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Properties of generalized $\delta p(\Lambda, s)$-closed sets

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Abstract. This paper deals with the concept of generalized $\delta p(\Lambda, s)$-closed sets. Especially, some properties of generalized $\delta p(\Lambda, s)$-closed sets are discussed. Moreover, we apply the notion of generalized $\delta p(\Lambda, s)$-closed sets to present and study new classes of spaces called $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-spaces and $\delta p(\Lambda, s)$-normal spaces. Several properties and characterizations concerning $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-spaces and $\delta p(\Lambda, s)$-normal spaces are established.

2020 Mathematics Subject Classifications: 54A05, 54D10

Key Words and Phrases: $\delta p(\Lambda, s)$-open set, generalized $\delta p(\Lambda, s)$-closed set

1. Introduction

In 1970, Levine [11] introduced the concept of generalized closed sets in topological spaces and defined the notion of a $T_{\frac{1}{2}}$-space to be one in which the closed sets and the generalized closed sets coincide. Dunham and Levine [9] investigated the further properties of generalized closed sets. The concept of generalized closed sets has been modified and studied by using weaker forms of open sets such as α-open sets [13], semi-open sets [10], preopen sets [12] and semi-preopen sets [1]. Levine [10] introduced the concept of semi-open sets which is weaker than the concept of open sets in topological spaces. Veličko [19] introduced δ-open sets, which are stronger than open sets. Park et al. [14] have offered new notion called δ-semiopen sets which are stronger than semi-open sets but weaker than δ-open sets and investigated the relationships between several types of these open sets. Caldas and Dontchev [4] introduced and investigated the notions of Λ_s-sets and V_s-sets in topological spaces. Moreover, Caldas et al. [7] investigated some weak separation axioms by utilizing δ-semiopen sets and the δ-semiclosure operator. Caldas et al. [6] investigated the notion of δ-Λ_s-semiclosed sets which is defined as the intersection of a δ-Λ_s-set and a δ-semiclosed set. Mashhour et al. [12] introduced and studied the concept of preopen sets. Raychaudhuri and Mukherjee [15] introduced the notions of δ-preopen sets and δ-preclosure. The class of δ-preopen sets is larger than that of preopen sets. Caldas et al. [5]

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introduced some weak separation axioms by utilizing the notions of δ-preopen sets and the δ-preclosure operator. Buadong et al. [2] introduced and studied some separation axioms in generalized topology and minimal structure spaces. Dungthaisong et al. [8] investigated some properties of pairwise μ-$T_{\frac{1}{2}}$-spaces. Torton et al. [18] introduced and studied the notions of $\mu_{(m,n)}$-regular spaces and $\mu_{(m,n)}$-normal spaces. Viriyapong and Boonpok [20] defined and investigated the notion of generalized (Λ, p)-closed sets in topological spaces. In [3], the present authors introduced and investigated the concept of (Λ, s)-closed sets by utilizing the notions of Λ_s-sets and semi-closed sets. In this paper, we introduce the concept of generalized $\delta p(\Lambda, s)$-closed sets. Moreover, some properties of generalized $\delta p(\Lambda, s)$-closed sets are discussed. In particular, we give several characterizations of $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-spaces and $\delta p(\Lambda, s)$-normal spaces by utilizing the concept of generalized $\delta p(\Lambda, s)$-closed sets.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is called *semi-open* [10] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semi-open set is called *semi-closed*. The family of all semi-open (resp. semi-closed) sets in a topological space (X, τ) is denoted by $SO(X, \tau)$ (resp. $SC(X, \tau)$). A subset A^{Λ_s} [4] (resp. A^{V_s}) is defined as follows: $A^{\Lambda_s} = \cap\{U \mid U \supseteq A, U \in SO(X, \tau)\}$ (resp. $A^{V_s} = \cup\{F \mid F \subseteq A, F \in SC(X, \tau)\}$). A subset A of a topological space (X, τ) is called a Λ_s*-set* (resp. V_s*-set*) [4] if $A = A^{\Lambda_s}$ (resp. $A = A^{V_s}$). A subset A of a topological space (X, τ) is called (Λ, s)*-closed* [3] if $A = T \cap C$, where T is a Λ_s-set and C is a semi-closed set. The complement of a (Λ, s)-closed set is called (Λ, s)*-open*. The family of all (Λ, s)-closed (resp. (Λ, s)-open) sets in a topological space (X, τ) is denoted by $\Lambda_s C(X, \tau)$ (resp. $\Lambda_s O(X, \tau)$). Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, s)*-cluster point* [3] of A if for every (Λ, s)-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all (Λ, s)-cluster points of A is called the (Λ, s)*-closure* [3] of A and is denoted by $A^{(\Lambda,s)}$. The union of all (Λ, s)-open sets contained in A is called the (Λ, s)*-interior* [3] of A and is denoted by $A_{(\Lambda,s)}$. Let A be a subset of a topological space (X, τ). A point x of X is called a $\delta(\Lambda, s)$*-cluster point* [16] of A if $A \cap [V^{(\Lambda,s)}]_{(\Lambda,s)} \neq \emptyset$ for every (Λ, s)-open set V of X containing x. The set of all $\delta(\Lambda, s)$-cluster points of A is called the $\delta(\Lambda, s)$*-closure* [16] of A and is denoted by $A^{\delta(\Lambda,s)}$. If $A = A^{\delta(\Lambda,s)}$, then A is said to be $\delta(\Lambda, s)$*-closed* [16]. The complement of a $\delta(\Lambda, s)$-closed set is said to be $\delta(\Lambda, s)$*-open*. The union of all $\delta(\Lambda, s)$-open sets contained in A is called the $\delta(\Lambda, s)$*-interior* [16] of A and is denoted by $A_{\delta(\Lambda,s)}$.

Definition 1. [16] *A subset A of a topological space (X, τ) is said to be $\delta p(\Lambda, s)$-open if $A \subseteq [A^{(\Lambda,s)}]_{\delta(\Lambda,s)}$. The complement of a $\delta p(\Lambda, s)$-open set is said to be $\delta p(\Lambda, s)$-closed.*

The family of all $\delta p(\Lambda, s)$-open (resp. $\delta p(\Lambda, s)$-closed) sets in a topological space (X, τ) is denoted by $\delta p(\Lambda, s)O(X, \tau)$ (resp. $\delta p(\Lambda, s)C(X, \tau)$). Let A be a subset of a topological space (X, τ). The intersection of all $\delta p(\Lambda, s)$-closed sets containing A is called the $\delta p(\Lambda, s)$-*closure* of A and is denoted by $A^{\delta p(\Lambda,s)}$.

Lemma 1. [16] *For the $\delta p(\Lambda, s)$-closure of subsets A, B in a topological space (X, τ), the following properties hold:*

(1) If $A \subseteq B$, then $A^{\delta p(\Lambda,s)} \subseteq B^{\delta p(\Lambda,s)}$.

(2) A is $\delta p(\Lambda, s)$-closed in (X, τ) if and only if $A = A^{\delta p(\Lambda,s)}$.

(3) $A^{\delta p(\Lambda,s)}$ is $\delta p(\Lambda, s)$-closed, that is, $A^{\delta p(\Lambda,s)} = [A^{\delta p(\Lambda,s)}]^{\delta p(\Lambda,s)}$.

(4) $x \in A^{\delta p(\Lambda,s)}$ if and only if $A \cap V \neq \emptyset$ for every $V \in \delta p(\Lambda, s)O(X, \tau)$ containing x.

Lemma 2. [16] *For a family $\{A_\gamma \mid \gamma \in \nabla\}$ of a topological space (X, τ), the following properties hold:*

(1) $[\cap\{A_\gamma \mid \gamma \in \nabla\}]^{\delta p(\Lambda,s)} \subseteq \cap\{A_\gamma^{\delta p(\Lambda,s)} \mid \gamma \in \nabla\}$.

(2) $[\cup\{A_\gamma \mid \gamma \in \nabla\}]^{\delta p(\Lambda,s)} \supseteq \cup\{A_\gamma^{\delta p(\Lambda,s)} \mid \gamma \in \nabla\}$.

Definition 2. *Let A be a subset of a topological space (X, τ). The union of all (Λ, sp)-open sets contained in A is called the $\delta p(\Lambda, s)$-interior of A and is denoted by $A_{\delta p(\Lambda,s)}$.*

Lemma 3. *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A_{\delta p(\Lambda,s)} \subseteq A$ and $[A_{\delta p(\Lambda,s)}]_{\delta p(\Lambda,s)} = A_{\delta p(\Lambda,s)}$.

(2) If $A \subseteq B$, then $A_{\delta p(\Lambda,s)} \subseteq B_{\delta p(\Lambda,s)}$.

(3) $A_{\delta p(\Lambda,s)}$ is $\delta p(\Lambda, s)$-open.

(4) A is $\delta p(\Lambda, s)$-open if and only if $A_{\delta p(\Lambda,s)} = A$.

(5) $[X - A]^{\delta p(\Lambda,s)} = X - A_{\delta p(\Lambda,s)}$.

(6) $[X - A]_{\delta p(\Lambda,s)} = X - A^{\delta p(\Lambda,s)}$.

3. Generalized $\delta p(\Lambda, s)$-closed sets

We begin this section by introducing the concept of generalized $\delta p(\Lambda, s)$-closed sets.

Definition 3. *A subset A of a topological space (X, τ) is said to be generalized $\delta p(\Lambda, s)$-closed (briefly, g-$\delta p(\Lambda, s)$-closed) if $A^{\delta p(\Lambda,s)} \subseteq U$ whenever $A \subseteq U$ and U is $\delta p(\Lambda, s)$-open in (X, τ). The complement of a generalized $\delta p(\Lambda, s)$-closed set is said to be generalized $\delta p(\Lambda, s)$-open (briefly, g-$\delta p(\Lambda, s)$-open).*

Theorem 1. *A subset A of a topological space (X, τ) is g-$\delta p(\Lambda, s)$-closed if and only if $A^{\delta p(\Lambda,s)} - A$ contains no nonempty $\delta p(\Lambda, s)$-closed set.*

Proof. Let F be a $\delta p(\Lambda, s)$-closed subset of $A^{\delta p(\Lambda,s)} - A$. Since $A \subseteq X - F$ and A is g-$\delta p(\Lambda, s)$-closed, $A^{\delta p(\Lambda,s)} \subseteq X - F$ and hence $F \subseteq X - A^{\delta p(\Lambda,s)}$. Thus,

$$F \subseteq A^{\delta p(\Lambda,s)} \cap [X - A^{\delta p(\Lambda,s)}] = \emptyset$$

and F is empty.

Conversely, suppose that $A \subseteq U$ and U is $\delta p(\Lambda, s)$-open. If $A^{\delta p(\Lambda,s)} \nsubseteq U$, then

$$A^{\delta p(\Lambda,s)} \cap (X - U)$$

is a nonempty $\delta p(\Lambda, s)$-closed subset of $A^{\delta p(\Lambda,s)} - A$.

Corollary 1. *Let A be a g-$\delta p(\Lambda, s)$-closed subset of a topological space (X, τ). Then, A is $\delta p(\Lambda, s)$-closed if and only if $A^{\delta p(\Lambda,s)} - A$ is $\delta p(\Lambda, s)$-closed.*

Proof. If A is a $\delta p(\Lambda, s)$-closed set, then $A^{\delta p(\Lambda,s)} - A = \emptyset$.

Conversely, suppose that $A^{\delta p(\Lambda,s)} - A$ is $\delta p(\Lambda, s)$-closed. Since A is g-$\delta p(\Lambda, s)$-closed and $A^{\delta p(\Lambda,s)} - A$ is a $\delta p(\Lambda, s)$-closed subset of itself, by Theorem 1, $A^{\delta p(\Lambda,s)} - A = \emptyset$ and hence $A^{\delta p(\Lambda,s)} = A$.

Theorem 2. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A is $\delta p(\Lambda, s)$-closed, then A is g-$\delta p(\Lambda, s)$-closed.

(2) If A is g-$\delta p(\Lambda, s)$-closed and $\delta p(\Lambda, s)$-open, then A is $\delta p(\Lambda, s)$-closed.

(3) If A is g-$\delta p(\Lambda, s)$-closed and $A \subseteq B \subseteq A^{\delta p(\Lambda,s)}$, then B is g-$\delta p(\Lambda, s)$-closed.

Proof. (1) Let A be $\delta p(\Lambda, s)$-closed and $A \subseteq U \in \delta p(\Lambda, s)O(X, \tau)$. Then, by Lemma 1, $A^{\delta p(\Lambda,s)} = A \subseteq U$ and hence A is g-$\delta p(\Lambda, s)$-closed.

(2) Let A be g-$\delta p(\Lambda, s)$-closed and $\delta p(\Lambda, s)$-open. Then, $A^{\delta p(\Lambda,s)} = A$ and by Lemma 1, A is $\delta p(\Lambda, s)$-closed.

(3) Let $B \subseteq U$ and $U \in \delta p(\Lambda, s)O(X, \tau)$. Since $A \subseteq U$ and A is g-$\delta p(\Lambda, s)$-closed, we have $A^{\delta p(\Lambda,s)} \subseteq U$. Since $A \subseteq B \subseteq A^{\delta p(\Lambda,s)}$, by Lemma 1, $A^{\delta p(\Lambda,s)} = B^{\delta p(\Lambda,s)}$ and hence $B^{\delta p(\Lambda,s)} \subseteq U$. Thus, B is g-$\delta p(\Lambda, s)$-closed.

Corollary 2. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A is $\delta p(\Lambda, s)$-open, then A is g-$\delta p(\Lambda, s)$-open.

(2) If A is g-$\delta p(\Lambda, s)$-open and $\delta p(\Lambda, s)$-closed, then A is $\delta p(\Lambda, s)$-open.

(3) If A is g-$\delta p(\Lambda, s)$-open and $A_{\delta p(\Lambda,s)} \subseteq B \subseteq A$, then B is g-$\delta p(\Lambda, s)$-open.

Proof. This follows from Theorem 2.

Definition 4. *Let A be a subset of a topological space (X,τ). The $\delta p(\Lambda,s)$-frontier of A, $\delta p(\Lambda,s)Fr(A)$, is defined as follows: $\delta p(\Lambda,s)Fr(A) = A^{\delta p(\Lambda,s)} \cap [X-A]^{\delta p(\Lambda,s)}$.*

Theorem 3. *Let A be a subset of a topological space (X,τ). If A is g-$\delta p(\Lambda,s)$-closed and $A \subseteq V \in \delta p(\Lambda,s)O(X,\tau)$, then $\delta p(\Lambda,s)Fr(V) \subseteq [X-A]_{\delta p(\Lambda,s)}$.*

Proof. Let A be g-$\delta p(\Lambda,s)$-closed and $A \subseteq V \in \delta p(\Lambda,s)O(X,\tau)$. Then, $A^{\delta p(\Lambda,s)} \subseteq V$. Let $x \in \delta p(\Lambda,s)Fr(V)$. Since $V \in \delta p(\Lambda,s)O(X,\tau)$, we have $\delta p(\Lambda,s)Fr(V) = V^{\delta p(\Lambda,s)} - V$. Thus, $x \notin V$ and hence $x \notin A^{\delta p(\Lambda,s)}$. Therefore, $x \in [X-A]_{\delta p(\Lambda,s)}$. This shows that $\delta p(\Lambda,s)Fr(V) \subseteq [X-A]_{\delta p(\Lambda,s)}$.

Theorem 4. *Let (X,τ) be a topological space. For each $x \in X$, either $\{x\}$ is $\delta p(\Lambda,s)$-closed or g-$\delta p(\Lambda,s)$-open.*

Proof. Suppose that $\{x\}$ is not $\delta p(\Lambda,s)$-closed. Then, $X-\{x\}$ is not $\delta p(\Lambda,s)$-open and the only $\delta p(\Lambda,s)$-open set containing $X-\{x\}$ is X itself. Therefore, $[X-\{x\}]^{\delta p(\Lambda,s)} \subseteq X$. Thus, $X-\{x\}$ is g-$\delta p(\Lambda,s)$-closed and hence $\{x\}$ is g-$\delta p(\Lambda,s)$-open.

Theorem 5. *Let A be a subset of a topological space (X,τ). Then, A is g-$\delta p(\Lambda,s)$-open if and only if $F \subseteq A_{\delta p(\Lambda,s)}$ whenever $F \subseteq A$ and F is $\delta p(\Lambda,s)$-closed.*

Proof. Suppose that A is a g-$\delta p(\Lambda,s)$-open set. Let F be a $\delta p(\Lambda,s)$-closed set and $F \subseteq A$. Then, $X-A \subseteq X-F \in \delta p(\Lambda,s)O(X,\tau)$ and $X-A$ is g-$\delta p(\Lambda,s)$-closed. Thus, $X-A_{\delta p(\Lambda,s)} = [X-A]^{\delta p(\Lambda,s)} \subseteq X-F$ and hence $F \subseteq A_{\delta p(\Lambda,s)}$.

Conversely, let $X-A \subseteq U$ and $U \in \delta p(\Lambda,s)O(X,\tau)$. Then, $X-U \subseteq A$ and $X-U$ is $\delta p(\Lambda,s)$-closed. By the hypothesis, $X-U \subseteq A_{\delta p(\Lambda,s)}$ and hence

$$[X-A]^{\delta p(\Lambda,s)} = X-A_{\delta p(\Lambda,s)} \subseteq U.$$

Thus, $X-A$ is g-$\delta p(\Lambda,s)$-closed. This shows that A is g-$\delta p(\Lambda,s)$-open.

Lemma 4. *Let A be a subset of a topological space (X,τ). If $G \in \delta p(\Lambda,s)O(X,\tau)$ and $A \cap G = \emptyset$, then $A^{\delta p(\Lambda,s)} \cap G = \emptyset$.*

Theorem 6. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is g-$\delta p(\Lambda,s)$-closed.

(2) $A^{\delta p(\Lambda,s)} - A$ contains no nonempty $\delta p(\Lambda,s)$-closed set.

(3) $A^{\delta p(\Lambda,s)} - A$ is g-$\delta p(\Lambda,s)$-open.

Proof. (1) $\Rightarrow$ (2): This follows from Theorem 1.

(2) $\Rightarrow$ (3): Let F be a $\delta p(\Lambda,s)$-closed set and $F \subseteq A^{\delta p(\Lambda,s)} - A$. By (2), we have $F = \emptyset$ and $F \subseteq [A^{\delta p(\Lambda,s)} - A]_{\delta p(\Lambda,s)}$. It follows from Theorem 5 that $A^{\delta p(\Lambda,s)} - A$ is g-$\delta p(\Lambda,s)$-open.

(3) $\Rightarrow$ (1): Suppose that $A \subseteq U$ and $U \in \delta p(\Lambda, s)O(X, \tau)$. Then,

$$A^{\delta p(\Lambda,s)} - U \subseteq A^{\delta p(\Lambda,s)} - A.$$

By (3), we have $A^{\delta p(\Lambda,s)} - A$ is g-$\delta p(\Lambda, s)$-open. Since $A^{\delta p(\Lambda,s)} - U$ is $\delta p(\Lambda, s)$-closed, by Theorem 5, $A^{\delta p(\Lambda,s)} - U \subseteq [A^{\delta p(\Lambda,s)} - A]_{\delta p(\Lambda,s)} = \emptyset$. Thus, $A^{\delta p(\Lambda,s)} \subseteq U$ and hence A is g-$\delta p(\Lambda, s)$-closed. Now, the proof of $[A^{\delta p(\Lambda,s)} - A]_{\delta p(\Lambda,s)} = \emptyset$ is given as follows. Suppose that $[A^{\delta p(\Lambda,s)} - A]_{\delta p(\Lambda,s)} \neq \emptyset$. Then, there exists $x \in [A^{\delta p(\Lambda,s)} - A]_{\delta p(\Lambda,s)}$ and hence there exists $G \in \delta p(\Lambda, s)O(X, \tau)$ such that $x \in G \subseteq A^{\delta p(\Lambda,s)} - A$. Since $G \subseteq X - A$, we have $G \cap A = \emptyset$, by Lemma 4, $G \cap A^{\delta p(\Lambda,s)} = \emptyset$ and hence $G \subseteq X - A^{\delta p(\Lambda,s)}$. Thus, $G \subseteq [X - A^{\delta p(\Lambda,s)}] \cap A^{\delta p(\Lambda,s)} = \emptyset$. This is a contradiction.

Theorem 7. *A subset A of a topological space (X, τ) is g-$\delta p(\Lambda, s)$-closed if and only if $F \cap A^{\delta p(\Lambda,s)} = \emptyset$ whenever $A \cap F = \emptyset$ and F is $\delta p(\Lambda, s)$-closed.*

Proof. Suppose that A is a $\delta p(\Lambda, s)$-closed set. Let F be a $\delta p(\Lambda, s)$-closed set and $A \cap F = \emptyset$. Then, $A \subseteq X - F \in \delta p(\Lambda, s)O(X, \tau)$ and $A^{\delta p(\Lambda,s)} \subseteq X - F$. Thus,

$$F \cap A^{\delta p(\Lambda,s)} = \emptyset.$$

Conversely, let $A \subseteq U$ and $U \in \delta p(\Lambda, s)O(X, \tau)$. Then, $A \cap (X - U) = \emptyset$ and $X - U$ is $\delta p(\Lambda, s)$-closed. By the hypothesis, $(X - U) \cap A^{\delta p(\Lambda,s)} = \emptyset$ and hence $A^{\delta p(\Lambda,s)} \subseteq U$. Thus, A is g-$\delta p(\Lambda, s)$-closed.

Theorem 8. *A subset A of a topological space (X, τ) is g-$\delta p(\Lambda, s)$-closed if and only if $A \cap \{x\}^{\delta p(\Lambda,s)} \neq \emptyset$ for every $x \in A^{\delta p(\Lambda,s)}$.*

Proof. Let A be a g-$\delta p(\Lambda, s)$-closed set and suppose that there exists $x \in A^{\delta p(\Lambda,s)}$ such that $A \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$. Thus, $A \subseteq X - \{x\}^{\delta p(\Lambda,s)}$ and hence $A^{\delta p(\Lambda,s)} \subseteq X - \{x\}^{\delta p(\Lambda,s)}$. Therefore, $x \notin A^{\delta p(\Lambda,s)}$, which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let U be any $\delta p(\Lambda, s)$-open set containing A. Let $x \in A^{\delta p(\Lambda,s)}$. By the hypothesis, $A \cap A^{\delta p(\Lambda,s)} \neq \emptyset$, so there exists $y \in A \cap \{x\}^{\delta p(\Lambda,s)}$ and hence $y \in A \subseteq U$. Thus, $\{x\} \cap U \neq \emptyset$. Therefore, $x \in U$, which implies that $A^{\delta p(\Lambda,s)} \subseteq U$. This shows that A is g-$\delta p(\Lambda, s)$-closed.

Corollary 3. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is g-$\delta p(\Lambda, s)$-open.

(2) $A - A_{\delta p(\Lambda,s)}$ does not contain any nonempty $\delta p(\Lambda, s)$-closed set.

(3) $(X - A) \cap \{x\}^{\delta p(\Lambda,s)} \neq \emptyset$ for every $x \in A - A_{\delta p(\Lambda,s)}$.

Theorem 9. *For a topological space (X, τ), the following properties are equivalent:*

(1) For every $\delta p(\Lambda, s)$-open set U of X, $U^{\delta p(\Lambda,s)} \subseteq U$.

(2) Every subset of X is g-$\delta p(\Lambda, s)$-closed.

Proof. (1) ⇒ (2): Let A be any subset of X and $A \subseteq U \in \delta(\Lambda, s)O(X, \tau)$. By (1), $U^{\delta p(\Lambda,s)} \subseteq U$ and hence $A^{\delta p(\Lambda,s)} \subseteq U^{\delta p(\Lambda,s)} \subseteq U$. Thus, A is g-$\delta p(\Lambda, s)$-closed.

(2) ⇒ (1): Let $U \in \delta p(\Lambda, s)O(X, \tau)$. By (2), U is g-$\delta p(\Lambda, s)$-closed and hence

$$U^{\delta p(\Lambda,s)} \subseteq U.$$

Theorem 10. *A subset A of a topological space (X, τ) is g-$\delta p(\Lambda, s)$-open if and only if $U = X$ whenever U is $\delta p(\Lambda, s)$-open and $(X - A) \cap A_{\delta p(\Lambda,s)} \subseteq U$.*

Proof. Suppose that A is g-$\delta p(\Lambda, s)$-open and $U \in \delta p(\Lambda, s)O(X, \tau)$ such that

$$(X - A) \cap A_{\delta p(\Lambda,s)} \subseteq U.$$

Thus, $X - U \subseteq [X - A_{\delta p(\Lambda,s)}] \cap A$ and hence $X - U \subseteq [X - A]^{\delta p(\Lambda,s)} - (X - A)$. Since $X - A$ is g-$\delta p(\Lambda, s)$-closed and $X - U$ is $\delta p(\Lambda, s)$-closed, by Theorem 1, $X - U = \emptyset$. This shows that $X = U$.

Conversely, suppose that $F \subseteq A$ and F is $\delta p(\Lambda, s)$-closed. By Lemma 3,

$$(X - A) \cup A_{\delta p(\Lambda,s)} \subseteq (X - F) \cup A_{\delta p(\Lambda,s)} \in \delta p(\Lambda, s)O(X, \tau).$$

By the hypothesis, we have $X = (X - F) \cup A_{\delta p(\Lambda,s)}$ and hence

$$F = F \cap [(X - F) \cup A_{\delta p(\Lambda,s)}] = F \cap A_{\delta p(\Lambda,s)} \subseteq A_{\delta p(\Lambda,s)}.$$

It follows from Theorem 5 that A is g-$\delta p(\Lambda, s)$-open.

Theorem 11. *Let A be a subset of a topological space (X, τ). If A is g-$\delta p(\Lambda, s)$-open and $A_{\delta p(\Lambda,s)} \subseteq B \subseteq A$, then B is g-$\delta p(\Lambda, s)$-open.*

Proof. We have $X - A \subseteq X - B \subseteq X - A_{\delta p(\Lambda,s)} = [X - A]^{\delta p(\Lambda,s)}$. Since $X - A$ is g-$\delta p(\Lambda, s)$-closed, it follows from Theorem 2 that $X - B$ is g-$\delta p(\Lambda, s)$-closed and hence B is g-$\delta p(\Lambda, s)$-open.

Definition 5. *A subset A of a topological space (X, τ) is said to be locally $\delta p(\Lambda, s)$-closed if $A = U \cap F$, where $U \in \delta p(\Lambda, s)O(X, \tau)$ and F is a $\delta p(\Lambda, s)$-closed set.*

Lemma 5. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is locally $\delta p(\Lambda, s)$-closed;

(2) $A = U \cap A^{\delta p(\Lambda,s)}$ for some $U \in \delta p(\Lambda, s)O(X, \tau)$;

(3) $A^{\delta p(\Lambda,s)} - A$ *is* $\delta p(\Lambda, s)$*-closed;*

(4) $[A \cup (X - A^{\delta p(\Lambda,s)})] \in \delta p(\Lambda, s)O(X, \tau)$*;*

(5) $A \subseteq [A \cup [X - A^{\delta p(\Lambda,s)}]]_{\delta p(\Lambda,s)}$.

Proof. (1) ⇒ (2): Let $A = U \cap F$, where $U \in \delta p(\Lambda, s)O(X, \tau)$ and F is $\delta p(\Lambda, s)$-closed. Since $A \subseteq F$, we have $A^{\delta p(\Lambda,s)} \subseteq F^{\delta p(\Lambda,s)} = F$. Since $A \subseteq U$, $A \subseteq U \cap A^{\delta p(\Lambda,s)} \subseteq U \cap F = A$. Thus, $A = U \cap A^{\delta p(\Lambda,s)}$.

(2) ⇒ (3): Suppose that $A = U \cap A^{\delta p(\Lambda,s)}$ for some $U \in \delta p(\Lambda, s)O(X, \tau)$. Then, we have $A^{\delta p(\Lambda,s)} - A = (X - [U \cap A^{\delta p(\Lambda,s)}]) \cap A^{\delta p(\Lambda,s)} = (X - U) \cap A^{\delta p(\Lambda,s)}$. Thus, $A^{\delta p(\Lambda,s)} - A$ is $\delta p(\Lambda, s)$-closed.

(3) ⇒ (4): Since $X - (A^{\delta p(\Lambda,s)} - A) = (X - A^{\delta p(\Lambda,s)}) \cup A$ and by (3), $A \cup (X - A^{\delta p(\Lambda,s)})$ is $\delta p(\Lambda, s)$-open.

(4) ⇒ (5): By (4), $A \subseteq A \cup (X - A^{\delta p(\Lambda,s)}) = [A \cup (X - A^{\delta p(\Lambda,s)})]_{\delta p(\Lambda,s)}$.

(5) ⇒ (1): We put $U = [A \cup (X - A^{\delta p(\Lambda,s)})]_{\delta p(\Lambda,s)}$. Then, U is $\delta p(\Lambda, s)$-open and $A = A \cap U \subseteq U \cap A^{\delta p(\Lambda,s)} \subseteq [A \cup (X - A^{\delta p(\Lambda,s)})] \cap A^{\delta p(\Lambda,s)} = A \cap A^{\delta p(\Lambda,s)} = A$. Thus, $A = U \cap A^{\delta p(\Lambda,s)}$, where $U \in \delta p(\Lambda, s)O(X, \tau)$ and $A^{\delta p(\Lambda,s)}$ is $\delta p(\Lambda, s)$-closed. This shows that A is locally $\delta p(\Lambda, s)$-closed.

Theorem 12. *A subset* A *of a topological space* (X, τ) *is* $\delta p(\Lambda, s)$*-closed if and only if* A *is locally* $\delta p(\Lambda, s)$*-closed and g-*$\delta p(\Lambda, s)$*-closed.*

Proof. Let A be a $\delta p(\Lambda, s)$-closed set. By Theorem 2, A is g-$\delta p(\Lambda, s)$-closed. Since X is $\delta p(\Lambda, s)$-open and $A = X \cap A$, A is locally $\delta p(\Lambda, s)$-closed.

Conversely, suppose that A is locally $\delta p(\Lambda, s)$-closed and g-$\delta p(\Lambda, s)$-closed. Since A is locally $\delta p(\Lambda, s)$-closed, by Lemma 5, $A \subseteq [A \cup [X - A^{\delta p(\Lambda,s)}]]_{\delta p(\Lambda,s)}$. Since

$$[A \cup [X - A^{\delta p(\Lambda,s)}]]_{\delta p(\Lambda,s)} \in \delta p(\Lambda, s)O(X, \tau)$$

and A is g-$\delta p(\Lambda, s)$-closed, we have $A^{\delta p(\Lambda,s)} \subseteq [A \cup [X - A^{\delta p(\Lambda,s)}]]_{\delta p(\Lambda,s)} \subseteq A \cup [X - A^{\delta p(\Lambda,s)}]$ and hence $A^{\delta p(\Lambda,s)} = A$. Thus, by Lemma 1, A is $\delta p(\Lambda, s)$-closed.

Definition 6. [17] *Let* A *be a subset of a topological space* (X, τ). *A subset* $\delta p(\Lambda, s)Ker(A)$ *is defined as follows:* $\delta p(\Lambda, s)Ker(A) = \cap\{U \mid A \subseteq U, U \in \delta p(\Lambda, s)O(X, \tau)\}$.

Lemma 6. *For subsets* A, B *of a topological space* (X, τ)*, the following properties hold:*

(1) $A \subseteq \delta p(\Lambda, s)Ker(A)$.

(2) If $A \subseteq B$*, then* $\delta p(\Lambda, s)Ker(A) \subseteq \delta p(\Lambda, s)Ker(B)$.

(3) $\delta p(\Lambda, s)Ker[\delta p(\Lambda, s)Ker(A)] = \delta p(\Lambda, s)Ker(A)$.

(4) If A *is* $\delta p(\Lambda, s)$*-open,* $\delta p(\Lambda, s)Ker(A) = A$.

A subset N_x of a topological space (X, τ) is said to be a $\delta p(\Lambda, s)$-*neighbourhood* [16] of a point $x \in X$ if there exists a $\delta p(\Lambda, s)$-open set U such that $x \in U \subseteq N_x$.

Lemma 7. *A subset A of a topological space (X, τ) is $\delta p(\Lambda, s)$-open in X if and only if A is a $\delta p(\Lambda, s)$-neighbourhood of each point of A.*

Definition 7. *Let (X, τ) be a topological space and $x \in X$. A subset $\langle x\rangle_{\delta p(\Lambda,s)}$ is defined as follows: $\langle x\rangle_{\delta p(\Lambda,s)} = \delta p(\Lambda, s)Ker(\{x\}) \cap \{x\}^{\delta p(\Lambda,s)}$.*

Theorem 13. *For a topological space (X, τ), the following properties hold:*

(1) $\Lambda_{\delta p(\Lambda,s)}(A) = \{x \in X \mid A \cap \{x\}^{\delta p(\Lambda,s)} \neq \emptyset\}$ for each subset A of X.

(2) For each $x \in X$, $\delta p(\Lambda, s)Ker(\langle x\rangle_{\delta p(\Lambda,s)}) = \delta p(\Lambda, s)Ker(\{x\})$.

(3) For each $x \in X$, $(\langle x\rangle_{\delta p(\Lambda,s)})^{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)}$.

(4) If U is $\delta p(\Lambda, s)$-open in X and $x \in U$, then $\langle x\rangle_{\delta p(\Lambda,s)} \subseteq U$.

(5) If F is $\delta p(\Lambda, s)$-closed in X and $x \in F$, then $\langle x\rangle_{\delta p(\Lambda,s)} \subseteq F$.

Proof. (1) Suppose that $A \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$. Then, $x \notin X - \{x\}^{\delta p(\Lambda,s)}$ which is a $\delta p(\Lambda, s)$-open set containing A. Thus, $x \notin \delta p(\Lambda, s)Ker(A)$ and hence

$$\delta p(\Lambda, s)Ker(A) \subseteq \{x \in X \mid A \cap \{x\}^{\delta p(\Lambda,s)} \neq \emptyset\}.$$

Next, let $x \in X$ such that $A \cap \{x\}^{\delta p(\Lambda,s)} \neq \emptyset$ and suppose that $x \notin \delta p(\Lambda, s)Ker(A)$. Then, there exists a $\delta p(\Lambda, s)$-open set U containing A and $x \notin U$. Let $y \in A \cap \{x\}^{\delta p(\Lambda,s)}$. Therefore, U is a $\delta p(\Lambda, s)$-neighbourhood of y which does not contain x. By this contradiction $x \in \delta p(\Lambda, s)Ker(A)$.

(2) Let $x \in X$. Then, we have $\{x\} \subseteq \{x\}^{\delta p(\Lambda,s)} \cap \delta p(\Lambda, s)Ker(\{x\}) = \langle x\rangle_{\delta p(\Lambda,s)}$. By Lemma 6, we obtain $\delta p(\Lambda, s)Ker(\{x\}) \subseteq \delta p(\Lambda, s)Ker(\langle x\rangle_{\delta p(\Lambda,s)})$. Next, we show the opposite implication. Suppose that $y \notin \delta p(\Lambda, s)Ker(\{x\})$. Then, there exists a $\delta p(\Lambda, s)$-open set V such that $x \in V$ and $y \notin V$. Since

$$\langle x\rangle_{\delta p(\Lambda,s)} \subseteq \delta p(\Lambda, s)Ker(\{x\}) \subseteq \delta p(\Lambda, s)Ker(V) = V,$$

we have $\delta p(\Lambda, s)Ker(\langle x\rangle_{\delta p(\Lambda,s)}) \subseteq V$. Since $y \notin V$, $y \notin \delta p(\Lambda, s)Ker(\langle x\rangle_{\delta p(\Lambda,s)})$. Thus, $\delta p(\Lambda, s)Ker(\langle x\rangle_{\delta p(\Lambda,s)}) \subseteq \delta p(\Lambda, s)Ker(\{x\})$ and hence

$$\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\langle x\rangle_{\delta p(\Lambda,s)}).$$

(3) By the definition of $\langle x\rangle_{\delta p(\Lambda,s)}$, we have $\{x\} \subseteq \langle x\rangle_{\delta p(\Lambda,s)}$ and

$$\{x\}^{\delta p(\Lambda,s)} \subseteq (\langle x\rangle_{\delta p(\Lambda,s)})^{\delta p(\Lambda,s)}$$

by Lemma 1. On the other hand, we have $\langle x\rangle_{\delta p(\Lambda,s)} \subseteq \{x\}^{\delta p(\Lambda,s)}$ and

$$(\langle x\rangle_{\delta p(\Lambda,s)})^{\delta p(\Lambda,s)} \subseteq (\{x\}^{\delta p(\Lambda,s)})^{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)}.$$

Thus, $(\langle x\rangle_{\delta p(\Lambda,s)})^{\delta p(\Lambda,s)} \subseteq \{x\}^{\delta p(\Lambda,s)}$.

(4) Since $x \in U$ and U is a $\delta p(\Lambda, s)$-open set, we have $\delta p(\Lambda, s)Ker(\{x\}) \subseteq U$. Thus, $\langle x\rangle_{\delta p(\Lambda,s)} \subseteq U$.

(5) Since $x \in F$ and F is a $\delta p(\Lambda, s)$-closed set,

$$\langle x\rangle_{\delta p(\Lambda,s)} = \{x\}^{\delta p(\Lambda,s)} \cap \delta p(\Lambda, s)Ker(\{x\}) \subseteq \{x\}^{\delta p(\Lambda,s)} \subseteq F^{\delta p(\Lambda,s)} = F.$$

Theorem 14. *For any points x and y in a topological space (X, τ), the following properties are equivalent:*

(1) $\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$.

(2) $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$.

Proof. (1) ⇒ (2): Suppose that $\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$. Then, there exists a point $z \in X$ such that $z \in \delta p(\Lambda, s)Ker(\{x\})$ and $z \notin \delta p(\Lambda, s)Ker(\{y\})$ or

$$z \in \delta p(\Lambda, s)Ker(\{y\})$$

and $z \notin \delta p(\Lambda, s)Ker(\{x\})$. We prove only the first case being the second analogous. From $z \in \delta p(\Lambda, s)Ker(\{x\})$ it follows that $\{x\} \cap \{z\}^{\delta p(\Lambda,s)} \neq \emptyset$ which implies $x \in \{z\}^{\delta p(\Lambda,s)}$. By $z \notin \delta p(\Lambda, s)Ker(\{y\})$, we have $\{y\} \cap \{z\}^{\delta p(\Lambda,s)} = \emptyset$. Since $x \in \{z\}^{\delta p(\Lambda,s)}$,

$$\{x\}^{\delta p(\Lambda,s)} \subseteq \{z\}^{\delta p(\Lambda,s)}$$

and $\{y\} \cap \{x\}^{\delta p(\Lambda,s)} = \emptyset$. Therefore, $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. Thus,

$$\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$$

implies that $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$.

(2) ⇒ (1): Suppose that $\{x\}^{\delta p(\Lambda,s)} \neq \{y\}^{\delta p(\Lambda,s)}$. There exists a point $z \in X$ such that $z \in \{x\}^{\delta p(\Lambda,s)}$ and $z \notin \{y\}^{\delta p(\Lambda,s)}$ or $z \in \{y\}^{\delta p(\Lambda,s)}$ and $z \notin \{x\}^{\delta p(\Lambda,s)}$. We prove only the first case being the second analogous. It follows that there exists a $\delta p(\Lambda, s)$-open set containing z and therefore x but not y, namely, $y \notin \delta p(\Lambda, s)Ker(\{x\})$ and thus $\delta p(\Lambda, s)Ker(\{x\}) \neq \delta p(\Lambda, s)Ker(\{y\})$.

Theorem 15. *Let (X, τ) be a topological space and $x, y \in X$. Then, the following properties hold:*

(1) $y \in \delta p(\Lambda, s)Ker(\{x\})$ *if and only if* $x \in \{y\}^{\delta p(\Lambda,s)}$.

(2) $\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\{y\})$ *if and only if* $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$.

Proof. (1) Let $x \notin \{y\}^{\delta p(\Lambda,s)}$. Then, there exists $U \in \delta p(\Lambda, s)O(X, \tau)$ such that $x \in U$ and $y \notin U$. Thus, $y \notin \delta p(\Lambda, s)Ker(\{x\})$. The converse is similarly shown.

(2) Suppose that $\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\{y\})$ for any $x, y \in X$. Since

$$x \in \delta p(\Lambda, s)Ker(\{x\}),$$

$x \in \delta p(\Lambda, s)Ker(\{y\})$, by (1), $y \in \{x\}^{\delta p(\Lambda,s)}$. By Lemma 1, $\{y\}^{\delta p(\Lambda,s)} \subseteq \{x\}^{\delta p(\Lambda,s)}$. Similarly, we have $\{x\}^{\delta p(\Lambda,s)} \subseteq \{y\}^{\delta p(\Lambda,s)}$ and hence $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$.

Conversely, suppose that $\{x\}^{\delta p(\Lambda,s)} = \{y\}^{\delta p(\Lambda,s)}$. Since $x \in \{x\}^{\delta p(\Lambda,s)}$, $x \in \{y\}^{\delta p(\Lambda,s)}$ and by (1), $y \in \delta p(\Lambda, s)Ker(\{x\})$. By Lemma 6,

$$\delta p(\Lambda, s)Ker(\{y\}) \subseteq \delta p(\Lambda, s)Ker(\delta p(\Lambda, s)Ker(\{x\})) = \delta p(\Lambda, s)Ker(\{x\}).$$

Similarly, we have $\delta p(\Lambda, s)Ker(\{x\}) \subseteq \delta p(\Lambda, s)Ker(\{y\})$ and hence

$$\delta p(\Lambda, s)Ker(\{x\}) = \delta p(\Lambda, s)Ker(\{y\}).$$

Definition 8. *A subset A of a topological space (X, τ) is called a $\Lambda_{\delta p(\Lambda,s)}$-set if $A = \delta p(\Lambda, s)Ker(A)$.*

The family of all $\Lambda_{\delta p(\Lambda,s)}$-sets of a topological space (X, τ) is denoted by $\Lambda_{\delta p(\Lambda,s)}(X, \tau)$ (or simply $\Lambda_{\delta p(\Lambda,s)}$).

Definition 9. *A subset A of a topological space (X, τ) is called a generalized $\Lambda_{\delta p(\Lambda,s)}$-set (briefly g-$\Lambda_{\delta p(\Lambda,s)}$-set) if $\delta p(\Lambda, s)Ker(A) \subseteq F$ whenever $A \subseteq F$ and F is a $\delta p(\Lambda, s)$-closed set.*

Definition 10. *A topological space (X, τ) is called a $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-space if every g-$\delta p(\Lambda, s)$-closed set of X is $\delta p(\Lambda, s)$-closed.*

Lemma 8. *For a topological space (X, τ), the following properties hold:*

(1) For each $x \in X$, the singleton $\{x\}$ is $\delta p(\Lambda, s)$-closed or $X - \{x\}$ is g-$\delta p(\Lambda, s)$-closed.

(2) For each $x \in X$, the singleton $\{x\}$ is $\delta p(\Lambda, s)$-open or $X - \{x\}$ is a g-$\Lambda_{\delta p(\Lambda,s)}$-set.

Proof. (1) Let $x \in X$ and the singleton $\{x\}$ be not $\delta p(\Lambda, s)$-closed. Then, $X - \{x\}$ is not $\delta p(\Lambda, s)$-open and X is the only $\delta p(\Lambda, s)$-open set which contains $X - \{x\}$ and hence $X - \{x\}$ is g-$\delta p(\Lambda, s)$-closed.

(2) Let $x \in X$ and the singleton $\{x\}$ be not $\delta p(\Lambda, s)$-open. Then, $X - \{x\}$ is not $\delta p(\Lambda, s)$-closed and X is the only $\delta p(\Lambda, s)$-closed set which contains $X - \{x\}$ and hence $X - \{x\}$ is a g-$\Lambda_{\delta p(\Lambda,s)}$-set.

Theorem 16. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is a $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-space.

(2) For each $x \in X$, the singleton $\{x\}$ is $\delta p(\Lambda, s)$-open or $\delta p(\Lambda, s)$-closed.

(3) Every g-$\Lambda_{\delta p(\Lambda,s)}$-set is a $\Lambda_{\delta p(\Lambda,s)}$-set.

Proof. (1) ⇒ (2): By Lemma 8, for each $x \in X$, the singleton $\{x\}$ is $\delta p(\Lambda, s)$-closed or $X - \{x\}$ is g-$\delta p(\Lambda, s)$-closed. Since (X, τ) is a $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-space, $X - \{x\}$ is $\delta p(\Lambda, s)$-closed and hence $\{x\}$ is $\delta p(\Lambda, s)$-open in the latter case. Thus, the singleton $\{x\}$ is $\delta p(\Lambda, s)$-open or $\delta p(\Lambda, s)$-closed.

(2) ⇒ (3): Suppose that there exists a g-$\Lambda_{\delta p(\Lambda,s)}$-set A which is not a $\Lambda_{\delta p(\Lambda,s)}$-set. There exists $x \in \delta p(\Lambda, s)Ker(A)$ such that $x \notin A$. In case the singleton $\{x\}$ is $\delta p(\Lambda, s)$-open, $A \subseteq X - \{x\}$ and $X - \{x\}$ is $\delta p(\Lambda, s)$-closed. Since A is a g-$\Lambda_{\delta p(\Lambda,s)}$-set,

$$\delta p(\Lambda, s)Ker(A) \subseteq X - \{x\}.$$

This is a contradiction. In case the singleton $\{x\}$ is $\delta p(\Lambda, s)$-closed, $A \subseteq X - \{x\}$ and $X - \{x\}$ is $\delta p(\Lambda, s)$-open. By Lemma 6,

$$\delta p(\Lambda, s)Ker(A) \subseteq \delta p(\Lambda, s)Ker(X - \{x\}) = X - \{x\}.$$

This is a contradiction. Thus, every g-$\Lambda_{\delta p(\Lambda,s)}$-set is a $\Lambda_{\delta p(\Lambda,s)}$-set.

(3) ⇒ (1): Suppose that (X, τ) is not a $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-space. Then, there exists a g-$\delta p(\Lambda, s)$-closed set A which is not $\delta p(\Lambda, s)$-closed. Since A is not $\delta p(\Lambda, s)$-closed, there exists a point $x \in A^{\delta p(\Lambda,s)}$ such that $x \notin A$. By Lemma 8, the singleton $\{x\}$ is $\delta p(\Lambda, s)$-open or $X - \{x\}$ is a $\Lambda_{\delta p(\Lambda,s)}$-set. (a) In case $\{x\}$ is $\delta p(\Lambda, s)$-open, since $x \in A^{\delta p(\Lambda,s)}$, $\{x\} \cap A \neq \emptyset$ and $x \in A$. This is a contradiction. (b) In case $X - \{x\}$ is a $\Lambda_{\delta p(\Lambda,s)}$-set, if $\{x\}$ is not $\delta p(\Lambda, s)$-closed, $X - \{x\}$ is not $\delta p(\Lambda, s)$-open and $\delta p(\Lambda, s)Ker(X - \{x\}) = X$. Thus, $X - \{x\}$ is not a $\Lambda_{\delta p(\Lambda,s)}$-set. This contradicts (3). If $\{x\}$ is $\delta p(\Lambda, s)$-closed,

$$A \subseteq X - \{x\} \in \delta p(\Lambda, s)O(X, \tau)$$

and A is g-$\delta p(\Lambda, s)$-closed. Hence, we have $A^{\delta p(\Lambda,s)} \subseteq X - \{x\}$. This contradicts that $x \in A^{\delta p(\Lambda,s)}$. This shows that (X, τ) is a $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-space.

Definition 11. *A topological space (X, τ) is said to be $\delta p(\Lambda, s)$-normal if for any pair of disjoint $\delta p(\Lambda, s)$-closed sets F and H, there exist disjoint $\delta p(\Lambda, s)$-open sets U and V such that $F \subseteq U$ and $H \subseteq V$.*

Lemma 9. *Let (X, τ) be a topological space. If U is $\delta p(\Lambda, s)$-open in X, then*

$$U^{\delta p(\Lambda,s)} \cap A \subseteq [U \cap A]^{\delta p(\Lambda,s)}$$

for every subset A of X.

Theorem 17. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is $\delta p(\Lambda, s)$-normal.

(2) For every pair of $\delta p(\Lambda,s)$-open sets U and V whose union is X, there exist $\delta p(\Lambda,s)$-closed sets F and H such that $F\subseteq U$, $H\subseteq V$ and $F\cup H=X$.

(3) For every $\delta p(\Lambda,s)$-closed set F and every $\delta p(\Lambda,s)$-open set G containing F, there exists a $\delta p(\Lambda,s)$-open set U such that $F\subseteq U\subseteq U^{\delta p(\Lambda,s)}\subseteq G$.

(4) For every pair of disjoint $\delta p(\Lambda,s)$-closed sets F and H, there exist disjoint $\delta p(\Lambda,s)$-open sets U and V such that $F\subseteq U$ and $H\subseteq V$ and $U^{\delta p(\Lambda,s)}\cap V^{\delta p(\Lambda,s)}=\emptyset$.

Proof. (1) $\Rightarrow$ (2): Let U and V be any pair of $\delta p(\Lambda,s)$-open sets in X such that $X=U\cup V$. Then, $X-U$ and $X-V$ are disjoint $\delta p(\Lambda,s)$-closed sets. Since (X,τ) is $\delta p(\Lambda,s)$-normal, there exist disjoint $\delta p(\Lambda,s)$-open sets G and W such that $X-U\subseteq G$ and $X-V\subseteq W$. Put $F=X-G$ and $H=X-W$. Then, F and H are $\delta p(\Lambda,s)$-closed sets such that $F\subseteq U$, $H\subseteq V$ and $F\cup H=X$.

(2) $\Rightarrow$ (3): Let F be a $\delta p(\Lambda,s)$-closed set and G be a $\delta p(\Lambda,s)$-open set containing F. Then, $X-F$ and G are $\delta p(\Lambda,s)$-open sets whose union is X. Then by (2), there exist $\delta p(\Lambda,s)$-closed sets M and N such that $M\subseteq X-F$, $N\subseteq G$ and $M\cup N=X$. Then, $F\subseteq X-M$, $X-G\subseteq X-N$ and $(X-M)\cap(X-N)=\emptyset$. Put $U=X-M$ and $V=X-N$. Then, U and V are disjoint $\delta p(\Lambda,s)$-open sets such that $F\subseteq U\subseteq X-V\subseteq G$. As $X-V$ is a $\delta p(\Lambda,s)$-closed set, we have $U^{\delta p(\Lambda,s)}\subseteq X-V$ and $F\subseteq U\subseteq U^{\delta p(\Lambda,s)}\subseteq G$.

(3) $\Rightarrow$ (4): Let F and H be two disjoint $\delta p(\Lambda,s)$-closed sets of X. Then, $F\subseteq X-H$ and $X-H$ is $\delta p(\Lambda,s)$-open, by (3), there exists a $\delta p(\Lambda,s)$-open set U of X such that $F\subseteq U\subseteq U^{\delta p(\Lambda,s)}\subseteq X-H$. Put $V=X-U^{\delta p(\Lambda,s)}$. Then, U and V are disjoint $\delta p(\Lambda,s)$-open sets of X such that $F\subseteq U$, $H\subseteq V$ and $U^{\delta p(\Lambda,s)}\cap V^{\delta p(\Lambda,s)}=\emptyset$.

(4) $\Rightarrow$ (1): The proof is obvious.

Theorem 18. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) is $\delta p(\Lambda,s)$-normal.

(2) For any pair of disjoint $\delta p(\Lambda,s)$-closed sets F and H, there exist disjoint g-$\delta p(\Lambda,s)$-open sets U and V such that $F\subseteq U$ and $H\subseteq V$.

(3) For each $\delta p(\Lambda,s)$-closed set F and each $\delta p(\Lambda,s)$-open set G containing F, there exists a g-$\delta p(\Lambda,s)$-open set U such that $F\subseteq U\subseteq U^{\delta p(\Lambda,s)}\subseteq G$.

(4) For each $\delta p(\Lambda,s)$-closed set F and each g-$\delta p(\Lambda,s)$-open set G containing F, there exists a $\delta p(\Lambda,s)$-open set U such that $F\subseteq U\subseteq U^{\delta p(\Lambda,s)}\subseteq G_{\delta p(\Lambda,s)}$.

(5) For each $\delta p(\Lambda,s)$-closed set F and each g-$\delta p(\Lambda,s)$-open set G containing F, there exists a g-$\delta p(\Lambda,s)$-open set U such that $F\subseteq U\subseteq U^{\delta p(\Lambda,s)}\subseteq G_{\delta p(\Lambda,s)}$.

(6) For each g-$\delta p(\Lambda,s)$-closed set F and each $\delta p(\Lambda,s)$-open set G containing F, there exists a $\delta p(\Lambda,s)$-open set U such that $F^{\delta p(\Lambda,s)}\subseteq U\subseteq U^{\delta p(\Lambda,s)}\subseteq G$.

(7) For each g-$\delta p(\Lambda,s)$-closed set F and each $\delta p(\Lambda,s)$-open set G containing F, there exists a g-$\delta p(\Lambda,s)$-open set U such that $F^{\delta p(\Lambda,s)}\subseteq U\subseteq U^{\delta p(\Lambda,s)}\subseteq G$.

Proof. (1) ⇒ (2): The proof is obvious.

(2) ⇒ (3): Let F be a $\delta p(\Lambda, s)$-closed set and G be a $\delta p(\Lambda, s)$-open set containing F. Then, F and $X - G$ are two disjoint $\delta p(\Lambda, s)$-closed sets. Hence by (2), there exist disjoint g-$\delta p(\Lambda, s)$-open sets U and V of X such that $F \subseteq U$ and $X - G \subseteq V$. Since V is g-$\delta p(\Lambda, s)$-open and $X - G$ is $\delta p(\Lambda, s)$-closed, by Theorem 5, $X - G \subseteq V_{\delta p(\Lambda,s)}$. Thus, $[X - V]^{\delta p(\Lambda,s)} = X - V_{\delta p(\Lambda,s)} \subseteq G$ and hence $F \subseteq U \subseteq U^{\delta p(\Lambda,s)} \subseteq G$.

(3) ⇒ (5): Let F be a $\delta p(\Lambda, s)$-closed set and G be a g-$\delta p(\Lambda, s)$-open set containing F. Since G is g-$\delta p(\Lambda, s)$-open and F is $\delta p(\Lambda, s)$-closed, by Theorem 5, $F \subseteq G_{\delta p(\Lambda,s)}$. Thus, by (3), there exists a g-$\delta p(\Lambda, s)$-open set U such that $F \subseteq U \subseteq U^{\delta p(\Lambda,s)} \subseteq G_{\delta p(\Lambda,s)}$.

(5) ⇒ (6): Let F be a g-$\delta p(\Lambda, s)$-closed set and G be a $\delta p(\Lambda, s)$-open set containing F. Then, we have $F^{\delta p(\Lambda,s)} \subseteq G$. Since G is g-$\delta p(\Lambda, s)$-open and $F^{\delta p(\Lambda,s)}$ is $\delta p(\Lambda, s)$-closed, by (5), there exists a g-$\delta p(\Lambda, s)$-open set U such that $F^{\delta p(\Lambda,s)} \subseteq U \subseteq U^{\delta p(\Lambda,s)} \subseteq G$. Since U is g-$\delta p(\Lambda, s)$-open and $F^{\delta p(\Lambda,s)} \subseteq U$, by Theorem 5, $F^{\delta p(\Lambda,s)} \subseteq U_{\delta p(\Lambda,s)}$. Put $V = U_{\delta p(\Lambda,s)}$. Then, V is $\delta p(\Lambda, s)$-open and $F^{\delta p(\Lambda,s)} \subseteq V \subseteq V^{\delta p(\Lambda,s)} = [U_{\delta p(\Lambda,s)}]^{\delta p(\Lambda,s)} \subseteq U^{\delta p(\Lambda,s)} \subseteq G$.

(6) ⇒ (4): Let F be a $\delta p(\Lambda, s)$-closed set and G be a g-$\delta p(\Lambda, s)$-open set containing F. Thus, by Theorem 5, $F^{\delta p(\Lambda,s)} = F \subseteq G_{\delta p(\Lambda,s)}$. Since F is g-$\delta p(\Lambda, s)$-closed and $G_{\delta p(\Lambda,s)}$ is $\delta p(\Lambda, s)$-open, by (6), there exists a $\delta p(\Lambda, s)$-open set U such that

$$F^{\delta p(\Lambda,s)} \subseteq U \subseteq U^{\delta p(\Lambda,s)} \subseteq G_{\delta p(\Lambda,s)}.$$

(4) ⇒ (5): The proof is obvious.

(6) ⇒ (7) and (7) ⇒ (3): The proofs are obvious.

(3) ⇒ (1): Let F and H be two disjoint $\delta p(\Lambda, s)$-closed sets of X. Then, F is a $\delta p(\Lambda, s)$-closed set and $X - H$ is a $\delta p(\Lambda, s)$-open set containing F, by (3), there exists a g-$\delta p(\Lambda, s)$-open set U such that $F \subseteq U \subseteq U^{\delta p(\Lambda,s)} \subseteq X - H$. Thus, by Theorem 5, $F \subseteq U_{\delta p(\Lambda,s)}$, $H \subseteq X - U^{\delta p(\Lambda,s)}$, where $U_{\delta p(\Lambda,s)}$ and $X - U^{\delta p(\Lambda,s)}$ are two disjoint $\delta p(\Lambda, s)$-open sets. This shows that (X, τ) is $\delta p(\Lambda, s)$-normal.

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