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Two-dimensional inverse boundary value problem for a third-order pseudo-hyperbolic equation with an additional integral condition

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Abstract. In this paper we study an inverse boundary value problem with an unknown time-dependent coefficient for a third-order pseudo-hyperbolic equation with an additional integral condition. The definition of the classical solution of the problem is given. The essence of the problem is that it is required together with the solution to determine the unknown coefficient. The problem is considered in a rectangular area. When solving the original inverse boundary value problem, the transition from the original inverse problem to some auxiliary inverse problem is carried out. The existence and uniqueness of a solution to an auxiliary problem are proved with the help of contracted mappings. Then the transition to the original inverse problem is again made, as a result, a conclusion is made about the solvability of the original inverse problem.

2020 Mathematics Subject Classifications: 31A25, 35L35

Key Words and Phrases: Inverse boundary value problem, third-order pseudo-hyperbolic equation, Fourier method, classical solution

1. Introduction and Problem Statement

It is known that the practical requirements often lead to the problem of determining the coefficients or the right hand side of the differential equations for some known data about their solutions. Such problems are called inverse problems in mathematical physics. Inverse problems arise in various fields of human activity, such as seismology, mineral exploration, biology, medical visualization, computed tomography, Earth remote sensing, spectral analysis, nondestructive control, etc.

Fundamentals of the theory and practice of research of inverse problems were established and developed in the works published by Tikhonov [22], Lavrent'ev [16], Ivanov [10], Romanov [21], Isakov [6], and so on.

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A more detailed bibliography and a classification of recent works connected with the investigation of inverse problems for partial differential equations can be found in monographs and in articles [2, 3, 5, 7–9, 11, 13, 20] and references therein.

It should be noted that pseudo-hyperbolic equations arise in the theory of unsteady flow of a viscous gas during the propagation of initial densifications in a viscous gas [23], in the theory of solutions [17] when describing the process of electron motion in the system "superconductor – dielectric with tunneling conductivity – superconductor". The solvability of inverse problems in certain formulations, with certain overdetermination conditions for pseudohyperbolic equations, was the subject of study in [1, 4, 14, 15, 18, 19] and references therein.

In this work we study a two-dimensional inverse boundary value problem for a third-order pseudo-hyperbolic equation with an additional integral condition. In the paper using the Fourier method and the contraction mappings principle, the existence and uniqueness of a classical solution to the considered nonlinear inverse boundary value problem is proved.

Consider for the equation

$$u_{tt}(x,y,t) - \alpha\Delta u_t(x,y,t) - \beta\Delta u(x,y,t) = a(t)u(x,y,t) + f(x,y,t) \ \ (x,y,t) \in D_T, \quad (1)$$

in the domain $D_T = Q_{xy} \times \{0 < t \le T\}$, where $Q_{xy} = \{(x,y) : 0 < x < 1,\ 0 < y < 1\}$ an inverse boundary problem with initial conditions

$$u(x,y,0) = \phi(x,y), \quad u_t(x,y,0) = \psi(x,y), \ \ 0 \le x,y \le 1, \quad (2)$$

with boundary conditions

$$u_x(0,y,t) = u(1,y,t) = 0, \ \ 0 \le y \le 1, \ \ 0 \le t \le T, \quad (3)$$

$$u(x,0,t) = u_y(x,1,t) = 0, \ \ 0 \le y \le 1, \ \ 0 \le t \le T, \quad (4)$$

and with additional condition

$$\int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy = h(t), \ \ 0 \le t \le T, \quad (5)$$

where $\alpha, \beta > 0$ are given numbers, $f(x,y,t)$, $\phi(x,y)$, $\psi(x,y)$, $\omega(x,y)$, and $h(t)$ $(i = 1,2)$ are given functions, $u(x,y,t)$, $a(t)$ are desired functions, and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Definition 1. *The pair $\{u(x,y,t),\, a(t)\}$ is said to be a classical solution of the inverse boundary value problem (1)-(5), if the following conditions are satisfied: the function $u(x,y,t) \in \tilde{C}^{2,2,2}(D_T) \cap C^{1,1,1}(D_T)$, $a(t) \in C[0,T]$, satisfying equation (1) in D_T, condition (2) in $\bar{Q}_{xy}$, condition (3) in $[0,1] \times [0,T]$, condition (4) in $[0,1] \times [0,T]$ and condition (5) in $[0,T]$.*

The following theorem holds:

Theorem 1. *Let* $\phi(x,y) \in C(\bar{Q}_{xy})$, $\psi(x,y) \in C(\bar{Q}_{xy})$, $f(x,y,t) \in C(D_T)$, $h(t) \in C^2[0,T]$, $h(t) \neq 0$, $0 \leq t \leq T$ *and the consistency condition*

$$\int_0^1 \int_0^1 \omega(x,y)\phi(x,y)dxdy = h(0), \quad \int_0^1 \int_0^1 \omega(x,y)\psi(x,y)dxdy = h'(0) \tag{6}$$

be satisfied. Then the problem of finding a classical solution to problem (1)-(5) is equivalent to the problem of determining the functions $u(x,y,t) \in \tilde{C}^{2,2,2}(\bar{D}_T)$, $a(t) \in C[0,T]$ *from (1)-(4) and*

$$h''(t) - \alpha \int_0^1 \int_0^1 \omega(x,y)\Delta u_t(x,y,t)dxdt - \beta \int_0^1 \int_0^1 \omega(x,y)\Delta u(x,y,t)dxdt =$$

$$= a(t)h(t) + \int_0^1 \int_0^1 \omega(x,t)f(x,y,t)dxdy \quad (0 \leq t \leq T). \tag{7}$$

Proof. Let $\{u(x,y,t), a(t)\}$ be a classical solution to problem (1)-(5), and $u(x,y,t) \in \tilde{C}^{2,2,2}(\bar{D}_T)$. Assuming $h(t) \in C^2[0,T]$ and differentiating two times (5), we get:

$$u_t(0,1,t) = h'(t)\,, \; u_{tt}(0,1,t) = h''(t) \quad (0 \leq t \leq T). \tag{8}$$

Further, multiplying Eq. (1) by the function $\omega(x,y)$, integrating the equation over x from 0 to 1, we have:

$$\frac{d^2}{dt^2} \int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy - \alpha \int_0^1 \int_0^1 \omega(x,y)\Delta u_t(x,y,t)dxdy -$$

$$-\beta \int_0^1 \int_0^1 \omega(x,y)\Delta u(x,y,t)dxdy =$$

$$= a(t) \int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy + \int_0^1 \int_0^1 \omega(x,y)f(x,y,t)dxdy \quad (0 \leq t \leq T). \tag{9}$$

From (9), taking into account (5) and (8), the fulfillment of (7) follows.

Now, suppose that $\{u(x,y,t), a(t)\}$ is a solution to the problem (1)-(4), (7). Then from (7) and (9) we find:

$$\frac{d^2}{dt^2} \left(\int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy - h(t) \right) =$$

$$= a(t)\left(\int_0^1\int_0^1 \omega(x,y)u(x,y,t)dxdy - h(t)\right) \quad (0 \le t \le T). \tag{10}$$

Due to (2) and (6), we have :

$$\begin{aligned} &\int_0^1\int_0^1 \omega(x,y)u(x,y,0)dxdy - h(0) = \int_0^1\int_0^1 \omega(x,y)\phi(x,y)dxdy - h(0) = 0, \\ &\int_0^1\int_0^1 \omega(x,y)u_t(x,y,0)dxdy - h'(0) = \int_0^1\int_0^1 \omega(x,y)\psi(x,y)dxdy - h'(0) = 0. \end{aligned} \tag{11}$$

From (10), (11) we conclude that

$$\int_0^1\int_0^1 \omega(x,y)u(x,y,t)dxdy - h(t) = 0 \quad (0 \le t \le T),$$

i.e. condition (5) is satisfied.

2. Solvability of the existence and uniqueness of the classical solution of the inverse boundary value problem

The first component of the solution $\{u(x,y,t), a(t)\}$ of the problem (1)-(4), (7) will be sought in the form:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x \sin\gamma_n y, \tag{12}$$

where

$$\lambda_k = \frac{\pi}{2}(2k-1), \;\; k = 1,2,..., \quad \gamma_n = \frac{\pi}{2}(2n-1), \quad n = 1,2,...\,,$$

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x \sin\gamma_n y dxdy, \;\; k,n = 1,2,....$$

Applying the method of separation of variables to determine the desired coefficients

$$u_{k,n}(t) \;\; (k = 1,2,...; n = 1,2,...),$$

of the function $u(x,y,t)$ from (1), (2), we get:

$$u''_{k,n}(t) + \alpha\mu_{k,n}^2 u'_{k.n}(t) + \beta\mu_{k,n}^2 u_{k,n}(t) = F_{k,n}(t;u,a), \; k,n = 1,2,..., \;\; 0 \le t \le T, \tag{13}$$

$$u_{k,n}(0) = \phi_{k,n}, \;\; u'_{k,n}(0) = \psi_{k,n}, \;\; k,n = 1,2,..., \tag{14}$$

where

$$\mu_{k,n}^2 = \lambda_k^2 + \gamma_n^2, \ \ k,n = 1,2,...,$$

$$F_{k,n}(t;u,a) = f_{k,n}(t) + a(t)u_{k,n}(t), \ \ k,n = 1,2,...,$$

$$f_{k,n}(t) = 4\int_0^1\int_0^1 f(x,y,t)\cos\lambda_k x \sin\gamma_n y dx dy, \ \ k,n = 1,2,...,$$

$$\phi_{k,n} = 4\int_0^1\int_0^1 \phi(x,y)\cos\lambda_k x \sin\gamma_n y dx dy, \ \ k,n = 1,2,...,$$

$$\psi_{k,n} = 4\int_0^1\int_0^1 \psi(x,y)\cos\lambda_k x \sin\gamma_n y dx dy, \ \ k,n = 1,2,....$$

Let's assume that

$$\frac{\alpha^2\pi^2}{4} - \beta > 0 \ .$$

Then solving problem (13), (14), we find:

$$u_{k,n}(t) = \frac{1}{\gamma_{k,n}}\left[\left(\mu_{2,k,n}e^{\mu_{1,k,n}t} - \mu_{1,k,n}e^{\mu_{2,k,n}t}\right)\phi_{k,n} + \left(e^{\mu_{2,k,n}t} - e^{\mu_{1,k,n}t}\right)\psi_{k,n} + \right.$$

$$\left. + \int_0^t F_{k,n}(\tau;u,a)\left(e^{\mu_{2,k,n}(t-\tau)} - e^{\mu_{1,k,n}(t-\tau)}\right)d\tau\right], \tag{15}$$

where

$$\mu_{i,k,n} = -\frac{\alpha\mu_{k,n}^2}{2} + (-1)^i\mu_{k,n}\sqrt{\frac{\alpha^2\mu_{k,n}^2}{4} - \beta} \qquad (i = 1,2),$$

$$\gamma_{k,n} = \mu_{2,k,n} - \mu_{1,k,n} = 2\mu_{k,n}\sqrt{\frac{\alpha^2\mu_{k,n}^2}{4} - \beta}\,.$$

After substituting the expression from (15) into (12), to determine the component of the solution to problem (1)-(3), (7), we obtain:

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{1}{\gamma_{k,n}}\left[\left(\mu_{2,k,n}e^{\mu_{1,k,n}t} - \mu_{1,k,n}e^{\mu_{2,k,n}t}\right)\phi_{k,n} + \left(e^{\mu_{2,k,n}t} - e^{\mu_{1,k,n}t}\right)\psi_{k,n} + \right.\right.$$

$$\left.\left. + \int_0^t F_{k,n}(\tau;u,a)\left(e^{\mu_{2,k,n}(t-\tau)} - e^{\mu_{2,k,n}(t-\tau)}\right)d\tau\right]\right\}\cos\lambda_k x\sin\gamma_n y. \tag{16}$$

From (6),(7), we have:

$$a(t)h(t) = h''(t) - \int\limits_0^1 \int\limits_0^1 \omega(x,y) f(x,y,t) dxdy +$$

$$+ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} p_{k,n} \left(\alpha u'_{k,n}(t) + \beta u_{k,n}(t) \right) \quad (0 \le t \le T), \tag{17}$$

where

$$p_{k,n} = \int\limits_0^1 \int\limits_0^1 \omega(x,y) \cos \lambda_k x \sin \gamma_n y dxdy. \tag{18}$$

Differentiating (15) two times, we get:

$$u'_{k,n}(t) = \frac{1}{\gamma_{k,n}} \left[\mu_{1,k,n} \mu_{2,k,n} \left(e^{\mu_{1,k,n} t} - e^{\mu_{2k,n} t} \right) \phi_{k,n} + \right.$$

$$+ \left(\mu_{2,k,n} e^{\mu_{2,k,n} t} - \mu_{1,k,n} e^{\mu_{1,k,n} t} \right) \psi_{k,n} +$$

$$\left. + \int\limits_0^t F_{k,n}(\tau; u, a) \left(\mu_{2,k,n} e^{\mu_{2,k,n}(t-\tau)} - \mu_{1,k,n} e^{\mu_{1,k,n}(t-\tau)} \right) d\tau \right], \tag{19}$$

$$u''_{k,n}(t) = \frac{1}{\gamma_{k,n}} \left[\mu_{1,k,n} \mu_{2,k,n} \left(\mu_{1,k,n} e^{\mu_{1,k,n} t} - \mu_{2,k,n} e^{\mu_{2,k,n} t} \right) \phi_{k,n} + \right. ,$$

$$+ \left(\mu_{2,k,n}^2 e^{\mu_{2,k,n} t} - \mu_{1,k,n}^2 e^{\mu_{1,k,n} t} \right) \psi_{k,n} +$$

$$\left. + \int\limits_0^t F_{k,n}(\tau; u, a) \left(\mu_{2,k,n}^2 e^{\mu_{2,k,n}(t-\tau)} - \mu_{1,k,n}^2 e^{\mu_{1,k,n}(t-\tau)} \right) d\tau \right] + F_{k,n}(t; u, a). \tag{20}$$

Due to (13) and (20) we have :

$$\alpha \mu_{k,n}^2 u'_{k,n}(t) + \beta \mu_{k,n}^2 u_{k,n}(t) = F_{k,n}(t; u, a) - u''_{k,n}(t) =$$

$$= -\frac{1}{\gamma_{k,n}} \left[\mu_{1,k,n} \mu_{2,k,n} \left(\mu_{1,k,n} e^{\mu_{1,k,n} t} - \mu_{2,k,n} e^{\mu_{2,k,n} t} \right) \phi_{k,n} + \right.$$

$$+ \left(\mu_{2,k,n}^2 e^{\mu_{2,k,n} t} - \mu_{1,k,n}^2 e^{\mu_{1,k,n} t} \right) \psi_{k,n} +$$

$$\left. + \int\limits_0^t F_{k,n}(\tau; u, a) \left(\mu_{2,k,n}^2 e^{\mu_{2,k,n}(t-\tau)} - \mu_{1,k,n}^2 e^{\mu_{1,k,n}(t-\tau)} \right) d\tau \right]. \tag{21}$$

In order to obtain an equation for the second component $a(t)$ of the solution $\{u(x,y,t),a(t)\}$ of problem (1)-(4), (7), we substitute expression (21) into (17):

$$a(t)=[h(t)]^{-1}\left\{h''(t)-\int_0^1\int_0^1\omega(x,y)f(x,y,t)dxdy+\right.$$

$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_{k,n}}{\mu_{k,n}^2\gamma_{k,n}}\left[\mu_{1,k,n}\mu_{2,k,n}\left(\mu_{1,k,n}e^{\mu_{1,k,n}t}-\mu_{2,k,n}e^{\mu_{2,k,n}t}\right)\phi_{k,n}+\right.$$

$$+\left(\mu_{2,k,n}^2e^{\mu_{2,k,n}t}-\mu_{1,k,n}^2e^{\mu_{1,k,n}t}\right)\psi_{k,n}+$$

$$\left.\left.+\int_0^t F_{k,n}(\tau;u,a)\left(\mu_{2,k,n}^2e^{\mu_{2,k,n}(t-\tau)}-\mu_{1,k,n}^2e^{\mu_{1,k,n}(t-\tau)}\right)d\tau\right]\right\}. \tag{22}$$

Thus, the solution of problem (1)-(4),(7) is reduced to the solution of system (15), (22) with respect to unknown functions $u(x,y,t)$ and $a(t)$.

To study the question of the uniqueness of the solution of problem (1)-(4), (7), the following lemma plays an important role.

Lemma 1. *If $\{u(x,y,t),a(t)\}$ is any classical solution to the problem (1)-(4),(7), then the functions*

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy,\quad k,n=1,2,....$$

satisfy of the system (15).

Proof. Let $\{u(x,t),a(t)\}$ be any solution to the problem (1)-(4), (7). Then multiplying both sides of equation (1), by the function

$$4\cos\lambda_k x\sin\gamma_n y\,,(k=1,2,...;\ n=1,2,...),$$

integrating the resulting equality over x and y from 0 to 1, and using the relations

$$4\int_0^1\int_0^1 u_{tt}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\ =$$

$$=\frac{d^2}{dt^2}\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=u''_{k,n}(t)\qquad(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{xx}(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy \;=$$

$$=-\lambda_k^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy\right)=-\lambda_k^2 u_{k,n}(t)\quad (k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{yy}(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy \;=$$

$$-\gamma_n^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy\right)=-\gamma_n^2 u_{k,n}(t)\quad (k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{txx}(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy \;=$$

$$=-\lambda_k^2\left(4\int_0^1\int_0^1 u_t(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy\right)=-\lambda_k^2 u'_{k,n}(t)\quad (k=1,2,...;\ n=1,2,...),$$

$$\int_0^1\int_0^1 u_{tyy}(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy \;=$$

$$=-\gamma_n^2\left(4\int_0^1\int_0^1 u_t(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy\right)=-\gamma_n^2 u''_{k,n}(t)\quad (k=1,2,...;\ n=1,2,...),$$

we obtain that equation (13) is satisfied.

Similarly, from (2) we conclude at condition (14).

Thus, $u_{k,n}(t)$ $(k=1,2,...;\ n=1,2,...)$ are a solution to problem (13), (14).

Then from this, it directly follows that the functions

$$u_{k,n}(t)\ (k=1,2,...;\ n=1,2,...)$$

satisfy on $[0,T]$ of the system (15).

It is obvious that if

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy, \quad (k=1,2,...;\ n=1,2,...)$$

are a solution of system (15), then the pair $\{u(x,t),a(t)\}$ of the functions

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y$$

and $a(t)$ is a solution of system (15), (21).

From Lemma 1 it follows that:

Remark 1. *Let system (15), (22) have a unique solution. Then problem (1)-(4), (7) cannot have more than one solution, i.e. if problem (1)-(4), (7) has a solution, then it is unique.*

1. We denote by $B^3_{2,T}$ [12], a consisting of all functions $u(x,y,t)$ of the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y,$$

considered in D_T, where

$$u_{k,n}(t)\ (k=1,2,...;\ n=1,.2,..)$$

is continuous on $[0,T]$ and

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu^3_{k,n}\|u_{k,n}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} < +\infty,$$

where

$$\mu_{k,n} = \sqrt{\lambda_k^2+\gamma_n^2} \quad (k=1,2,...;n=1,2,...).$$

The norm in this set is defined as follows:

$$\|u(x,y,t)\|_{B^3_{2,T}} = \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu^3_{k,n}\|u_{k,n}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}.$$

2. The spaces E^3_T denote the space consisting of the topological product $B^3_{2,T}\times C[0,T]$. The norm of element $z=\{u,a\}$ is determined by the formula

$$\|z\|_{E^3_T} = \|u(x,y,t)\|_{B^3_{2,T}} + \|a(t)\|_{C[0,T]}.$$

It is obvious that $B^3_{2,T}$ and E^3_T are Banach spaces.

Now consider in the space E_T^3 the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\},$$

where

$$\Phi_1(u,a) = \tilde{u}(x,y,t) \equiv \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{u}_{k,n}(t)\cos\lambda_k x \sin\gamma_n y, \quad \Phi_2(u,a) = \tilde{a}(t),$$

and $\tilde{u}_{k,n}(t)$ $(k = 1,2,...;\ n = 1,2,...)$ and $\tilde{a}(t)$ are equal to, respectively, the right sides of (15), and (21).

It is easy to get that

$$\mu_{k,n}^3 \le (\lambda_k^2 + \gamma_k^2)(\lambda_k + \gamma_n) = \lambda_k^3 + \lambda_k^2\gamma_n + \gamma_n^2\lambda_k + \gamma_n^3,$$

$$|\gamma_{k,n}| > \frac{\alpha}{\sqrt{2}}\mu_{k,n}^2, |\mu_{i,k,n}| \le \alpha\mu_{k,n}^2 \quad (i = 1,2),$$

$$|\mu_{1,k,n}\mu_{2,k,n}| = \beta\mu_{k,n}^2 \quad (i = 1,2),$$

$$|p_{k,n}| \le \|\omega(x,y)\|_{C(\bar{Q}_{xy})}.$$

Taking into account this ratio, we have:

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\|\tilde{u}_{k,n}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \le 4\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\ |\phi_{k,n}|\right)^2\right)^{\frac{1}{2}} +$$

$$+4\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\ |\phi_{k,n}|\right)^2\right)^{\frac{1}{2}} + 4\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\ |\phi_{k,n}|\right)^2\right)^{\frac{1}{2}} +$$

$$+4\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\ |\phi_{k,n}|\right)^2\right)^{\frac{1}{2}} +$$

$$+\frac{4}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\ |\psi_{k,n}|\right)^2\right)^{\frac{1}{2}} + \frac{4}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n\ |\psi_{k,n}|\right)^2\right)^{\frac{1}{2}} +$$

$$+\frac{4\sqrt{T}}{\alpha}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\,|f_{k,n}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} + \left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n\,|f_{k,n}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right) +$$

$$+\frac{4T}{\alpha}\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\,\|u_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}, \tag{23}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| h''(t) - \int\limits_0^1 \int\limits_0^1 \omega(x,y) f(x,y,t) dxdy \right\|_{C[0,T]} + \right.$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_{k,n}^{-2} \right)^{\frac{1}{2}} \|\omega(x,y)\|_{C[\bar{Q}_{xy}]} \left[\beta \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k^3 \ |\phi_{k,n}| \right)^2 \right)^{\frac{1}{2}} \right.$$

$$+\beta \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k^2 \gamma_n \ |\phi_{k,n}| \right)^2 \right)^{\frac{1}{2}} + \beta \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k \gamma_n^2 \ |\phi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+ \beta \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\gamma_n^3 \ |\phi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+\alpha \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k \ |\psi_{k,n}| \right)^2 \right)^{\frac{1}{2}} + \alpha \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\gamma_n \ |\psi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+\alpha\sqrt{T} \left(\left(\int\limits_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k \left| f_{k,n}(\tau) \right| \right)^2 d\tau \right)^{\frac{1}{2}} + \left(\int\limits_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\gamma_n \left| f_{k,n}(\tau) \right| \right)^2 d\tau \right)^{\frac{1}{2}} \right) +$$

$$\left. \left. +\alpha T \|a(t)\|_{C[0,T]} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\mu_{k,n}^3 \left\| u_{k,n}(t) \right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \right] \right\}. \tag{24}$$

Let us assume that the data of problem (1)-(4), (7) satisfy the following conditions:

1. $\alpha > 0, \ \beta > 0, \ \frac{\alpha^2}{8} - \beta > 0;$
2. $\phi(x,y), \phi_x(x,y), \phi_{xx}(x,y), \phi_y(x,y), \phi_{xy}(x,y), \phi_{yy}(x,y) \in C(\bar{Q}_{xy}),$
   $\phi_{xxy}(x,y), \phi_{xyy}(x,y), \phi_{xxx}(x,y), \phi_{yyy}(x,y) \in L_2(Q_{xy}),$
   $\phi_x(0,y) = \phi(1,y) = \phi_{xx}(1,y) = 0, \ \ 0 \leq y \leq 1,$
   $\phi(x,0) = \phi_y(x,1) = \phi_{yy}(x,0) = 0, \ \ 0 \leq x \leq 1;$
3. $\psi(x,y), \psi_x(x,y), \psi_y(x,y), \psi_{xx}(x,y), \psi_{xy}(x,y), \psi_{yy}(x,y) \in C(\bar{Q}_{xy}),$
   $\psi_{xxy}(x,y), \ \psi_{xyy}(x,y), \psi_{xxx}(x,y), \ \psi_{yyy}(x,y) \in L_2(Q_{xy}),$
   $\psi_x(0,y) = \psi(1,y) = \psi_{xx}(1,y) = 0, \ \ 0 \leq y \leq 1,$
   $\psi(x,0) = \psi_y(x,1) = \psi_{yy}(x,1) = 0, \ \ 0 \leq x \leq 1;$
4. $f(x,y,t), f_x(x,y,t), f_y(x,y,t), f_{xx}(x,y,t), f_{xy}(x,y,t), f_{yy}(x,y,t) \in C(D_T),$
   $f_{xxx}(x,y,t), f_{xxy}(x,y,t), f_{xyy}(x,y,t), f_{yyy}(x,y,t) \in L_2(D_T),$
   $f_x\left(0,y,t\right) = f\left(1,y,t\right) = f_{xx}\left(0,y,t\right) = 0, \ \ 0 \leq y \leq 1, \ \ 0 \leq t \leq T,$
   $f\left(x,0,t\right) = f_y\left(x,1,t\right) = f_{yy}\left(x,1,t\right) = 0, \ \ 0 \leq x \leq 1, \ \ 0 \leq t \leq T;$

5. $h(t) \in C^2[0,T],\ h(t) \neq 0\ (0 \leq t \leq T)$.

Then from (26) - (28), respectively, we obtain:

$$\|u(x,y,t)\|_{B^3_{2,T}} \leq A_1(T) + B_1(T)\|a(t)\|_{C[0,T]}\|u(x,y,t)\|_{B^3_{2,T}} + C_1(T)\|b(t)\|_{C[0,T]}, \tag{25}$$

$$\|\tilde{a}\,(t)\|_{C[0,T]} \leq A_2\,(T) + B_2\,(T)\,\|a\,(t)\|_{C[0,T]}\|u\,(x,y,t)\|_{B^3_{2,T}} + C_2\,(T)\,\|b\,(t)\|_{C[0,T]}, \tag{26}$$

where

$$A_1(T) = 4\|\phi_{xxx}(x,y)\|_{L_2(Q_{xy})} + 4\|\phi_{xxy}(x,y)\|_{L_2(Q_{xy})} + 4\,\|\phi_{xyy}(x,y)\|_{\ L_2(Q_{xy})} +$$

$$+4\|\phi_{yyy}(x,y)\|_{L_2(Q_{xy})} + \frac{4}{\alpha}\|\psi_x(x,y)\|_{L_2(Q_{xy})} + \frac{4}{\alpha}\|\psi_y(x,y)\|_{L_2(Q_{xy})} +$$

$$+\frac{4\sqrt{T}}{\alpha}\left(\|f_x(x,y,t)\|_{L_2(D_T)} + \|f_y(x,y,t)\|_{L_2(D_T)}\right),$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) - \int_0^1\int_0^1 \omega(x,y)f(x,y,t)dxdy\right\|_{C[0,T]} +\right.$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\|\omega(x,y)\|_{C[\bar{Q}_{xy}]}\beta\|\phi_{xxx}(x,y)\|_{L_2(Q_{xy})} +$$

$$+\beta\|\phi_{xxy}(x,y)\|_{L_2(Q_{xy})} + \beta\,\|\phi_{xyy}(x,y)\|_{\ L_2(Q_{xy})} + \beta\|\phi_{yyy}(x,y)\|_{L_2(Q_{xy})} +$$

$$+\alpha\|\psi_x(x,y)\|_{L_2(Q_{xy})} + \alpha\|\phi_y(x,y)\|_{L_2(Q_{xy})} +$$

$$\left.+\alpha\sqrt{T}\left(\|f_x(x,y,t)\|_{L_2(D_T)} + \|f_y(x,y,t)\|_{L_2(D_T)}\right)\Big]\right\},$$

$$B_2\,(T) = 2\sqrt{2}\alpha\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\|\omega(x,y)\|_{C[\bar{Q}_{xy}]}T.$$

From inequalities (25)-(26), we conclude:

$$\|u(x,y,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T)\|a(t)\|_{C[0,T]}\|u(x,y,t)\|_{B^3_{2,T}}, \tag{27}$$

where

$$A(T) = A_1(T) + A_2(T),\ \ B(T) = B_1(T) + B_2(T).$$

So, we can prove the following theorem:

Theorem 2. *Let conditions 1-5 be satisfied and*

$$B(T)(A(T)+2)^2 < 1.$$

Then problem (1)-(4), (7) has unique solution in the ball $K = K_R(\|z\|_{E_T^3} \leq R = A(T)+2)$ *of the spaces* E_T^3.

Proof. In the space E_T^3 consider the equation

$$z = \Phi z,$$

where $z = \{u, a\}$, the components $\Phi_i(u,a)(i = 1, 2)$ of the operator (u, a) are defined by the right-hand sides of equations (15), (22).

Consider the operator (u, a) in the sphere $K = K_R(\|z\|_{E_T^3} \leq R = A(T) + 2)$ from E_T^3.

Similarly to (27), we obtain that for any $z, z_1 \in K_R$ fair estimates:

$$\|\Phi z\|_{E_T^3} \leq A(T) + B(T)\|a(t)\|_{C[0,T]} \|u(x, y, t)\|_{B_{2,T}^3} \leq A(T) + B(T) \left(A(T) + 2\right)^2, \quad (28)$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq B(T)R \left(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x, y, t) - u_2(x, y, t)\|_{B_{2,T}^3} \right). \quad (29)$$

Then estimates (30) and (31), taking into account (28), it follows that the operator Φ acts in the sphere $K = K_R$ and is contractive. Therefore, in the sphere $K = K_R$ the operator Φ has a unique fixed point $\{u, a\}$, that is a solution of equation (15),(22).

The function $u(x, y, t)$, as an element of space $B_{2,T}^3$, is continuous and has continuous derivatives $u_x(x, y, t)$, $u_{xx}(x, y, t)$, $u_y(x, y, t)$, $u_{xy}(x, y, t)$, $u_{yy}(x, y, t)$, $u_{xxx}(x, y, t)$, $u_{yyy}(x, y, t)$ in D_T.

Further, from (19), we find:

$$\left\{ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\mu_{k,n}^3 \|u'_{k,n}(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} \leq \frac{4\sqrt{2}\beta}{\alpha} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k^3 \ |\phi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+ \frac{4\sqrt{2}\beta}{\alpha} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k^2 \gamma_n \ |\phi_{k,n}| \right)^2 \right)^{\frac{1}{2}} + \frac{4\sqrt{2}\beta}{\alpha} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k \gamma_n^2 \ |\phi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+ \frac{4\sqrt{2}\beta}{\alpha} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\gamma_n^3 \ |\phi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+4\sqrt{2} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k^3 \ |\psi_{k,n}| \right)^2 \right)^{\frac{1}{2}} + 4\sqrt{2} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k^2 \gamma_n \ |\psi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+ 4\sqrt{2} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\lambda_k \gamma_n^2 \ |\psi_{k,n}| \right)^2 \right)^{\frac{1}{2}} + 4\sqrt{2} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\gamma_n^3 \ |\psi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+4\sqrt{2T}\left(\left(\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\,|f_{k,n}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}+\left(\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\,|f_{k,n}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.+\left(\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\,|f_{k,n}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}+\left(\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\,|f_{k,n}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right)+$$

$$+4\sqrt{2}\,T\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^2\,\|u_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},$$

or

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \le \frac{4\sqrt{2}\beta}{\alpha}\|\phi_{xxx}(x,y)\|_{L_2(Q_{xy})}+$$

$$4\|\phi_{xxx}(x,y)\|_{L_2(Q_{xy})}+4\|\phi_{xxy}(x,y)\|_{L_2(Q_{xy})}+4\,\|\phi_{xyy}(x,y)\|_{L_2(Q_{xy})}+$$

$$+\frac{4\sqrt{2}\beta}{\alpha}\|\phi_{xxy}(x,y)\|_{L_2(Q_{xy})}+\frac{4\sqrt{2}\beta}{\alpha}\|\phi_{xyy}(x,y)\|_{L_2(Q_{xy})}+$$

$$+\frac{4\sqrt{2}\beta}{\alpha}\|\phi_{yyy}(x,y)\|_{L_2(Q_{xy})}+$$

$$+4\sqrt{2}\|\psi_{xxx}(x,y)\|_{L_2(Q_{xy})}+4\sqrt{2}\|\psi_{xxy}(x,y)\|_{L_2(Q_{xy})}+$$

$$+4\sqrt{2}\|\psi_{xyy}(x,y)\|_{L_2(Q_{xy})}+4\sqrt{2}\|\psi_{yyy}(x,y)\|_{L_2(Q_{xy})}+$$

$$+4\sqrt{2T}\left(\|f_{xxx}(x,y,t)\|_{L_2(D_T)}+\|f_{xxy}(x,y,t)\|_{L_2(D_T)}+\right.$$

$$\left.+\|f_{xyy}(x,y,t)\|_{L_2(D_T)}+\|f_{yyy}(x,y,t)\|_{L_2(D_T)}\right)+$$

$$+4\sqrt{2}\,T\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\,\|u_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}.$$

From the last relation, it is clear that $u_t(x,y,t)$, $u_{tx}(x,y,t)$, $u_{ty}(x,y,t)$, $u_{txx}(x,y,t)$, $u_{tyy}(x,y,t)$ are continuous in D_T.

Now, from (13) it is easy to see that

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}\left\|u''_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \leq 2\left[\alpha\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}+\right.$$

$$+\beta\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\|u_{k,n}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}+\left\|\|f_x(x,y,t)+f_y(x,y,t)\|_{C[0,T]}\right\|_{L_2(Q_{xy})}+$$

$$\left.+\left\|\|a(t)\left(u_x(x,y,t)+u_y(x,y,t)\right)\|_{C[0,T]}\right\|_{L_2(Q_{xy})}\right].$$

It is easy to verify that $u_{tt}(x,y,t)$ is continuous in D_T. Obvious that equation (1) and conditions (2)–(4), (7) are satisfied in the usual sense. Thus, the solution to problem (1)-(4), (7) is a triple of functions $\{u(x,t), a(t)\}$ and by the corollary of Lemma 1, it is unique in the ball $K = K_R$.

Using Theorems 1 and 2, we obtain the unique solvability of problem (1)–(5).

Theorem 3. *Let all the conditions of Theorem 2 and the consistency condition (6) be satisfied. Then problem (1)-(5) has in the ball* $K = K_R(\|z\|_{E_T^3} \leq R = A(T)+2)$ *of the spaces* E_T^3 *a unique classical solution.*

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