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Characterizations of $\delta p(\Lambda, s)$-R_0 spaces

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Abstract. Our main purpose is to introduce the concept of $\delta p(\Lambda, s)$-R_0 spaces. Moreover, some characterizations of $\delta p(\Lambda, s)$-R_0 spaces are investigated.

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Key Words and Phrases: $\delta p(\Lambda, s)$-open set, $\delta p(\Lambda, s)$-R_0 space

1. Introduction

In 1943, Shanin [20] introduced the concept of R_0 topological spaces. Davis [11] introduced the concept of a separation axiom called R_1. These concepts are further investigated by Naimpally [16], Dube [13] and Dorsett [12]. Cammaroto and Noiri [10] introduce a weak separation axiom m-R_0 in m-spaces which are equivalent to generalized topological spaces due to Lugojan [15]. Noiri [17] introduced the notion of m-R_1 spaces and investigated several characterizations of m-R_0 spaces and m-R_1 spaces. In 1963, Levine [14] introduced the concept of semi-open sets which is weaker than the concept of open sets in topological spaces. Veličko [23] introduced δ-open sets, which are stronger than open sets. Park et al. [18] have offered new notion called δ-semiopen sets which are stronger than semi-open sets but weaker than δ-open sets and investigated the relationships between several types of these open sets. Caldas and Dontchev [6] introduced and investigated the notions of Λ_s-sets and V_s-sets in topological spaces. Moreover, Caldas et al. [9] investigated some weak separation axioms by utilizing δ-semiopen sets and the δ-semiclosure operator. Caldas et al. [8] investigated the notion of δ-Λ_s-semiclosed sets which is defined as the intersection of a δ-Λ_s-set and a δ-semiclosed set. In 1982, Mashhour et al. [1] introduced and studied the concept of preopen sets. Raychaudhuri and Mukherjee [19] introduced the notions of δ-preopen sets and δ-preclosure. The class of δ-preopen sets is larger than that of preopen sets. Caldas et al. [7] introduced some weak separation axioms by utilizing the notions of δ-preopen sets and the δ-preclosure operator. In [5], the present authors introduced and studied the concept of (Λ, s)-closed sets by utilizing the notions of Λ_s-sets

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and semi-closed sets. Furthermore, several characterizations of (Λ, s)-R_0 spaces and Λ_p-R_0 spaces were established in [5] and [4], respectively. Boonpok and Khampakdee [2] introduced and investigated the concepts of $\delta s(\Lambda, s)$-R_0 spaces and $\delta s(\Lambda, s)$-R_1 spaces. Quite recently, Srisarakham and Boonpok [21] defined and studied the notion of $\delta p(\Lambda, s)$-open sets in topological spaces. In this paper, we introduce the concept of $\delta p(\Lambda, s)$-R_0 spaces. Moreover, some characterizations of $\delta p(\Lambda, s)$-R_0 spaces are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is called *semi-open* [14] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semi-open set is called *semi-closed*. The family of all semi-open (resp. semi-closed) sets in a topological space (X, τ) is denoted by $SO(X, \tau)$ (resp. $SC(X, \tau)$). A subset A^{Λ_s} [6] (resp. A^{V_s}) is defined as follows: $A^{\Lambda_s} = \cap\{U \mid U \supseteq A, U \in SO(X, \tau)\}$ (resp. $A^{V_s} = \cup\{F \mid F \subseteq A, F \in SC(X, \tau)\}$). A subset A of a topological space (X, τ) is called a Λ_s*-set* (resp. V_s*-set*) [6] if $A = A^{\Lambda_s}$ (resp. $A = A^{V_s}$). A subset A of a topological space (X, τ) is called (Λ, s)*-closed* [5] if $A = T \cap C$, where T is a Λ_s-set and C is a semi-closed set. The complement of a (Λ, s)-closed set is called (Λ, s)*-open*. The family of all (Λ, s)-closed (resp. (Λ, s)-open) sets in a topological space (X, τ) is denoted by $\Lambda_s C(X, \tau)$ (resp. $\Lambda_s O(X, \tau)$). Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, s)*-cluster point* [5] of A if for every (Λ, s)-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all (Λ, s)-cluster points of A is called the (Λ, s)*-closure* [5] of A and is denoted by $A^{(\Lambda,s)}$. The union of all (Λ, s)-open sets contained in A is called the (Λ, s)*-interior* [5] of A and is denoted by $A_{(\Lambda,s)}$.

Let A be a subset of a topological space (X, τ). A point x of X is called a $\delta(\Lambda, s)$*-cluster point* [21] of A if $A \cap [V^{(\Lambda,s)}]_{(\Lambda,s)} \neq \emptyset$ for every (Λ, s)-open set V of X containing x. The set of all $\delta(\Lambda, s)$-cluster points of A is called the $\delta(\Lambda, s)$*-closure* [21] of A and is denoted by $A^{\delta(\Lambda,s)}$. If $A = A^{\delta(\Lambda,s)}$, then A is said to be $\delta(\Lambda, s)$*-closed* [21]. The complement of a $\delta(\Lambda, s)$-closed set is said to be $\delta(\Lambda, s)$*-open* [21]. The union of all $\delta(\Lambda, s)$-open sets contained in A is called the $\delta(\Lambda, s)$*-interior* [21] of A and is denoted by $A_{\delta(\Lambda,s)}$.

Definition 1. [21] *A subset A of a topological space (X, τ) is said to be $\delta p(\Lambda, s)$-open if $A \subseteq [A^{(\Lambda,s)}]_{\delta(\Lambda,s)}$. The complement of a $\delta p(\Lambda, s)$-open set is said to be $\delta p(\Lambda, s)$-closed.*

The family of all $\delta p(\Lambda, s)$-open (resp. $\delta p(\Lambda, s)$-closed) sets in a topological space (X, τ) is denoted by $\delta p(\Lambda, s)O(X, \tau)$ (resp. $\delta p(\Lambda, s)C(X, \tau)$). Let A be a subset of a topological space (X, τ). The intersection of all $\delta p(\Lambda, s)$-closed sets containing A is called the $\delta p(\Lambda, s)$-*closure* [22] of A and is denoted by $A^{\delta p(\Lambda,s)}$.

Lemma 1. [21] *For the $\delta p(\Lambda, s)$-closure of subsets A, B in a topological space (X, τ), the following properties hold:*

(1) If $A \subseteq B$, then $A^{\delta p(\Lambda,s)} \subseteq B^{\delta p(\Lambda,s)}$.

(2) A *is* $\delta p(\Lambda, s)$*-closed in* (X, τ) *if and only if* $A = A^{\delta p(\Lambda,s)}$.

(3) $A^{\delta p(\Lambda,s)}$ *is* $\delta p(\Lambda, s)$*-closed, that is,* $A^{\delta p(\Lambda,s)} = [A^{\delta p(\Lambda,s)}]^{\delta p(\Lambda,s)}$.

(4) $x \in A^{\delta p(\Lambda,s)}$ *if and only if* $A \cap V \neq \emptyset$ *for every* $V \in \delta p(\Lambda, s)O(X, \tau)$ *containing* x.

Lemma 2. [21] *For a family* $\{A_\gamma \mid \gamma \in \nabla\}$ *of a topological space* (X, τ)*, the following properties hold:*

(1) $[\cap\{A_\gamma \mid \gamma \in \nabla\}]^{\delta p(\Lambda,s)} \subseteq \cap\{A_\gamma^{\delta p(\Lambda,s)} \mid \gamma \in \nabla\}$.

(2) $[\cup\{A_\gamma \mid \gamma \in \nabla\}]^{\delta p(\Lambda,s)} \supseteq \cup\{A_\gamma^{\delta p(\Lambda,s)} \mid \gamma \in \nabla\}$.

3. Some characterizations of $\delta p(\Lambda, s)$-R_0 spaces

In this section, we introduce the notion of $\delta p(\Lambda, s)$-R_0 spaces. Moreover, several characterizations of $\delta p(\Lambda, s)$-R_0 spaces are discussed.

Definition 2. *A topological space* (X, τ) *is called* $\delta p(\Lambda, s)$-R_0 *if, for each* $\delta p(\Lambda, s)$*-open set* U *and each* $x \in U$, $\{x\}^{\delta p(\Lambda,s)} \subseteq U$.

Theorem 1. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) (X, τ) *is* $\delta p(\Lambda, s)$-R_0.

(2) For each $\delta p(\Lambda, s)$*-closed set* F *and each* $x \in X - F$*, there exists* $U \in \delta p(\Lambda, s)O(X, \tau)$ *such that* $F \subseteq U$ *and* $x \notin U$.

(3) For each $\delta p(\Lambda, s)$*-closed set* F *and each* $x \in X - F$, $F \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$.

(4) For any distinct points x, y *in* X, $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$ *or* $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let F be a $\delta p(\Lambda, s)$-closed set and $x \in X - F$. Since (X, τ) is $\delta p(\Lambda, s)$-R_0, we have $\{x\}^{\delta p(\Lambda,s)} \subseteq X - F$. Put $U = X - \{x\}^{\delta p(\Lambda,s)}$. Thus, by Lemma 1, $U \in \delta p(\Lambda, s)O(X, \tau)$, $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3): Let F be a $\delta p(\Lambda, s)$-closed set and $x \in X - F$. Thus, by (2), there exists $U \in \delta p(\Lambda, s)O(X, \tau)$ such that $F \subseteq U$ and $x \notin U$. Since $U \in \delta p(\Lambda, s)O(X, \tau)$, $U \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$ and hence $F \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$.

(3) $\Rightarrow$ (4): Let x and y be distinct points of X. Suppose that $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} \neq \emptyset$. By (3), $x \in \{y\}^{\delta p(\Lambda,s)}$ and $y \in \{x\}^{\delta p(\Lambda,s)}$. By Lemma 1, $\{x\}^{\delta p(\Lambda,s)} \subseteq \{y\}^{\delta p(\Lambda,s)} \subseteq \{x\}^{\delta p(\Lambda,s)}$ and hence $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$.

(4) $\Rightarrow$ (1): Let $V \in \delta p(\Lambda, s)O(X, \tau)$ and $x \in V$. For each $y \notin V$, $V \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$ and hence $x \notin \{y\}^{\delta p(\Lambda,s)}$. Thus, $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. By (4), for each $y \notin V$,

$$\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset.$$

Since $X - V$ is $\delta p(\Lambda, s)$-closed, $y \in \{y\}^{\delta p(\Lambda,s)} \subseteq X - V$ and $\cup_{y\in X-V}\{y\}^{\delta p(\Lambda,s)} = X - V$. Thus,

$$\begin{aligned}\{x\}^{\delta p(\Lambda,s)} \cap (X - V) &= \{x\}^{\delta p(\Lambda,s)} \cap [\cup_{y\in X-V}\{y\}^{\delta p(\Lambda,s)}] \\ &= \cup_{y\in X-V}[\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)}] \\ &= \emptyset\end{aligned}$$

and hence $\{x\}^{\delta p(\Lambda,s)} \subseteq V$. This shows that (X, τ) is $\delta p(\Lambda, s)$-R_0.

Corollary 1. *A topological space (X, τ) is $\delta p(\Lambda, s)$-R_0 if and only if for any points x and y in X, $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$ implies $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$.*

Proof. This is obvious by Theorem 1.

Conversely, let $U \in \delta p(\Lambda, s)O(X, \tau)$ and $x \in U$. If $y \notin U$, then $U \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$. Thus, $x \notin \{y\}^{\delta p(\Lambda,s)}$ and $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. By the hypothesis, $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$ and hence $y \notin \{x\}^{\delta p(\Lambda,s)}$. Therefore, $\{x\}^{\delta p(\Lambda,s)} \subseteq U$. This shows that (X, τ) is $\delta p(\Lambda, s)$-R_0.

Definition 3. [22] *Let A be a subset of a topological space (X, τ). The $\delta p(\Lambda, s)$-kernel of A, denoted by $\delta p(\Lambda, s)Ker(A)$, is defined to be the set*

$$\delta p(\Lambda, s)Ker(A) = \cap\{U \in \delta p(\Lambda, s)O(X, \tau) \mid A \subseteq U\}.$$

Lemma 3. [3] *For subsets A, B of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \delta p(\Lambda, s)Ker(A)$.

(2) If $A \subseteq B$, then $\delta p(\Lambda, s)Ker(A) \subseteq \delta p(\Lambda, s)Ker(B)$.

(3) $\delta p(\Lambda, s)Ker(\delta p(\Lambda, s)Ker(A)) = \delta p(\Lambda, s)Ker(A)$.

(4) If A is $\delta p(\Lambda, s)$-open, $\delta p(\Lambda, s)Ker(A) = A$.

Theorem 2. *For any points x and y in a topological space (X, τ), the following properties are equivalent:*

(1) $\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$.

(2) $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$.

Proof. (1) ⇒ (2): Suppose that $\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$. Then, there exists a point $z \in X$ such that $z \in \delta p(\Lambda, s)Ker(\{x\})$ and $z \notin \delta p(\Lambda, s)Ker(\{y\})$ or

$$z \in \delta p(\Lambda, s)Ker(\{y\})$$

and $z \notin \delta p(\Lambda, s)Ker(\{x\})$. We prove only the first case being the second analogous. From $z \in \delta p(\Lambda, s)Ker(\{x\})$ it follows that $\{x\} \cap \{z\}^{\delta p(\Lambda,s)} \neq \emptyset$ which implies $x \in \{z\}^{\delta p(\Lambda,s)}$. By $z \notin \delta p(\Lambda, s)Ker(\{y\})$, we have $\{y\} \cap \{z\}^{\delta p(\Lambda,s)} = \emptyset$. Since $x \in \{z\}^{\delta p(\Lambda,s)}$,

$$\{x\}^{\delta p(\Lambda,s)} \subseteq \{z\}^{\delta p(\Lambda,s)}$$

and $\{y\} \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$. Therefore, $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. Thus,

$$\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$$

implies that $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$.

(2) $\Rightarrow$ (1): Suppose that $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. There exists a point $z \in X$ such that $z \in \{x\}^{\delta p(\Lambda,s)}$ and $z \notin \{y\}^{\delta p(\Lambda,s)}$ or $z \in \{y\}^{\delta p(\Lambda,s)}$ and $z \notin \{x\}^{\delta p(\Lambda,s)}$. We prove only the first case being the second analogous. It follows that there exists a $\delta p(\Lambda, s)$-open set containing z and therefore x but not y, namely, $y \notin \delta p(\Lambda, s)Ker(\{x\})$ and thus $\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$.

Lemma 4. *Let (X, τ) be a topological space and $x, y \in X$. Then, the following properties hold:*

(1) $y \in \delta p(\Lambda, s)Ker(\{x\})$ if and only if $x \in \{y\}^{\delta p(\Lambda,s)}$.

(2) $\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\{y\})$ if and only if $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$.

Proof. (1) Let $x \notin \{y\}^{\delta p(\Lambda,s)}$. Then, there exists $U \in \delta p(\Lambda, s)O(X, \tau)$ such that $x \in U$ and $y \notin U$. Thus, $y \notin \delta p(\Lambda, s)Ker(\{x\})$. The converse is similarly shown.

(2) Suppose that $\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\{y\})$ for any $x, y \in X$. Since

$$x \in \delta p(\Lambda, s)Ker(\{x\}),$$

$x \in \delta p(\Lambda, s)Ker(\{y\})$, by (1), $y \in \{x\}^{\delta p(\Lambda,s)}$. By Lemma 1, $\{y\}^{\delta p(\Lambda,s)} \subseteq \{x\}^{\delta p(\Lambda,s)}$. Similarly, we have $\{x\}^{\delta p(\Lambda,s)} \subseteq \{y\}^{\delta p(\Lambda,s)}$ and hence $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$.

Conversely, suppose that $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$. Since $x \in \{x\}^{\delta p(\Lambda,s)}$, $x \in \{y\}^{\delta p(\Lambda,s)}$ and by (1), $y \in \delta p(\Lambda, s)Ker(\{x\})$. By Lemma 3,

$$\delta p(\Lambda, s)Ker(\{y\}) \subseteq \delta p(\Lambda, s)Ker(\delta p(\Lambda, s)Ker(\{x\})) = \delta p(\Lambda, s)Ker(\{x\}).$$

Similarly, we have $\delta p(\Lambda, s)Ker(\{x\}) \subseteq \delta p(\Lambda, s)Ker(\{y\})$ and hence

$$\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\{y\}).$$

Theorem 3. *A topological space (X, τ) is $\delta p(\Lambda, s)$-R_0 if and only if, for each points x and y in X, $\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$ implies*

$$\delta p(\Lambda, s)Ker(\{x\}) \cap \delta p(\Lambda, s)Ker(\{y\}) = \emptyset.$$

Proof. Let (X, τ) be $\delta p(\Lambda, s)$-R_0. Suppose that

$$\delta p(\Lambda, s)Ker(\{x\}) \cap \delta p(\Lambda, s)Ker(\{y\}) \neq \emptyset.$$

Let $z \in \delta p(\Lambda, s)Ker(\{x\}) \cap \delta p(\Lambda, s)Ker(\{y\})$. Then, $z \in \delta p(\Lambda, s)Ker(\{x\})$ and by Lemma 4, $x \in \{z\}^{\delta p(\Lambda,s)}$. Thus, $x \in \{z\}^{\delta p(\Lambda,s)} \cap \{x\}^{\delta p(\Lambda,s)}$ and by Corollary 1,

$$\{z\}^{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)}.$$

Similarly, we have $\{z\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$ and hence $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$, by Lemma 4, $\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\{y\})$.

Conversely, we show the sufficiency by using Corollary 1. Suppose that

$$\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}.$$

By Lemma 4, $\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$ and hence

$$\delta p(\Lambda, s)Ker(\{x\}) \cap \delta p(\Lambda, s)Ker(\{y\}) = \emptyset.$$

Thus, $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$. In fact, assume that $z \in \{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)}$. Then,

$$z \in \{x\}^{\delta p(\Lambda,s)}$$

implies $x \in \delta p(\Lambda, s)Ker(\{z\})$ and hence $x \in \delta p(\Lambda, s)Ker(\{z\}) \cap \delta p(\Lambda, s)Ker(\{x\})$. By the hypothesis, $\delta p(\Lambda, s)Ker(\{z\}) = \delta p(\Lambda, s)Ker(\{x\})$ and by Lemma 4,

$$\{z\}^{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)}.$$

Similarly, we have $\{z\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$ and hence $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$. This contradicts that $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. Thus, $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$. This shows that (X, τ) is $\delta p(\Lambda, s)$-R_0.

Theorem 4. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is* $\delta p(\Lambda, s)$-R_0.

(2) $x \in \{y\}^{\delta p(\Lambda,s)}$ *if and only if* $y \in \{x\}^{\delta p(\Lambda,s)}$.

Proof. (1) ⇒ (2): Suppose that $x \in \{y\}^{\delta p(\Lambda,s)}$. By Lemma 4, $y \in \delta p(\Lambda, s)Ker(\{x\})$ and hence $\delta p(\Lambda, s)Ker(\{x\}) \cap \delta p(\Lambda, s)Ker(\{y\}) \neq \emptyset$. By Theorem 3,

$$\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\{y\})$$

and hence $x \in \delta p(\Lambda, s)Ker(\{y\})$. Thus, by Lemma 4, $y \in \{x\}^{\delta p(\Lambda,s)}$. The converse is similarly shown.

(2) ⇒ (1): Let $U \in \delta p(\Lambda, s)O(X, \tau)$ and $x \in U$. If $y \notin U$, then $U \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$. Thus, $x \notin \{y\}^{\delta p(\Lambda,s)}$ and $y \notin \{x\}^{\delta p(\Lambda,s)}$. This implies that $\{x\}^{\delta p(\Lambda,s)} \subseteq U$. Therefore, (X, τ) is $\delta p(\Lambda, s)$-R_0.

Theorem 5. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is* $\delta p(\Lambda, s)$-R_0.

(2) For each nonempty subset A *of* X *and each* $U \in \delta p(\Lambda, s)O(X, \tau)$ *such that* $A \cap U \neq \emptyset$, *there exists a* $\delta p(\Lambda, s)$*-closed set* F *such that* $A \cap F \neq \emptyset$ *and* $F \subseteq U$.

(3) $F = \delta p(\Lambda, s)Ker(F)$ *for each* $\delta p(\Lambda, s)$*-closed set* F.

(4) $\{x\}^{\delta p(\Lambda,s)} = \delta p(\Lambda, s)Ker(\{x\})$ *for each* $x \in X$.

(5) $\{x\}^{\delta p(\Lambda,s)} \subseteq \delta p(\Lambda, s)Ker(\{x\})$ *for each* $x \in X$.

Proof. (1) ⇒ (2): Let A be a nonempty subset of X and $U \in \delta p(\Lambda, s)O(X, \tau)$ such that $A \cap U \neq \emptyset$. Then, there exists $x \in A \cap U$ and hence $\{x\}^{\delta p(\Lambda,s)} \subseteq U$. Put $F = \{x\}^{\delta p(\Lambda,s)}$. Then, F is $\delta p(\Lambda, s)$-closed such that $A \cap F \neq \emptyset$ and $F \subseteq U$.

(2) ⇒ (3): Let F be any $\delta p(\Lambda, s)$-closed set of X. By Lemma 3, we have

$$F \subseteq \delta p(\Lambda, s)Ker(F).$$

Next, we show $F \supseteq \delta p(\Lambda, s)Ker(F)$. Let $x \notin F$. Then, $x \in X - F \in \delta p(\Lambda, s)O(X, \tau)$ and by (2), there exists a $\delta p(\Lambda, s)$-closed set K such that $x \in K$ and $K \subseteq X - F$. Now, put $U = X - K$. Then, $F \subseteq U \in \delta p(\Lambda, s)O(X, \tau)$ and $x \notin U$. Thus, $x \notin \delta p(\Lambda, s)Ker(F)$. This shows that $F \supseteq \delta p(\Lambda, s)Ker(F)$.

(3) ⇒ (4): Let $x \in X$ and $y \notin \delta p(\Lambda, s)Ker(\{x\})$. There exists $U \in \delta p(\Lambda, s)O(X, \tau)$ such that $x \in U$ and $y \notin U$. Thus, $U \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$. By (3),

$$U \cap \delta p(\Lambda, s)Ker(\{y\}^{\delta p(\Lambda,s)}) = \emptyset.$$

Since $x \notin \delta p(\Lambda, s)Ker(\{y\}^{\delta p(\Lambda,s)})$, there exists $V \in \delta p(\Lambda, s)O(X, \tau)$ such that

$$\{y\}^{\delta p(\Lambda,s)} \subseteq V$$

and $x \notin V$. Thus, $V \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$. Since $y \in V$, we have $y \notin \{x\}^{\delta p(\Lambda,s)}$ and hence $\{x\}^{\delta p(\Lambda,s)} \subseteq \delta p(\Lambda, s)Ker(\{x\})$. Moreover,

$$\{x\}^{\delta p(\Lambda,s)} \subseteq \delta p(\Lambda, s)Ker(\{x\}) \subseteq \delta p(\Lambda, s)Ker(\{x\}^{\delta p(\Lambda,s)}) = \{x\}^{\delta p(\Lambda,s)}.$$

This shows that $\{x\}^{\delta p(\Lambda,s)} = \delta p(\Lambda, s)Ker(\{x\})$.

(4) ⇒ (5): The proof is obvious.

(5) ⇒ (1): Let $U \in \delta p(\Lambda, s)O(X, \tau)$ and $x \in U$. If $y \notin U$, then $U \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$ and $x \notin \{y\}^{\delta p(\Lambda,s)}$. By Lemma 4, $y \notin \delta p(\Lambda, s)Ker(\{x\})$ and by (5), $y \notin \{x\}^{\delta p(\Lambda,s)}$. Thus, $\{x\}^{\delta p(\Lambda,s)} \subseteq U$ and hence (X, τ) is $\delta p(\Lambda, s)$-R_0.

Corollary 2. *A topological space* (X, τ) *is* $\delta p(\Lambda, s)$-R_0 *if and only if*

$$\delta p(\Lambda, s)Ker(\{x\}) \subseteq \{x\}^{\delta p(\Lambda,s)}$$

for each $x \in X$.

Proof. This is obvious by Theorem 5.

Conversely, let $x \in \{y\}^{\delta p(\Lambda,s)}$. Thus, by Lemma 4, $y \in \delta p(\Lambda,s)Ker(\{x\})$ and hence $y \in \{x\}^{\delta p(\Lambda,s)}$. Similarly, if $y \in \{x\}^{\delta p(\Lambda,s)}$, then $x \in \{y\}^{\delta p(\Lambda,s)}$. It follows from Theorem 4 that (X,τ) is $\delta p(\Lambda,s)$-R_0.

Definition 4. [3] *Let (X,τ) be a topological space and $x \in X$. A subset $\langle x\rangle_{\delta p(\Lambda,s)}$ is defined as follows: $\langle x\rangle_{\delta p(\Lambda,s)} = \delta p(\Lambda,s)Ker(\{x\}) \cap \{x\}^{\delta p(\Lambda,s)}$.*

Theorem 6. *A topological space (X,τ) is $\delta p(\Lambda,s)$-R_0 if and only if $\langle x\rangle_{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)}$ for each $x \in X$.*

Proof. Let $x \in X$. By Theorem 5, $\delta p(\Lambda,s)Ker(\{x\}) = \{x\}^{\delta p(\Lambda,s)}$. Thus,

$$\langle x\rangle_{\delta p(\Lambda,s)} = \delta p(\Lambda,s)Ker(\{x\}) \cap \{x\}^{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)}.$$

Conversely, let $x \in X$. By the hypothesis,

$$\{x\}^{\delta p(\Lambda,s)} = \langle x\rangle_{\delta p(\Lambda,s)} = \delta p(\Lambda,s)Ker(\{x\}) \cap \{x\}^{\delta p(\Lambda,s)} \subseteq \delta p(\Lambda,s)Ker(\{x\}).$$

It follows from Theorem 5 that (X,τ) is $\delta p(\Lambda,s)$-R_0.

Definition 5. *A topological space (X,τ) is said to be $\delta p(\Lambda,s)$-R_1 if for each points x,y in X with $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$, there exist disjoint $\delta p(\Lambda,s)$-open sets U and V such that $\{x\}^{\delta p(\Lambda,s)} \subseteq U$ and $\{y\}^{\delta p(\Lambda,s)} \subseteq V$.*

Theorem 7. *A topological space (X,τ) is $\delta p(\Lambda,s)$-R_1 if and only if for any points x,y in X with $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$, there exist $\delta p(\Lambda,s)$-closed sets F and K such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$.*

Proof. Let x and y be any points in X with $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. Then, there exist disjoint $U,V \in \delta p(\Lambda,s)O(X,\tau)$ such that $\{x\}^{\delta p(\Lambda,s)} \subseteq U$ and $\{y\}^{\delta p(\Lambda,s)} \subseteq V$. Now, put $F = X - V$ and $K = X - U$. Then, F and K are $\delta p(\Lambda,s)$-closed sets of X such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$.

Conversely, let x and y be any points in X such that $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. Then, $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$. In fact, if $z \in \{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)}$, then $\{z\}^{\delta p(\Lambda,s)} \neq \{x\}^{\delta p(\Lambda,s)}$ or $\{z\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. In case $\{z\}^{\delta p(\Lambda,s)} \neq \{x\}^{\delta p(\Lambda,s)}$, by the hypothesis, there exists a $\delta p(\Lambda,s)$-closed set F such that $x \in F$ and $z \notin F$. Then, $z \in \{x\}^{\delta p(\Lambda,s)} \subseteq F$. This contradicts that $z \notin F$. In case $\{z\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$, similarly, this leads to the contradiction. Thus, $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$, by Corollary 1, (X,τ) is $\delta p(\Lambda,s)$-R_0. By the hypothesis, there exist $\delta p(\Lambda,s)$-closed sets F and K such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$. Put $U = X - K$ and $V = X - F$. Then, $x \in U \in \delta p(\Lambda,s)O(X,\tau)$ and

$$y \in V \in \delta p(\Lambda,s)O(X,\tau).$$

Since (X,τ) is $\delta p(\Lambda,s)$-R_0, we have $\{x\}^{\delta p(\Lambda,s)} \subseteq U$, $\{y\}^{\delta p(\Lambda,s)} \subseteq V$ and also $U \cap V = \emptyset$. This shows that (X,τ) is $\delta p(\Lambda,s)$-R_1.

Definition 6. *Let A be a subset of a topological space (X, τ). The $\theta\delta p(\Lambda, s)$-closure of A, $A^{\theta\delta p(\Lambda,s)}$, is defined as follows:*

$$A^{\theta\delta p(\Lambda,s)} = \{x \in X \mid A \cap \ U^{\delta p(\Lambda,s)} \neq \emptyset \text{ for each } U \in \delta p(\Lambda, s)O(X, \tau) \text{ containing } x\}.$$

Lemma 5. *If a topological space (X, τ) is $\delta p(\Lambda, s)$-R_1, then (X, τ) is $\delta p(\Lambda, s)$-R_0.*

Proof. Let $U \in \delta p(\Lambda, s)O(X, \tau)$ and $x \in U$. If $y \notin U$, then $U \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$ and $x \notin \{y\}^{\delta p(\Lambda,s)}$. Thus, $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. Since (X, τ) is $\delta p(\Lambda, s)$-R_1, there exists $V \in \delta p(\Lambda, s)O(X, \tau)$ such that $\{y\}^{\delta p(\Lambda,s)} \subseteq V$ and $x \notin V$. Thus, $V \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$ and hence $y \notin \{x\}^{\delta p(\Lambda,s)}$. Therefore, $\{x\}^{\delta p(\Lambda,s)} \subseteq U$. This shows that (X, τ) is $\delta p(\Lambda, s)$-R_0.

Theorem 8. *A topological space (X, τ) is $\delta p(\Lambda, s)$-R_1 if and only if $\langle x\rangle_{\delta p(\Lambda,s)} = \{x\}^{\theta\delta p(\Lambda,s)}$ for each $x \in X$.*

Proof. Let (X, τ) be $\delta p(\Lambda, s)$-R_1. By Lemma 5, (X, τ) is $\delta p(\Lambda, s)$-R_0 and by Theorem 6, $\langle x\rangle_{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)} \subseteq \{x\}^{\theta\delta p(\Lambda,s)}$ for each $x \in X$. Thus, $\langle x\rangle_{\delta p(\Lambda,s)} \subseteq \{x\}^{\theta\delta p(\Lambda,s)}$ for each $x \in X$. In order to show the opposite inclusion, suppose that $y \notin \langle x\rangle_{\delta p(\Lambda,s)}$. Then, $\langle x\rangle_{\delta p(\Lambda,s)} \neq \langle y\rangle_{\delta p(\Lambda,s)}$. Since (X, τ) is $\delta p(\Lambda, s)$-R_0, by Theorem 6, $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. Since (X, τ) is $\delta p(\Lambda, s)$-R_1, there exist disjoint $\delta p(\Lambda, s)$-open sets U and V of X such that $\{x\}^{\delta p(\Lambda,s)} \subseteq U$ and $\{y\}^{\delta p(\Lambda,s)} \subseteq V$. Since $\{x\} \cap V^{\delta p(\Lambda,s)} \subseteq U \cap V^{\delta p(\Lambda,s)} = \emptyset$, $y \notin \{x\}^{\theta\delta p(\Lambda,s)}$. Thus, $\{x\}^{\theta\delta p(\Lambda,s)} \subseteq \langle x\rangle_{\delta p(\Lambda,s)}$ and hence $\{x\}^{\theta\delta p(\Lambda,s)} = \langle x\rangle_{\delta p(\Lambda,s)}$.

Conversely, suppose that $\{x\}^{\theta\delta p(\Lambda,s)} = \langle x\rangle_{\delta p(\Lambda,s)}$ for each $x \in X$. Then,

$$\langle x\rangle_{\delta p(\Lambda,s)} = \{x\}^{\theta\delta p(\Lambda,s)} \supseteq \{x\}^{\delta p(\Lambda,s)} \supseteq \langle x\rangle_{\delta p(\Lambda,s)}$$

and $\langle x\rangle_{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)}$ for each $x \in X$. By Theorem 6, (X, τ) is $\delta p(\Lambda, s)$-R_0. Suppose that $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. Thus, by Corollary 1, $\{x\}^{\delta p(\Lambda,s)} \cap \{y\}^{\delta p(\Lambda,s)} = \emptyset$. By Theorem 6, $\langle x\rangle_{\delta p(\Lambda,s)} \cap \langle y\rangle_{\delta p(\Lambda,s)} = \emptyset$ and hence $\{x\}^{\theta\delta(\Lambda,s)} \cap \{y\}^{\theta\delta p(\Lambda,s)} = \emptyset$. Since $y \notin \{x\}^{\theta\delta p(\Lambda,s)}$, there exists a $\delta p(\Lambda, s)$-open set U of X such that $y \in U \subseteq U^{\delta p(\Lambda,s)} \subseteq X - \{x\}$. Let

$$V = X - U^{\delta p(\Lambda,s)},$$

then $x \in V \in \delta p(\Lambda, s)O(X, \tau)$. Since (X, τ) is $\delta p(\Lambda, s)$-R_0, $\{y\}^{\delta p(\Lambda,s)} \subseteq U$, $\{x\}^{\delta p(\Lambda,s)} \subseteq V$ and $U \cap V = \emptyset$. This shows that (X, τ) is $\delta p(\Lambda, s)$-R_1.

Corollary 3. *A topological space (X, τ) is $\delta p(\Lambda, s)$-R_1 if and only if $\{x\}^{\delta p(\Lambda,s)} = \{x\}^{\theta\delta p(\Lambda,s)}$ for each $x \in X$.*

Proof. Let (X, τ) be a $\delta p(\Lambda, s)$-R_1 space. By Theorem 8, we have

$$\{x\}^{\delta p(\Lambda,s)} \supseteq \langle x\rangle_{\delta p(\Lambda,s)} = \{x\}^{\theta\delta p(\Lambda,s)} \supseteq \{x\}^{\delta p(\Lambda,s)}$$

and hence $\{x\}^{\delta p(\Lambda,s)} = \{x\}^{\theta\delta p(\Lambda,s)}$ for each $x \in X$.

Conversely, suppose that $\{x\}^{\delta p(\Lambda,s)} = \{x\}^{\theta\delta p(\Lambda,s)}$ for each $x \in X$. First, we show that (X,τ) is $\delta p(\Lambda,s)$-R_0. Let $U \in \delta p(\Lambda,s)O(X,\tau)$ and $x \in U$. Let $y \notin U$. Then,

$$U \cap \{y\}^{\delta p(\Lambda,s)} = U \cap \{y\}^{\theta\delta p(\Lambda,s)} = \emptyset.$$

Thus, $x \notin \{y\}^{\theta\delta p(\Lambda,s)}$. There exists $V \in \delta p(\Lambda,s)O(X,\tau)$ such that $x \in V$ and $y \notin V^{\delta p(\Lambda,s)}$. Since $\{x\}^{\delta p(\Lambda,s)} \subseteq V^{\delta p(\Lambda,s)}$, $y \notin \{x\}^{\delta p(\Lambda,s)}$. This shows that $\{x\}^{\delta p(\Lambda,s)} \subseteq U$ and hence (X,τ) is $\delta p(\Lambda,s)$-R_0. By Theorem 6, $\langle x\rangle_{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)} = \{x\}^{\theta\delta p(\Lambda,s)}$ for each $x \in X$. Thus, by Theorem 8, (X,τ) is $\delta p(\Lambda,s)$-R_1.

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