EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 997-1004
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On direct product of d-Algebras

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Abstract. The main aim of this work is to introduce and study the notions of ideal direct product d-algebras, d-ideal direct product d-algebras, sub-direct product d-algebras, edge direct product and positive implicative direct product d-algebras and investigate their characterizations.

2020 Mathematics Subject Classifications: 06F35

Key Words and Phrases: d-algebras, direct product d-algebras, ideal direct product d-algebras, d-ideal direct product d-algebras, ideal, edge direct product d-algebras, positive implicative, direct product d-algebra

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1. Introduction

The concept of d-algebras was first introduced by J. Neggers and H. S. Kim ([9]). A d-algebra $X = (X, *, 0)$ is an algebra of type $(2, 0)$, that is, a nonempty set X together with a binary operation $*$ and a constant 0 satisfying some axioms In [1], they introduced and investigated several relations between d-algebras and BCK-algebras and showed that the class of oriented digraphs corresponds in a simple way to the class of edge d-algebras and that arbitrary d-algebras also determine unique edge d-algebras in a natural manner. In 1999, J. Neggers, Y. B. Jun and H. S. Kim ([8]), introduced the notions of a d-subalgebra, d-ideal, and a $d*$-ideal in d-algebras, and investigated relations among them. Furthermore, they are able to define the ideal of a quotient d-algebra and to prove a fundamental theorem of d-morphisms for d-algebras as a consequence. S. S. Ahn and K. S. So ([1], defined left-regular maps on d-algebras.These mappings show behaviors reminiscent or homomorphisms on d-algebras. In particular, they have introduced the kernels, annihilators, co-annihilators and some of their properties for these mappings, especially in the setting of positive implicative d-algebras. The study of multipliers have been made by various researchers in the context of C*-algebras, rings and semigroups in ([6]). In 2012, M. A. Chaudhry and F. Ali ([3]) introduced the concept of a multiplier on d-algebra and obtain some properties of multipliers of d-algebras.

DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4738

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The concept of the direct product, was first defined in groups and obtained the properties that a direct product of groups is also a group. In 1999, J. Neggers and H. S. Kim ([9]) introduced the concept of a direct product of d-algebras, they investigate several relations between projection mappings and d-morphisms on a direct sum of edge d-algebras, In 2020, A. Setiani, S. Gemawati and L. Deswita ([10]) introduced the notions of a direct product of BP-algebra and some of related properties are investigated. Also, the notion of direct product of 0-commutative BP-algebra and BP-homomorphisms were studied. In 2022, C. Chanmanee, R. Chinram, R. Prasertpong, P. Julatha, and A. Iampan ([2]) gave the concept an external direct produc and a weak direct product of B-algebras and they provided several fundamental theorems of (anti-)B-homomorphisms in view of the external direct product B-algebras.

In this paper, we introduce the concept of an ideal direct product d-algebra, a d-ideal direct product d-algebra, sub-direct product d-algebra, an edge direct product and a positive implicative direct product d-algebra.

2. Preliminaries

First, we will review some essential notations and definitions of d-algebras and ordinary senses that are needed for this study in this section.

Definition 1. *[9] A d-algebras is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:*

(*i*) $x * x = 0$,

(*ii*) $0 * x = 0$,

(*iii*) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$ *for all* $x, y \in X$.

A nonempty subset S of a d-algebra X is said to be a sub-algebra of X if $x * y \in S$ for all $x, y \in S$.

Definition 2. *[1] A d-algebras $(X, *, 0)$ is said to be a positive implicative if $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.*

Example 1. *[1] Let $X = \{0, a, b, c\}$ be a set with a binary operation $*$ on X defined by the following table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

Then $(X, *, 0)$ is a positive implicative d-algebra.

Example 2. *[5] Let $X = \{0, a, b, c\}$ be a set with a binary operation $*$ on X defined by the following table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Then $(X, *, 0)$ is a d-algebra but not positive implicative because $(a*b)*c = 0*c = 0 \neq b = b*0 = (a*c)*(b*c)$. The set $S_1 = \{b, c\}$ is not a sub-algebra of X whereas $S_2 = \{0, a, b\}$ is a sub-algebra of X.

Definition 3. *[7] Let $(X, *, 0)$ be a d-algebra and $x \in X$. Define $x * X := \{x * a \mid a \in X\}$. We say that X is edge if $x * X = \{x, 0\}$ for all $x \in X$.*

Example 3. *[7] Let $X = \{0, 1, 2, 3\}$ be a set with the binary operation $*$ on X defined by the following table:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ is an edge d-algebra.

Example 4. *[4] Let $X = \{0, 1, 2, 3\}$ be a set with the following table:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Since $3 * X = \{3, 1, 0\} \neq \{3, 0\}$, then $(X, *, 0)$ is not an edge d-algebra.

Theorem 1. *[7] Let $(X, *, 0)$ be an edge d-algebra. Then the following conditions are satisfiesd :*

(i) $x * 0 = x$,

(ii) $(x * y) * z = (x * z) * y$,

(iii) $x * (x * y) = y$, *for any* $x, y, z \in X$.

Definition 4. *[3] Let $(X, *, 0)$ be a d-algebra and I a subset of X, then I is called an ideal of X if it satisfies the following conditions:*

(i) $0 \in I$,

(ii) If $x * y \in I$ *and* $y \in I$ *imply* $x \in I$.

Definition 5. *[3] Let $(X, *, 0)$ be a d-algebra and I a nonempty subset of X, then I is called a d-ideal of X if it satisfies the following conditions :*

(i) If $x * y \in I$ *and* $y \in I$ *imply* $x \in I$,

(ii) If $x \in I$ *and* $y \in X$ *imply* $x * y \in I$.

Clealy, If I is a d-ideal of a d-algebra X, then $x * x = 0 \in I$ for any $x \in I$ and then I is an ideal of X, but the converse need not be true as the following example:

Example 5. *[9] Let $X = \{0, a, b, c\}$ be a set with binary operation $*$ on X defined by the following table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Then $(X, *, 0)$ is a d-algebra and $I := \{0, a\}$ is an ideal of X, but not a d- ideal of X, since $a * c = b \notin I$.

Theorem 2. *[9] Let I be a d-ideal of a d-algebra X. If $x \in I$ and $y \in X$ such that $y * x = 0$, then $y \in I$.*

3. Direct product d-Algebras

J. Neggers and H. S. Kim ([9]) introduced the concept of a direct product of d-algebras as follows. Let $\{(X_i, *, 0) \mid i \in I\}$ be a non-empty family of d-algebras and $\prod_{i \in I} X_i = \{(x_i)_{i \in I} \mid x_i \in X_i\}$. Then $(0_i)_{i \in I}$ where $0_i \in X_i$. serves as 0 of $\prod_{i \in I} X_i$. Define a binary operation $\odot$ on $\prod_{i\in I} X_i$ by $(x_i)_{i\in I} \odot (y_i)_{i\in I} = (x_i * y_i)_{i\in I}$ for all $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Then $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ is a d-algebra, called a direct product d-algebra. That is a direct product d-algebra $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ is satisfies the following conditions :

(i) $(x_i)_{i\in I} \odot (x_i)_{i\in I} = (0_i)_{i\in I}$,

(ii) $(0_i)_{i\in I}) \odot (x_i)_{i\in I} = (0_i)_{i\in I}$,

(iii) $(x_i)_{i\in I} \odot (y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i)_{i\in I} \odot (x_i)_{i\in I} = (0_i)_{i\in I}$ implies $(x_i)_{i\in I} = (y_i)_{i\in I}$ for all $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$.

Definition 6. *Let $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ be a direct product d-algebra. A non-empty subset $\prod_{i\in I} N_i$ of $\prod_{i\in I} X_i$ is said to be an ideal direct product d-algebra if it satisfies the following conditions :*

(I1) $(0_i)_{i\in I} \in \prod_{i\in I} N_i$,

(I2) $(x_i)_{i\in I} * (y_i)_{i\in I} \in \prod_{i\in I} N_i$ *and* $(y_i)_{i\in I} \in \prod_{i\in I} N_i$ *implies* $(x_i)_{i\in I} \in \prod_{i\in I} N_i$.

Definition 7. *Let $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ be a direct product d-algebra. A non-empty subset $\prod_{i\in I} N_i$ of $\prod_{i\in I} X_i$ is said to be a d-ideal direct product d-algebras if it satisfies the following conditions:*

(D1) $(x_i)_{i\in I} \odot (y_i)_{i\in I} \in \prod_{i\in I} N_i$ *and* $(y_i)_{i\in I} \in \prod_{i\in I} N_i$ *implies* $(x_i)_{i\in I} \in \prod_{i\in I} N_i$,

(D2) $(x_i)_{i\in I} \in \prod_{i\in I} N_i$ *and* $(y_i)_{i\in I} \in \prod_{i\in I} X_i$ *implies* $(x_i)_{i\in I} \odot (y_i)_{i\in I} \in \prod_{i\in I} N_i$.

Example 6. *[1], [9] Let $X_1 = \{0, 1, 2, 3\}$ and $X_2 = \{0', a, b, c\}$. Define binary operations $*$ on X_1 and $*'$ on X_2. defined by the following two tables, respectively.*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

$*'$	$0'$	a	b	c
$0'$	$0'$	$0'$	$0'$	$0'$
a	a	$0'$	$0'$	b
b	b	b	$0'$	$0'$
c	c	c	c	$0'$

By example 4 and example 5, $(X_1, *, 0)$ and $(X_2, *', 0')$ are d-algebras. Consider an ideal $N_1 = \{0, 1\}$ of $(X_1$ and ideal $N_2 = \{0', a\}$ of X_2, we have $N_1 \times N_2 = \{(0, 0'), (0, a), (1, 0')\}, (1, a), \}$ is an ideal direct product d-algebra but not a d-ideal direct product d-algebra, since then $(0, a) \odot (2, c) = (0 * 2, a *' c) = (0, b) \notin N_1 \times N_2$.

Definition 8. *Let* $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ *be a direct product d-algebra, a nonempty subset* $\prod_{i\in I} N_i$ *of* $\prod_{i\in I} X_i$ *is said to be a sub-direct product of* $\prod_{i\in I} X_i$ *if* $(x_i)_{i\in I} \odot (y_i)_{i\in I} \in \prod_{i\in I} N_i$ *for all* $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} N_i$.

Theorem 3. *Every d-ideal direct product d-algebra is an ideal direct product d-algebra .*

Proof. *Let* $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ *be a direct product d-algebra and* $\prod_{i\in I} N_i$ *be a d-ideal of* $\prod_{i\in I} X_i$. *Since* $x_i * x_i = 0_i$ *for all* $i \in I$ *implies that* $(x_i)_{i\in I} \odot (x_i)_{i\in I} = (0_i)_{i\in I} \in \prod_{i\in I} N_i$ *for any* $(x_i)_{i\in I} \in \prod_{i\in I} N_i$. *Thus* $\prod_{i\in I} N_i$ *is an ideal of* $\prod_{i\in I} X_i$.

Theorem 4. *Every d-ideal a direct product d-algebra is a sub-direct product d-algebra.*

Proof. It is Clear by definition 7 and 8

Definition 9. *Let* $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ *be a direct product d-algebra and* $(a_i)_{i\in I} \in \prod_{i\in I} X_i$. *Define the set* $(a_i)_{i\in I} \odot \prod_{i\in I} X_i := \{(a_i)_{i\in I} \odot (x_i)_{i\in I} \mid (x_i)_{i\in I} \in \prod_{i\in I} X_i\}$. *We say that* $\prod_{i\in I} X_i$ *is to be an edge direct product of d-algebra if* $(a_i)_{i\in I} \odot \prod_{i\in I} X_i := \{(a_i)_{i\in I}, (0_i)_{i\in I}\}$.

Example 7. *[9],[7] Let* $X_1 = \{0, 1, 2, 3\}$ *and* $X_2 = \{0', a, b, c\}$ *be the set with a binary operation* $*$ *and* $*'$ *respectively that following 2 of tables :*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

$*'$	$0'$	a	b	c
$0'$	$0'$	$0'$	$0'$	$0'$
a	a	$0'$	$0'$	b
b	b	b	$0'$	$0'$
c	c	c	c	$0'$

Then $(X_1, *, 0)$ *and* $(X_2, *', 0)$ *are edge d-algebras. But* $X_1 \times X_2$ *is not an edge direct product d-algebra, because of* $(2, a) \odot (X_1 \times X_2) = \{(2, a), (2, 0), (0, a), (0, 0')\} \neq \{(0, 0'), (2, a)\}$.

Theorem 5. *Let* $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ *be an edge direct product d-algebra and* $\prod_{i\in I} N_i$ *be an ideal direct product of* $\prod_{i\in I} X_i$. *If* $(n_i)_{i\in I} \in \prod_{i\in I} N_i$ *and* $(x_i)_{i\in I} \in \prod_{i\in I} X_i$, *then* $(x_i)_{i\in I} \odot ((x_i)_{i\in I} \odot (n_i)_{i\in I}) \in \prod_{i\in I} N_i$.

Proof. *Consider* $((x_i)_{i\in I} \odot ((x_i)_{i\in I} \odot (n_i)_{i\in I})) \odot (n_i)_{i\in I} = ((x_i)_{i\in I} \odot (n_i)_{i\in I})) \odot ((x_i)_{i\in I} \odot (n_i)_{i\in I})) = (0_i)_{i\in I}$, *by definition 7 and theorem 1,* $(x_i)_{i\in I} \odot ((x_i)_{i\in I} \odot (n_i)_{i\in I}) \in \prod_{i\in I} N_i$.

Definition 10. *A direct product d-algebra* $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ *is said to be positive implicative if* $((x_i)_{i\in I} \odot (y_i)_{i\in I}) \odot (z_i)_{i\in I} = ((x_i)_{i\in I} \odot (z_i)_{i\in I}) \odot ((y_i)_{i\in I} \odot (z_i)_{i\in I})$ *for all* $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$.

Theorem 6. *Let $\{(X_i, *, 0_i) \mid i \in I\}$ be a non-empty family of positive implicative d-algebra, then $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ is a positive implicative direct product d-algebra.*

Proof. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}((x_i)_{i\in I} \odot (y_i)_{i\in I}) \odot (z_i)_{i\in I} &= (x_i * y_i)_{i\in I} * (z_i))_{i\in I} \\ &= (x_i * z_i)_{i\in I} * (y_i * z_i)_{i\in I} \\ &= ((x_i)_{i\in I} \odot (z_i)_{i\in I}) \odot ((y_i)_{i\in I} \odot (z_i)_{i\in I}).\end{aligned}$$

Thus $\prod_{i\in I} X_i$ is a positive implicative direct product d-algebra.

Theorem 7. *Every ideal of a positive implicative direct product d-algebra is a d-ideal direct product d-algebra.*

Proof. Let $(\prod_{i\in I} X_i, \odot, (0_i)_{i\in I})$ be a positive implicative direct product d-algebra and $\prod_{i\in I} N_i$ is an ideal of $\prod_{i\in I} X_i$. By Definition 10, we have

$$\begin{aligned}(n_i)_{i\in I} \odot (x_i)_{i\in I}) \odot (n_i)_{i\in I} &= (n_i * x_i)_{i\in I} \odot (n_i)_{i\in I} \\ &= ((n_i * x_i) * ((n_i))_{i\in I} \\ &= ((n_i * n_i) * (x_i * n_i))_{i\in I} \\ &= (0_i * (x_i * n_i))_{i\in I} \\ &= (0_i)_{i\in I} \in I.\end{aligned}$$

Hence $((n_i)_{i\in I} \odot (x_i)_{i\in I}) \in I$, implies that $\prod_{i\in I} N_i$ is a d-ideal direct product of d-algebras.

4. Conclusion

In this paper, we give the concept of ideal, d-ideal, sub-direct product and edge in a direct product d-algebra and we prove relationship between ideal direct product and d-ideal direct product of d-algebras. Moreover, we shown that a direct product of edge d-algebras is not an edge direct product d-algebra.

Acknowledgements

This research project was financially supported by Mahasarakham University.

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