, p)$-$\mathscr{D}_1$ space, $\delta p(\Lambda, p)$-$R_0$ space

## 1. Introduction

The concept of $\delta$-open sets was first introduced by Veličko [10]. In 1982, Mashhour et al. [7] introduced and investigated the notion of preopen sets which is weaker than the notion of open sets in topological spaces. Raychaudhuri and Mukherjee [8] introduced and studied the notions of $\delta$-preopen sets and $\delta$-closure. The class of $\delta$-preopen sets is larger than that of preopen sets. In 1996, Raychaudhuri and Mukherjee [9] introduced and investigated the concept of $\delta_p$-closed spaces. Caldas et al. [3] introduced some weak separation axioms by utilizing the notions of $\delta$-preopen sets and the $\delta$-preclosure operator. Furthermore, Caldas et al. [3] showed that $(\delta, p)$-$T_1$ spaces, $(\delta, p)$-$R_0$ spaces and $(\delta, p)$-symmetric spaces are all equivalent. In 2003, Caldas et al. [5] investigated some weak separation axioms by utilizing $\delta$-semiopen sets and the $\delta$-semiclosure operator. In 2005, Caldas et al. [4] investigated the notion of $\delta$-$\Lambda_s$-semiclosed sets which is defined as the intersection of a $\delta$-$\Lambda_s$-set and a $\delta$-semiclosed set. In [2], the present authors introduced the notions of $(\Lambda, p)$-open sets and $(\Lambda, p)$-closed sets which are defined by utilizing the notions of $\Lambda_p$-sets and preclosed sets. Quite recently, Boonpok and Viriyapong [1] investigated some characterizations of $(\Lambda, s)$-$R_0$ topological spaces. In this paper, we introduced the concept of $\delta p(\Lambda, p)$-open sets. Moreover, some properties of $\delta p(\Lambda, p)$-open sets and $\delta p(\Lambda, p)$-closed sets are discussed. In particular, several characterizations of $\delta p(\Lambda, p)$-$\mathscr{D}_1$ spaces and $\delta p(\Lambda, p)$-$R_0$ spaces are investigated.

*Corresponding author.
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*Email addresses:* chawalit.b@msu.ac.th (C. Boonpok), montri.t@msu.ac.th (M. Thongmoon)

 

## 2. Preliminaries

Throughout the present paper, spaces $(X,\tau)$ and $(Y,\sigma)$ (or simply $X$ and $Y$) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a topological space $(X,\tau)$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, represent the closure and the interior of $A$, respectively. A subset $A$ of a topological space $(X,\tau)$ is said to be *preopen* [7] if $A\subseteq \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a preopen set is called *preclosed*. The family of all preopen sets of a topological space $(X,\tau)$ is denoted by $PO(X,\tau)$. A subset $\Lambda_p(A)$ [6] is defined as follows: $\Lambda_p(A)=\cap\{U \mid A\subseteq U, U\in PO(X,\tau)\}$. A subset $A$ of a topological space $(X,\tau)$ is called a $\Lambda_p$*-set* [1] (*pre-*$\Lambda$*-set* [6]) if $A=\Lambda_p(A)$. A subset $A$ of a topological space $(X,\tau)$ is called $(\Lambda,p)$*-closed* [1] if $A=T\cap C$, where $T$ is a $\Lambda_p$-set and $C$ is a preclosed set. The complement of a $(\Lambda,p)$-closed set is called $(\Lambda,p)$*-open*. The family of all $(\Lambda,p)$-open (resp. $(\Lambda,p)$-closed) sets in a topological space $(X,\tau)$ is denoted by $\Lambda_pO(X,\tau)$ (resp. $\Lambda_pC(X,\tau)$). Let $A$ be a subset of a topological space $(X,\tau)$. A point $x\in X$ is called a $(\Lambda,p)$*-cluster point* [1] of $A$ if $A\cap U\neq\emptyset$ for every $(\Lambda,p)$-open set $U$ of $X$ containing $x$. The set of all $(\Lambda,p)$-cluster points of $A$ is called the $(\Lambda,p)$*-closure* [1] of $A$ and is denoted by $A^{(\Lambda,p)}$. The union of all $(\Lambda,p)$-open sets contained in $A$ is called the $(\Lambda,p)$*-interior* [1] of $A$ and is denoted by $A_{(\Lambda,p)}$. A subset $A$ of a topological space $(X,\tau)$ is said to be $p(\Lambda,p)$*-open* [1] if $A\subseteq [A^{(\Lambda,p)}]_{(\Lambda,p)}$. The complement of a $p(\Lambda,p)$-open set is said to be $p(\Lambda,p)$*-closed*.

## 3. $\delta p(\Lambda,p)$-open sets

In this section, we introduced the concept of $\delta p(\Lambda,p)$-open sets. Moreover, some properties of $\delta p(\Lambda,p)$-open sets and $\delta p(\Lambda,p)$-closed sets are investigated. Furthermore, several characterizations of $\delta p(\Lambda,p)$-$\mathscr{D}_1$ spaces and $\delta p(\Lambda,p)$-$R_0$ spaces are discussed.

**Definition 1.** *Let $A$ be a subset of a topological space $(X,\tau)$. A point $x$ of $X$ is called a $\delta(\Lambda,p)$-cluster point of $A$ if $A\cap[V^{(\Lambda,p)}]_{(\Lambda,p)}\neq\emptyset$ for every $(\Lambda,p)$-open set $V$ of $X$ containing $x$. The set of all $\delta(\Lambda,p)$-cluster points of $A$ is called the $\delta(\Lambda,p)$-closure of $A$ and is denoted by $A^{\delta(\Lambda,p)}$. If $A=A^{\delta(\Lambda,p)}$, then $A$ is said to be $\delta(\Lambda,p)$-closed. The complement of a $\delta(\Lambda,p)$-closed set is said to be $\delta(\Lambda,p)$-open. The union of all $\delta(\Lambda,p)$-open sets contained in $A$ is called the $\delta(\Lambda,p)$-interior of $A$ and is denoted by $A_{\delta(\Lambda,p)}$.*

**Definition 2.** *A subset $A$ of a topological space $(X,\tau)$ is said to be $\delta p(\Lambda,p)$-open if $A\subseteq [A^{(\Lambda,p)}]_{\delta(\Lambda,p)}$. The complement of a $\delta p(\Lambda,p)$-open set is said to be $\delta p(\Lambda,p)$-closed.*

The family of all $\delta p(\Lambda,p)$-open (resp. $\delta p(\Lambda,p)$-closed) sets in a topological space $(X,\tau)$ is denoted by $\delta p(\Lambda,p)O(X,\tau)$ (resp. $\delta p(\Lambda,p)C(X,\tau)$). Let $A$ be a subset of a topological space $(X,\tau)$. The intersection of all $\delta p(\Lambda,p)$-closed sets containing $A$ is called the $\delta p(\Lambda,p)$-*closure* of $A$ and is denoted by $A^{\delta p(\Lambda,p)}$.

**Lemma 1.** *For the $\delta p(\Lambda,p)$-closure of subsets $A$, $B$ in a topological space $(X,\tau)$, the following properties hold:*

*(1) If* $A \subseteq B$*, then* $A^{\delta p(\Lambda,p)} \subseteq B^{\delta p(\Lambda,p)}$.

*(2)* $A$ *is* $\delta p(\Lambda,p)$*-closed in* $(X,\tau)$ *if and only if* $A = A^{\delta p(\Lambda,p)}$.

*(3)* $A^{\delta p(\Lambda,p)}$ *is* $\delta p(\Lambda,p)$*-closed, that is,* $A^{\delta p(\Lambda,p)} = [A^{\delta p(\Lambda,p)}]^{\delta p(\Lambda,p)}$.

*(4)* $x \in A^{\delta p(\Lambda,p)}$ *if and only if* $A \cap V \neq \emptyset$ *for every* $V \in \delta p(\Lambda,p)O(X,\tau)$ *containing* $x$.

**Lemma 2.** *For a family* $\{A_\gamma \mid \gamma \in \nabla\}$ *of a topological space* $(X,\tau)$*, the following properties hold:*

*(1)* $[\cap\{A_\gamma \mid \gamma \in \nabla\}]^{\delta p(\Lambda,p)} \subseteq \cap\{A_\gamma^{\delta p(\Lambda,p)} \mid \gamma \in \nabla\}$.

*(2)* $[\cup\{A_\gamma \mid \gamma \in \nabla\}]^{\delta p(\Lambda,p)} \supseteq \cup\{A_\gamma^{\delta p(\Lambda,p)} \mid \gamma \in \nabla\}$.

**Definition 3.** *A subset* $A$ *of a topological space* $(X,\tau)$ *is called a* $\delta p(\Lambda,p)\mathscr{D}$*-set if there exist* $\delta p(\Lambda,p)$*-open sets* $U$ *and* $V$ *such that* $U \neq X$ *and* $A = U - V$.

**Definition 4.** *A topological space* $(X,\tau)$ *is said to be:*

*(i)* $\delta p(\Lambda,p)$*-*$T_1$ *if for any distinct pair of points* $x$ *and* $y$ *of* $X$*, there exist a* $\delta p(\Lambda,p)$*-open set* $U$ *of* $X$ *containing* $x$ *but not* $y$ *and a* $\delta p(\Lambda,p)$*-open set* $V$ *of* $X$ *containing* $y$ *but not* $x$*;*

*(ii)* $\delta p(\Lambda,p)$*-*$\mathscr{D}_1$ *if for any distinct pair of points* $x$ *and* $y$ *of* $X$*, there exist a* $\delta p(\Lambda,p)\mathscr{D}$*-set* $U$ *of* $X$ *containing* $x$ *but not* $y$ *and a* $\delta p(\Lambda,p)\mathscr{D}$*-set* $V$ *of* $X$ *containing* $y$ *but not* $x$.

**Definition 5.** *A subset* $N$ *of a topological space* $(X,\tau)$ *is called a* $\delta p(\Lambda,p)$*-neighborhood of a point* $x \in X$ *if there exists a* $\delta p(\Lambda,p)$*-open set* $U$ *such that* $x \in U \subseteq N$.

**Definition 6.** *Let* $(X,\tau)$ *be a topological space. A point* $x \in X$ *which has only* $X$ *as the* $\delta p(\Lambda,p)$*-neighbourhood is called a* $\delta p(\Lambda,p)$*-neat point.*

**Lemma 3.** *Let* $(X,\tau)$ *be a topological space. For each point* $x \in X$*,* $\{x\}$ *is* $p(\Lambda,p)$*-open or* $p(\Lambda,p)$*-closed.*

**Theorem 1.** *For a topological space* $(X,\tau)$*, the following properties are equivalent:*

*(1)* $(X,\tau)$ *is* $\delta p(\Lambda,p)$*-*$\mathscr{D}_1$*;*

*(2)* $(X,\tau)$ *has no* $\delta p(\Lambda,p)$*-neat point.*

*Proof.* (1) $\Rightarrow$ (2): Since $(X,\tau)$ is $\delta p(\Lambda,p)$-$\mathscr{D}_1$, so each point $x$ of $X$ is contained in a $\delta p(\Lambda,p)\mathscr{D}$-set $G = U - V$ and thus in $U$, where $U$ and $V$ are $\delta p(\Lambda,p)$-open sets. By definition $U \neq X$. This implies that $x$ is not a $\delta p(\Lambda,p)$-neat point.

(2) $\Rightarrow$ (1): By Lemma 3 for each distinct pair of points $x, y \in X$, at least one of them, $x$(say) has a $\delta p(\Lambda,p)$-neighborhood $U$ containing $x$ and not $y$. Thus, $U$ which is different from $X$ is a $\delta p(\Lambda,p)\mathscr{D}$-set. If $X$ has no $\delta p(\Lambda,p)$-neat point, then $y$ is not a $\delta p(\Lambda,p)$-neat point. This means that there exists a $\delta p(\Lambda,p)$-neighborhood $V$ of $y$ such that $V \neq X$. Thus, $y \in V - U$ but not $y$ and $V - U$ is a $\delta p(\Lambda,p)\mathscr{D}$-set. This shows that $(X,\tau)$ is $\delta p(\Lambda,p)$-$\mathscr{D}_1$.

**Definition 7.** *A function $f : (X, \tau) \to (Y, \sigma)$ is called $\delta p(\Lambda, p)$-continuous if, for each $x \in X$ and each $\delta p(\Lambda, p)$-open set $V$ of $Y$ containing $f(x)$, there exists a $\delta p(\Lambda, p)$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$.*

**Lemma 4.** *A function $f : (X, \tau) \to (Y, \sigma)$ is $\delta p(\Lambda, p)$-continuous if and only if $f^{-1}(V)$ is $\delta p(\Lambda, p)$-open in $X$ for every $\delta p(\Lambda, p)$-open set $V$ of $Y$.*

**Theorem 2.** *If $f : (X, \tau) \to (Y, \sigma)$ is a $\delta p(\Lambda, p)$-continuous surjective function and $B$ is a $\delta p(\Lambda, p)\mathscr{D}$-set in $Y$, then $f^{-1}(B)$ is a $\delta p(\Lambda, p)\mathscr{D}$-set in $X$.*

*Proof.* Let $B$ be a $\delta p(\Lambda, p)\mathscr{D}$-set in $Y$. Then, there exist $\delta p(\Lambda, p)$-open sets $U$ and $V$ in $Y$ such that $B = U - V$ and $U \neq Y$. By the $\delta p(\Lambda, p)$-continuity of $f$, $f^{-1}(U)$ and $f^{-1}(V)$ are $\delta p(\Lambda, p)$-open in $X$. Since $U \neq Y$, we have $f^{-1}(U) \neq X$. Thus, $f^{-1}(B) = f^{-1}(U) - f^{-1}(V)$ is a $\delta p(\Lambda, p)\mathscr{D}$-set.

**Theorem 3.** *If $(Y, \sigma)$ is a $\delta p(\Lambda, p)$-$\mathscr{D}_1$ space and $f : (X, \tau) \to (Y, \sigma)$ is a $\delta p(\Lambda, p)$-continuous bijection, then $(X, \tau)$ is $\delta p(\Lambda, p)$-$\mathscr{D}_1$.*

*Proof.* Suppose that $(Y, \sigma)$ is a $\delta p(\Lambda, p)$-$\mathscr{D}_1$ space. Let $x$ and $y$ be any pair of distinct points in $X$. Since $f$ is injective and $(Y, \sigma)$ is $\delta p(\Lambda, p)$-$\mathscr{D}_1$, there exist $\delta p(\Lambda, p)\mathscr{D}$-sets $U$ and $V$ of $Y$ containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin U$ and $f(x) \notin V$. By Theorem 2, $f^{-1}(U)$ and $f^{-1}(V)$ are $\delta p(\Lambda, p)\mathscr{D}$-sets in $X$ containing $x$ and $y$, respectively, such that $y \notin f^{-1}(U)$ and $x \notin f^{-1}(V)$. This shows that $(X, \tau)$ is $\delta p(\Lambda, p)$-$\mathscr{D}_1$.

**Definition 8.** *A topological space $(X, \tau)$ is called $\delta p(\Lambda, p)$-$R_0$ if for each $\delta p(\Lambda, p)$-open set $U$ and each $x \in U$, $\{x\}^{\delta p(\Lambda,p)} \subseteq U$.*

**Theorem 4.** *For a topological space $(X, \tau)$, the following properties are equivalent:*

*(1) $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$.*

*(2) For each $\delta p(\Lambda, p)$-closed set $F$ and each $x \in X - F$, there exists $U \in \delta p(\Lambda, p)O(X, \tau)$ such that $F \subseteq U$ and $x \notin U$.*

*(3) For each $\delta p(\Lambda, p)$-closed set $F$ and each $x \in X - F$, $F \cap \{x\}^{\delta p(\Lambda,p)} = \emptyset$.*

*(4) For any distinct points $x, y$ in $X$, $\{x\}^{\delta p(\Lambda,p)} = \{y\}^{\delta p(\Lambda,p)}$ or $\{x\}^{\delta p(\Lambda,p)} \cap \{y\}^{\delta p(\Lambda,p)} = \emptyset$.*

*Proof.* (1) $\Rightarrow$ (2): Let $F$ be a $\delta p(\Lambda, p)$-closed set and $x \in X - F$. Since $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$, we have $\{x\}^{\delta p(\Lambda,p)} \subseteq X - F$. Put $U = X - \{x\}^{\delta p(\Lambda,p)}$. Thus, by Lemma 1, $U \in \delta p(\Lambda, p)O(X, \tau)$, $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3): Let $F$ be a $\delta p(\Lambda, p)$-closed set and $x \in X - F$. By (2), there exists

$$U \in \delta p(\Lambda, p)O(X, \tau)$$

such that $F \subseteq U$ and $x \notin U$. Since $U \in \delta p(\Lambda, p)O(X, \tau)$, $U \cap \{x\}^{\delta p(\Lambda,p)} = \emptyset$ and hence $F \cap \{x\}^{\delta p(\Lambda,p)} = \emptyset$.

(3) $\Rightarrow$ (4): Let $x$ and $y$ be distinct points of $X$. Suppose that $\{x\}^{\delta p(\Lambda,p)}\cap\{y\}^{\delta p(\Lambda,p)}\neq\emptyset$. By (3), we have $x\in\{y\}^{\delta p(\Lambda,p)}$ and $y\in\{x\}^{\delta p(\Lambda,p)}$. By Lemma 1,

$$\{x\}^{\delta p(\Lambda,p)}\subseteq\{y\}^{\delta p(\Lambda,p)}\subseteq\{x\}^{\delta p(\Lambda,p)}$$

and hence $\{x\}^{\delta p(\Lambda,p)}=\{y\}^{\delta p(\Lambda,p)}$.

(4) $\Rightarrow$ (1): Let $V\in\delta p(\Lambda,p)O(X,\tau)$ and $x\in V$. For each $y\notin V$,

$$V\cap\{y\}^{\delta p(\Lambda,p)}=\emptyset$$

and hence $x\notin\{y\}^{\delta p(\Lambda,p)}$. Thus, $\{x\}^{\delta p(\Lambda,p)}\neq\{y\}^{\delta p(\Lambda,p)}$. By (4), for each $y\notin V$, $\{x\}^{\delta p(\Lambda,p)}\cap\{y\}^{\delta p(\Lambda,p)}=\emptyset$. Since $X-V$ is $\delta p(\Lambda,p)$-closed, $y\in\{y\}^{\delta p(\Lambda,p)}\subseteq X-V$ and $\cup_{y\in X-V}\{y\}^{\delta p(\Lambda,p)}=X-V$. Thus,

$$\begin{aligned}\{x\}^{\delta p(\Lambda,p)}\cap(X-V)&=\{x\}^{\delta p(\Lambda,p)}\cap[\cup_{y\in X-V}\{y\}^{\delta p(\Lambda,p)}]\\&=\cup_{y\in X-V}[\{x\}^{\delta p(\Lambda,p)}\cap\{y\}^{\delta p(\Lambda,p)}]\\&=\emptyset\end{aligned}$$

and hence $\{x\}^{\delta p(\Lambda,p)}\subseteq V$. This shows that $(X,\tau)$ is $\delta p(\Lambda,p)$-$R_0$.

**Corollary 1.** *A topological space $(X,\tau)$ is $\delta p(\Lambda,p)$-$R_0$ if and only if, for any points $x$ and $y$ in $X$, $\{x\}^{\delta p(\Lambda,p)}\neq\{y\}^{\delta p(\Lambda,p)}$ implies $\{x\}^{\delta p(\Lambda,p)}\cap\{y\}^{\delta p(\Lambda,p)}=\emptyset$.*

*Proof.* This is obvious by Theorem 4.

Conversely, let $U\in\delta p(\Lambda,p)O(X,\tau)$ and $x\in U$. If $y\notin U$, then $U\cap\{y\}^{\delta p(\Lambda,p)}=\emptyset$. Thus, $x\notin\{y\}^{\delta p(\Lambda,p)}$ and $\{x\}^{\delta p(\Lambda,p)}\neq\{y\}^{\delta p(\Lambda,p)}$. By the hypothesis, $\{x\}^{\delta p(\Lambda,p)}\cap\{y\}^{\delta p(\Lambda,p)}=\emptyset$ and hence $y\notin\{x\}^{\delta p(\Lambda,p)}$. This shows that $\{x\}^{\delta p(\Lambda,p)}\subseteq U$. Thus, $(X,\tau)$ is $\delta p(\Lambda,p)$-$R_0$.

**Definition 9.** *Let $A$ be a subset of a topological space $(X,\tau)$. The $\delta p(\Lambda,p)$-kernel of $A$, denoted by $\delta p(\Lambda,p)Ker(A)$, is defined to be the set*

$$\delta p(\Lambda,p)Ker(A)=\cap\{U\in\delta p(\Lambda,p)O(X,\tau)\mid A\subseteq U\}.$$

**Lemma 5.** *For subsets $A,B$ of a topological space $(X,\tau)$, the following properties hold:*

*(1) $A\subseteq\delta p(\Lambda,p)Ker(A)$.*

*(2) If $A\subseteq B$, then $\delta p(\Lambda,p)Ker(A)\subseteq\delta p(\Lambda,p)Ker(B)$.*

*(3) $\delta p(\Lambda,p)Ker(\delta p(\Lambda,sp)Ker(A))=\delta p(\Lambda,p)Ker(A)$.*

*(4) If $A$ is $\delta p(\Lambda,p)$-open, $\delta p(\Lambda,p)Ker(A)=A$.*

**Theorem 5.** *For any points $x$ and $y$ in a topological space $(X,\tau)$, the following properties are equivalent:*

*(1)* $\delta p(\Lambda,p)Ker(\{x\}) \neq \delta p(\Lambda,p)Ker(\{y\})$.

*(2)* $\{x\}^{\delta p(\Lambda,p)} \neq \{y\}^{\delta p(\Lambda,p)}$.

*Proof.* (1) $\Rightarrow$ (2): Suppose that $\delta p(\Lambda,p)Ker(\{x\}) \neq \delta p(\Lambda,p)Ker(\{y\})$. Then, there exists a point $z \in X$ such that $z \in \delta p(\Lambda,p)Ker(\{x\})$ and $z \notin \delta p(\Lambda,p)Ker(\{y\})$ or

$$z \in \delta p(\Lambda,p)Ker(\{y\})$$

and $z \notin \delta p(\Lambda,p)Ker(\{x\})$. We prove only the first case being the second analogous. From $z \in \delta p(\Lambda,p)Ker(\{x\})$ it follows that $\{x\} \cap \{z\}^{\delta p(\Lambda,p)} \neq \emptyset$ which implies

$$x \in \{z\}^{\delta p(\Lambda,p)}.$$

By $z \notin \delta p(\Lambda,p)Ker(\{y\})$, we have $\{y\} \cap \{z\}^{\delta p(\Lambda,p)} = \emptyset$. Since $x \in \{z\}^{\delta p(\Lambda,p)}$, $\{x\}^{\delta p(\Lambda,p)} \subseteq \{z\}^{\delta p(\Lambda,p)}$ and $\{y\} \cap \{x\}^{\delta p(\Lambda,p)} = \emptyset$. Therefore, $\{x\}^{\delta p(\Lambda,p)} \neq \{y\}^{\delta p(\Lambda,p)}$. Thus,

$$\delta p(\Lambda,p)Ker(\{x\}) \neq \delta p(\Lambda,p)Ker(\{y\})$$

implies that $\{x\}^{\delta p(\Lambda,p)} \neq \{y\}^{\delta p(\Lambda,p)}$.

(2) $\Rightarrow$ (1): Suppose that $\{x\}^{\delta p(\Lambda,p)} \neq \{y\}^{\delta p(\Lambda,p)}$. There exists a point $z \in X$ such that $z \in \{x\}^{\delta p(\Lambda,p)}$ and $z \notin \{y\}^{\delta p(\Lambda,p)}$ or $z \in \{y\}^{\delta p(\Lambda,p)}$ and $z \notin \{x\}^{\delta p(\Lambda,p)}$. We prove only the first case being the second analogous. It follows that there exists a $\delta p(\Lambda,p)$-open set containing $z$ and therefore $x$ but not $y$, namely, $y \notin \delta p(\Lambda,p)Ker(\{x\})$ and thus $\delta p(\Lambda,p)Ker(\{x\}) \neq \delta p(\Lambda,p)Ker(\{y\})$.

**Lemma 6.** *Let $(X,\tau)$ be a topological space and $x, y \in X$. Then, the following properties hold:*

*(1)* $y \in \delta p(\Lambda,p)Ker(\{x\})$ *if and only if* $x \in \{y\}^{\delta p(\Lambda,p)}$.

*(2)* $\delta p(\Lambda,p)Ker(\{x\}) = \delta p(\Lambda,p)Ker(\{y\})$ *if and only if* $\{x\}^{\delta p(\Lambda,p)} = \{y\}^{\delta p(\Lambda,p)}$.

*Proof.* (1) Let $x \notin \{y\}^{\delta p(\Lambda,p)}$. Then, there exists $U \in \delta p(\Lambda,p)O(X,\tau)$ such that $x \in U$ and $y \notin U$. Thus, $y \notin \delta p(\Lambda,p)Ker(\{x\})$. The converse is similarly shown.

(2) Suppose that $\delta p(\Lambda,p)Ker(\{x\}) = \delta p(\Lambda,p)Ker(\{y\})$ for any $x, y \in X$. Since

$$x \in \delta p(\Lambda,p)Ker(\{x\}),$$

$x \in \delta p(\Lambda,p)Ker(\{y\})$, by (1), $y \in \{x\}^{\delta p(\Lambda,p)}$. By Lemma 1, $\{y\}^{\delta p(\Lambda,p)} \subseteq \{x\}^{\delta p(\Lambda,p)}$. Similarly, we have $\{x\}^{\delta p(\Lambda,p)} \subseteq \{y\}^{\delta p(\Lambda,p)}$ and hence $\{x\}^{\delta p(\Lambda,p)} = \{y\}^{\delta p(\Lambda,p)}$.

Conversely, suppose that $\{x\}^{\delta p(\Lambda,p)} = \{y\}^{\delta p(\Lambda,p)}$. Since $x \in \{x\}^{\delta p(\Lambda,p)}$, we have

$$x \in \{y\}^{\delta p(\Lambda,p)}$$

and by (1), $y \in \delta p(\Lambda,p)Ker(\{x\})$. By Lemma 5,

$$\delta p(\Lambda,p)Ker(\{y\}) \subseteq \delta p(\Lambda,p)Ker(\delta p(\Lambda,p)Ker(\{x\})) = \delta p(\Lambda,p)Ker(\{x\}).$$

Similarly, we have $\delta p(\Lambda,p)Ker(\{x\}) \subseteq \delta p(\Lambda,p)Ker(\{y\})$ and hence

$$\delta p(\Lambda,p)Ker(\{x\}) = \delta p(\Lambda,p)Ker(\{y\}).$$

**Theorem 6.** *A topological space* $(X, \tau)$ *is* $\delta p(\Lambda, p)$-$R_0$ *if and only if for each points* $x$ *and* $y$ *in* $X$, $\delta p(\Lambda, p)Ker(\{x\}) \neq \delta p(\Lambda, p)Ker(\{y\})$ *implies*

$$\delta p(\Lambda, p)Ker(\{x\}) \cap \delta p(\Lambda, p)Ker(\{y\}) = \emptyset.$$

*Proof.* Let $(X, \tau)$ be $\delta p(\Lambda, p)$-$R_0$. Suppose that

$$\delta p(\Lambda, p)Ker(\{x\}) \cap \delta p(\Lambda, p)Ker(\{y\}) \neq \emptyset.$$

Let $z \in \delta p(\Lambda, p)Ker(\{x\}) \cap \delta p(\Lambda, p)Ker(\{y\})$. Then, $z \in \delta p(\Lambda, p)Ker(\{x\})$ and by Lemma 6, $x \in \{z\}^{\delta p(\Lambda,p)}$. Thus, $x \in \{z\}^{\delta p(\Lambda,p)} \cap \{x\}^{\delta p(\Lambda,p)}$ and by Corollary 1,

$$\{z\}^{\delta p(\Lambda,p)} = \{x\}^{\delta p(\Lambda,p)}.$$

Similarly, we have $\{z\}^{\delta p(\Lambda,p)} = \{y\}^{\delta p(\Lambda,p)}$ and hence $\{x\}^{\delta p(\Lambda,p)} = \{y\}^{\delta p(\Lambda,p)}$, by Lemma 6, $\delta p(\Lambda, p)Ker(\{x\}) = \delta p(\Lambda, p)Ker(\{y\})$.

Conversely, we show the sufficiency by using Corollary 1. Suppose that

$$\{x\}^{\delta p(\Lambda,p)} \neq \{y\}^{\delta p(\Lambda,p)}.$$

By Lemma 6, $\delta p(\Lambda, p)Ker(\{x\}) \neq \delta p(\Lambda, p)Ker(\{y\})$ and hence

$$\delta p(\Lambda, p)Ker(\{x\}) \cap \delta p(\Lambda, p)Ker(\{y\}) = \emptyset.$$

Thus, $\{x\}^{\delta p(\Lambda,p)} \cap \{y\}^{\delta p(\Lambda,p)} = \emptyset$. In fact, assume that $z \in \{x\}^{\delta p(\Lambda,p)} \cap \{y\}^{\delta p(\Lambda,p)}$. Then,

$$z \in \{x\}^{\delta p(\Lambda,p)}$$

implies $x \in \delta p(\Lambda, p)Ker(\{z\})$ and hence $x \in \delta p(\Lambda, p)Ker(\{z\}) \cap \delta p(\Lambda, p)Ker(\{x\})$. By the hypothesis, $\delta p(\Lambda, p)Ker(\{z\}) = \delta p(\Lambda, p)Ker(\{x\})$ and by Lemma 6,

$$\{z\}^{\delta p(\Lambda,p)} = \{x\}^{\delta p(\Lambda,p)}.$$

Similarly, we have $\{z\}^{\delta p(\Lambda,p)} = \{y\}^{\delta p(\Lambda,p)}$ and hence $\{x\}^{\delta p(\Lambda,p)} = \{y\}^{\delta p(\Lambda,p)}$. This contradicts that $\{x\}^{\delta p(\Lambda,p)} \neq \{y\}^{\delta p(\Lambda,p)}$. Thus, $\{x\}^{\delta p(\Lambda,p)} \cap \{y\}^{\delta p(\Lambda,p)} = \emptyset$. This shows that $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$.

**Theorem 7.** *For a topological space* $(X, \tau)$*, the following properties are equivalent:*

*(1)* $(X, \tau)$ *is* $\delta p(\Lambda, p)$-$R_0$.

*(2)* $x \in \{y\}^{\delta p(\Lambda,p)}$ *if and only if* $y \in \{x\}^{\delta p(\Lambda,p)}$.

*Proof.* (1) ⇒ (2): Suppose that $x \in \{y\}^{\delta p(\Lambda,p)}$. By Lemma 6, $y \in \delta p(\Lambda, p)Ker(\{x\})$ and hence $\delta p(\Lambda, p)Ker(\{x\}) \cap \delta p(\Lambda, p)Ker(\{y\}) \neq \emptyset$. By Theorem 6,

$$\delta p(\Lambda, p)Ker(\{x\}) = \delta p(\Lambda, p)Ker(\{y\})$$

and hence $x \in \delta p(\Lambda, p)Ker(\{y\})$. Thus, by Lemma 6, $y \in \{x\}^{\delta p(\Lambda,p)}$. The converse is similarly shown.

(2) ⇒ (1): Let $U \in \delta p(\Lambda, p)O(X, \tau)$ and $x \in U$. If $y \notin U$, then $U \cap \{y\}^{\delta p(\Lambda,p)} = \emptyset$. Thus, $x \notin \{y\}^{\delta p(\Lambda,p)}$ and $y \notin \{x\}^{\delta p(\Lambda,p)}$. This implies that $\{x\}^{\delta p(\Lambda,p)} \subseteq U$. Therefore, $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$.

**Theorem 8.** *For a topological space $(X, \tau)$, the following properties are equivalent:*

*(1)* $(X, \tau)$ *is* $\delta p(\Lambda, p)$-$R_0$.

*(2) For each nonempty subset $A$ of $X$ and each $U \in \delta p(\Lambda, p)O(X, \tau)$ such that $A \cap U \neq \emptyset$, there exists a $\delta p(\Lambda, p)$-closed set $F$ such that $A \cap F \neq \emptyset$ and $F \subseteq U$.*

*(3) $F = \delta p(\Lambda, p)Ker(F)$ for each $\delta p(\Lambda, p)$-closed set $F$.*

*(4) $\{x\}^{\delta p(\Lambda,p)} = \delta p(\Lambda, p)Ker(\{x\})$ for each $x \in X$.*

*(5) $\{x\}^{\delta p(\Lambda,p)} \subseteq \delta p(\Lambda, p)Ker(\{x\})$ for each $x \in X$.*

*Proof.* (1) ⇒ (2): Let $A$ be a nonempty subset of $X$ and $U \in \delta p(\Lambda, p)O(X, \tau)$ such that $A \cap U \neq \emptyset$. Then, there exists $x \in A \cap U$ and hence $\{x\}^{\delta p(\Lambda,p)} \subseteq U$. Put $F = \{x\}^{\delta p(\Lambda,p)}$. Then, $F$ is $\delta p(\Lambda, p)$-closed, $A \cap F \neq \emptyset$ and $F \subseteq U$.

(2) ⇒ (3): Let $F$ be any $\delta p(\Lambda, p)$-closed set of $X$. By Lemma 5, we have

$$F \subseteq \delta p(\Lambda, p)Ker(F).$$

Next, we show $F \supseteq \delta p(\Lambda, p)Ker(F)$. Let $x \notin F$. Then, $x \in X - F \in \delta p(\Lambda, p)O(X, \tau)$ and by (2), there exists a $\delta p(\Lambda, p)$-closed set $K$ such that $x \in K$ and $K \subseteq X - F$. Now, put $U = X - K$. Then, $F \subseteq U \in \delta p(\Lambda, p)O(X, \tau)$ and $x \notin U$. Thus, $x \notin \delta p(\Lambda, p)Ker(F)$. This shows that $F \supseteq \delta p(\Lambda, p)Ker(F)$.

(3) ⇒ (4): Let $x \in X$ and $y \notin \delta p(\Lambda, p)Ker(\{x\})$. There exists $U \in \delta p(\Lambda, p)O(X, \tau)$ such that $x \in U$ and $y \notin U$. Thus, $U \cap \{y\}^{\delta p(\Lambda,p)} = \emptyset$. By (3),

$$U \cap \delta p(\Lambda, p)Ker(\{y\}^{\delta p(\Lambda,p)}) = \emptyset.$$

Since $x \notin \delta p(\Lambda, p)Ker(\{y\}^{\delta p(\Lambda,p)})$, there exists $V \in \delta p(\Lambda, p)O(X, \tau)$ such that

$$\{y\}^{\delta p(\Lambda,p)} \subseteq V$$

and $x \notin V$. Thus, $V \cap \{x\}^{\delta p(\Lambda,p)} = \emptyset$. Since $y \in V$, $y \notin \{x\}^{\delta p(\Lambda,p)}$ and hence

$$\{x\}^{\delta p(\Lambda,p)} \subseteq \delta p(\Lambda, p)Ker(\{x\}).$$

Moreover, $\{x\}^{\delta p(\Lambda,p)} \subseteq \delta p(\Lambda, p)Ker(\{x\}) \subseteq \delta p(\Lambda, p)Ker(\{x\}^{\delta p(\Lambda,p)}) = \{x\}^{\delta p(\Lambda,p)}$. This shows that $\{x\}^{\delta p(\Lambda,p)} = \delta p(\Lambda, p)Ker(\{x\})$.

(4) ⇒ (5): The proof is obvious.

(5) ⇒ (1): Let $U \in \delta p(\Lambda, p)O(X, \tau)$ and $x \in U$. If $y \notin U$, then $U \cap \{y\}^{\delta p(\Lambda,p)} = \emptyset$ and $x \notin \{y\}^{\delta p(\Lambda,p)}$. By Lemma 6, $y \notin \delta p(\Lambda, p)Ker(\{x\})$ and by (5), $y \notin \{x\}^{\delta p(\Lambda,p)}$. Thus, $\{x\}^{\delta p(\Lambda,p)} \subseteq U$ and hence $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$.

**Corollary 2.** *A topological space $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$ if and only if*

$$\delta p(\Lambda, p)Ker(\{x\}) \subseteq \{x\}^{\delta p(\Lambda,p)}$$

*for each $x \in X$.*

*Proof.* This is obvious by Theorem 8.

Conversely, let $x \in \{y\}^{\delta p(\Lambda,p)}$. Thus, by Lemma 6, $y \in \delta p(\Lambda, p)Ker(\{x\})$ and hence $y \in \{x\}^{\delta p(\Lambda,p)}$. Similarly, if $y \in \{x\}^{\delta p(\Lambda,p)}$, then $x \in \{y\}^{\delta p(\Lambda,p)}$. It follows from Theorem 7 that $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$.

**Definition 10.** *Let $(X, \tau)$ be a topological space and $x \in X$. A subset $\langle x \rangle_{\delta p(\Lambda,p)}$ is defined as follows: $\langle x \rangle_{\delta p(\Lambda,p)} = \delta p(\Lambda, p)Ker(\{x\}) \cap \{x\}^{\delta p(\Lambda,p)}$.*

**Theorem 9.** *A topological space $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$ if and only if $\langle x \rangle_{\delta p(\Lambda,p)} = \{x\}^{\delta p(\Lambda,p)}$ for each $x \in X$.*

*Proof.* Let $x \in X$. By Theorem 8, $\delta p(\Lambda, p)Ker(\{x\}) = \{x\}^{\delta p(\Lambda,sp)}$. Thus,

$$\langle x \rangle_{\delta p(\Lambda,p)} = \delta p(\Lambda, p)Ker(\{x\}) \cap \{x\}^{\delta p(\Lambda,p)} = \{x\}^{\delta p(\Lambda,p)}.$$

Conversely, let $x \in X$. By the hypothesis,

$$\{x\}^{\delta p(\Lambda,p)} = \langle x \rangle_{\delta p(\Lambda,p)} = \delta p(\Lambda, p)Ker(\{x\}) \cap \{x\}^{\delta p(\Lambda,p)} \subseteq \delta p(\Lambda, p)Ker(\{x\}).$$

It follows from Theorem 8 that $(X, \tau)$ is $\delta p(\Lambda, p)$-$R_0$.

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