EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 1318-1325
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Bounds on Intersection Number in the Join and Corona of Graphs

Jesrael B. Palco[1,*], Rolando N. Paluga[2]

[1] *Department of Physical Sciences and Mathematics, College of Marine and Allied Sciences, Mindanao State University at Naawan, 9023, Philippines*
[2] *Department of Mathematics, College of Mathematics and Natural Sciences, Caraga State University , 8600, Philippines*

Abstract. In this paper, we provide an upper bound for the intersection number in the join and corona of graphs. Moreover, we give formulas for the intersection number of $K_n \circ G$, $P_n \circ G$, $C_n \circ G$ and Cr_n.

2020 Mathematics Subject Classifications: 05C69
Key Words and Phrases: Intersection number, extreme intersection graph, join and corona

---

1. Introduction

Let S be a set and $F = \{S_1, S_2, \cdots, S_p\}$, for some integer p, a nonempty family of distinct nonempty subsets of S whose union is S. The intersection graph of F is denoted by $\Omega(F)$ and defined by $V(\Omega(F)) = F$, with S_i and S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. A graph G is an intersection graph on S if there exists a family F of subsets of S for which $G \cong \Omega(F)$. The intersection number $\omega(G)$ of a given graph G is the minimum number of elements in a set S such that G is an intersection graph on S. The intersection number has been studied by [1]. They obtained the best possible upper bound for the intersection number of a graph with a given number of points. In [2], Frank Harary provided an upper bound for the intersection number of a graph G. He showed that $\omega(G) \leq |E(G)|$. In [3], the authors provided a lower bound for the intersection number of a graph G. They showed that $\log_2(|V(G)| + 1) \leq \omega(G)$. Moreover, the authors provided formulas for the intersection numbers of P_n, C_n, W_n, F_n, K_n, and $G + K_1$ for any connected graph G. They also defined the concept of an extreme intersection graph. A graph G is an *extreme intersection graph* if for any family F of subsets of $S = \{1, 2, 3, ..., \omega(G)\}$ such that $\Omega(F) \cong G$, then $S \in F$.

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4744

Email addresses: jesrael.palco@msunaawan.edu.ph (Jesrael B. Palco), rnpaluga@carsu.edu.ph (Rolando N. Paluga)

2. Results

The *join* of two graphs G and H, denoted by $G + H$, is the graph with $V(G{+}H) = V(G) \cup V(H)$ and $E(G{+}H) = E(G) \cup E(H) \cup \{a_ib_j : a_i \in V(G) \text{ and } b_j \in V(H)\}$.

Theorem 1. Suppose G is not an extreme intersection graph. Then for any graph H, $\omega(G + H) \leq \omega(G)\omega(H)$.

Proof. Let G be not an extreme intersection graph. Then there exists a family F_1 of nonempty subsets of a set S_1 such that $S_1 \notin F_1$ and $\Omega(F_1) \cong G$. That is, there is an isomorphism $\phi_1 : V(G) \to F_1$ such that $\phi_1(x) \neq S_1$, for all $x \in V(G)$. Let H be any graph and suppose $\omega(H) = m$. Let $S_2 = \{1, 2, .., m\}$ and F_2 be a nonempty subset of a set S_2 for which $\Omega(F_2) \cong H$. That is, there is an isomorphism $\phi_2 : V(H) \to F_2$. Let $S = S_1 \times S_2$, and $F = (\cup\{A \times S_2 : A \in F_1\}) \cup (\cup\{S_1 \times B : B \in F_2\})$. Let $\phi : V(G + H) \to F$ be a mapping defined by

$$\phi(x) = \begin{cases} \phi_1(x) \times S_2, & \text{if } x \in V(G) \\ S_1 \times \phi_2(x), & \text{if } x \in V(H). \end{cases}$$

Let $x_1, x_2 \in V(G + H)$ such that $\phi(x_1) = \phi(x_2)$. The case $x_1 \in V(G)$ and $x_2 \in V(H)$ is not possible. Since $\phi(x_1) = \phi_1(x_1) \times S_2$ and $\phi(x_2) = S_1 \times \phi_2(x_2)$. Consider the following cases:

Case 1. Suppose $x_1,\ x_2 \in V(G)$. Then $\phi(x_1) = \phi_1(x_1) \times S_2$ and $\phi(x_2) = \phi_1(x_2) \times S_2$. Note that $\phi(x_1) = \phi(x_2)$, so we have $\phi_1(x_1) = \phi_1(x_2)$. Since ϕ_1 is one to one, $x_1 = x_2$.

Case 2. Suppose $x_1,\ x_2 \in V(H)$. Then $\phi(x_2) = S_1 \times \phi_2(x_1)$ and $\phi(x_2) = S_1 \times \phi_2(x_2)$. Note that $\phi(x_1) = \phi(x_2)$, so we have $\phi_2(x_1) = \phi_2(x_2)$. Since ϕ_2 is one to one, $x_1 = x_2$. Therefore, ϕ is one to one.

Let $u \in F$. If $u = S_1 \times B$, $B \in F_2$. Since ϕ_2 is onto, there exists $x \in V(H) \subseteq V(G{+}H)$ such that $\phi_2(x) = B$. Thus, $\phi(x) = S_1 \times \phi_2(x) = S_1 \times B = u$. Therefore, ϕ is onto.

If $u = A \times S_2$, $A \in F_1$. Since ϕ_1 is onto, there exists $x \in V(G) \subseteq V(G + H)$ such that $\phi_1(x) = A$. Thus, $\phi(x) = \phi_1(x) \times S_2 = A \times S_2 = u$. Therefore, ϕ is onto.

Let x_1 and x_2 be adjacent in $G + H$. Consider the following cases:

Case 1. Suppose x_1 and x_2 are adjacent in G. Then $\phi(x_1) = \phi_1(x_1) \times S_2$ and $\phi(x_2) = \phi_1(x_2) \times S_2$. Now,

$$\begin{aligned} \phi(x_1) \cap \phi(x_2) &= (\phi_1(x_1) \times S_2) \cap (\phi_1(x_2) \times S_2) \\ &= (\phi_1(x_1) \cap \phi_1(x_2)) \times S_2 \\ &\neq \varnothing, \text{since } \phi_1 \text{ preserves adjacency.} \end{aligned}$$

Therefore, $\phi(x_1)$ and $\phi(x_2)$ are adjacent in $\Omega(F)$.

Case 2. Suppose x_1 and x_2 are adjacent in H. Then $\phi(x_1) = S_1 \times \phi_2(x_1)$ and $\phi(x_2) = S_1 \times \phi_2(x_2)$. Now,

$$\begin{aligned} \phi(x_1) \cap \phi(x_2) &= (S_1 \times \phi_2(x_1)) \cap (S_1 \times \phi_2(x_2)) \\ &= S_1 \times (\phi_2(x_1) \cap \phi_2(x_2)) \end{aligned}$$

$$\neq \varnothing, \text{since } \phi_2 \text{ preserves adjacency.}$$

Therefore, $\phi(x_1)$ and $\phi(x_2)$ are adjacent $\Omega(F)$.
Case 3. Suppose $x_1 \in V(G)$ and $x_2 \in V(H)$. Then $\phi(x_1) = \phi_1(x_1) \times S_2$ and $\phi(x_2) = S_1 \times \phi_2(x_2)$. Now,

$$\begin{aligned}\phi(x_1) \cap \phi(x_2) &= (\phi_1(x_1) \times S_2) \cap (S_1 \times \phi_2(x_2))\\ &= (\phi_1(x_1) \cap S_1) \times (S_2 \cap \phi_2(x_2))\\ &= \phi_1(x_1) \times \phi_2(x_2), \text{since } \phi_1(x_1) \subseteq S_1 \text{ and } \phi_2(x_2) \subseteq S_2\\ &\neq \varnothing.\end{aligned}$$

Therefore, $\phi(x_1)$ and $\phi(x_2)$ are adjacent $\Omega(F)$.

Let $u, v \in F$. If $u = A \times S_2$ and $v = S_1 \times B$ for some $A \in F_1$ and $B \in F_2$, then $u = \phi_1(x) \times S_2$ and $v = S_1 \times \phi_2(y)$ for some $x \in V(G)$ and $y \in V(H)$. Consequently, $\phi^{-1}(u) = x \in V(G)$ and $\phi^{-1}(v) = y \in V(H)$. It follows that x and y are adjacent in $G + H$.

If $u = A_1 \times S_2$ and $v = A_2 \times S_2$, for some $A_1, A_2 \in F_1$ then $u = \phi_1(x_1) \times S_2 = \phi(x_1)$ and $v = \phi_1(x_2) \times S_2 = \phi(x_2)$, for some $x_1, x_2 \in V(G)$. Consequently, $\phi^{-1}(u) = x_1 \in V(G)$ and $\phi^{-1}(v) = x_2 \in V(G)$. Thus, x_1 and x_2 are adjacent in G.

If $u = S_1 \times B_1$ and $v = S_1 \times B_2$, for some $B_1, B_2 \in F_2$ then $u = S_1 \times \phi_1(y_1) = \phi(y_1)$ and $v = S_1 \times \phi_1(y_2) = \phi(y_2)$ for some $y_1, y_2 \in V(H)$. Consequently, $\phi^{-1}(u) = y_1 \in V(H)$ and $\phi^{-1}(v) = y_2 \in V(H)$. Thus, y_1 and y_2 are adjacent in H. Therefore, ϕ preserves adjacency.

Hence, $\Omega(F) \cong G + H$

Accordingly, $\omega(G + H) \leq |S|$, since $S = S_1 \times S_2$. Then $|S| = |S_1||S_2| = \omega(G)\omega(H)$.

Hence, $\omega(G + H) \leq \omega(G)\omega(H)$. □

Let G be a connected graph. A subset S of $V(G)$ is a **clique** if $\langle S \rangle$ is a complete graph. A clique M is **maximal** if $a \in V(G) - M$, then $M \cup \{a\}$ is no longer a clique in G. The **clique graph** of G, denoted by $\zeta(G)$, is the intersection graph of the set of all maximal cliques of G. The **clique order** of G, denoted by $co(G)$, is $|V(\zeta(G))|$. That is, $co(G)$ is the number of maximal cliques in G.

Theorem 2. Let K_n, P_n and C_n be a complete graph, path and cycle, respectively. Then

(i) $co(K_n) = 1$, $n \geq 1$

(ii) $co(P_n) = n - 1$, $n \geq 2$

(iii) $co(C_n) = \begin{cases} 1, & if\ n = 3 \\ n, & if\ n \geq 4 \end{cases}$

The *corona* $G \circ H$ of two graphs G and H, is the graph obtained by making n copies (n is the ordered of G) of H and joining every vertex of the ith copy of H with the vertex v_i of G. For each $a \in V(G)$, we denote by H^a the copy of H corresponding to the vertex a.

Theorem 3. Let G be a connected graph and H be any graph. Then

$$\omega(G \circ H) \leq co(G) + |V(G)| \cdot \omega(H).$$

Proof. Let $V(G) = \{a_1, a_2, a_3, ..., a_n\}$ and $V(\zeta(G)) = \{B_1, B_2, ..., B_{co(G)}\}$. For each $i = 1, 2, ..., n$, let F_i be a collection of nonempty subsets of $S_i = \{(i, j) : 1 \leq j \leq \omega(H)\}$ such that $\Omega(F_i) \cong Ha_i$. For each $i = 1, 2, ..., n$, let $\phi_i : V(Ha_i) \to F_i$ be an isomorphism. Let $S_o = \{(0, j) : 1 \leq j \leq co(G)\}$ and $S = \bigcup_{i=0}^{n} S_i$. For each $i = 1, 2, ..., n$, let $T_i = \{(0, j) : a_i \in B_j$, for some $j\}$. Let $F = (\bigcup_{i=1}^{n} F_i) \bigcup \{S_i \bigcup T_i : 1 \leq i \leq n\}$.
Define a mapping $\phi : V(G \circ H) \to F$ as follows

$$\phi(x) = \begin{cases} \phi_i(x), & \text{if } x \in V(Ha_i), \text{ for some } i \\ S_i \cup T_i, & \text{for some } i. \end{cases}$$

Let $x_1, x_2 \in V(G \circ H)$ such that $\phi(x_1) = \phi(x_2)$. Suppose $x_1 \in V(G)$ and $x_2 \in V(Ha_i)$ for some i. Then $x_1 \in B_j$ for some j. Thus, $(0, j) \in \phi(x_1)$. Now, $\phi(x_2) = \phi_i(x_2) \subseteq S_i$, so $(0, j) \notin S_j$. This is a contradiction. Therefore, the case $x_1 \in V(G)$ and $x_2 \in V(H_{a_i})$ is not possible. Consider the following cases:
Case 1. Suppose $x_1, x_2 \in V(G)$. Then $x_1 = a_i$ and $x_2 = a_j$. Thus, $\phi(x_1) = S_i \cup T_i$ and $\phi(x_2) = S_j \cup T_j$. Note that $(i, 1) \in S_i \subseteq \phi(x_1) = \phi(x_2)$. It follows that $(i, 1) \in S_j = \{(j, 1), (j, 2), ..., (j, \omega(H))\}$. Consequently, $i = j$. In effect $x_1 = x_2$.
Case 2. Suppose $x_1 \in V(H_{a_i})$ and $x_2 \in V(H_{a_j})$. Suppose $i \neq j$. Then $\phi(x_1) \cap \phi(x_2) = \phi_i(x_1) \cap \phi_j(x_2) \subseteq S_i \cap S_j \neq \emptyset$. This is a contradiction. Hence, $i = j$. Consequently, $\phi_i(x_1) = \phi(x_1) = \phi(x_2) = \phi_j(x_2) = \phi_i(x_2)$. Since ϕ_i is one to one, $x_1 = x_2$. Therefore, ϕ is one to one.

Suppose $B \in F_i$ for some i. Since $\phi_i : V(H_{a_i}) \to F_i$ is onto, there exists $x \in V(H_{a_i})$ such that $\phi_i(x) = B$. Consequently, $\phi(x) = \phi_i(x) = B$. Suppose $B = S_i \cup T_i$, for some i. Take $x = a_i$. Then $\phi(x) = \phi(a_i) = B$. Hence, ϕ is onto.

Let x_1 and x_2 be adjacent in $G \circ H$. Consider the following cases:
Case 1. Suppose x_1 and x_2 are adjacent in G. Then $x_1 = a_i$ and $x_2 = a_j$, for some i and j. In effect, $\phi(x_1) = S_i \cup T_i$ and $\phi(x_2) = S_j \cup T_j$. Since a_i and a_j are adjacent in G, there exists k such that $a_i, a_j \in B_k$. This implies that $(0, k) \in T_i$ and $(0, k) \in T_j$. It follows $\phi(x_1) \cap \phi(x_2) \neq \emptyset$. Therefore, $\phi(x_1)$ and $\phi(x_2)$ are adjacent in $\Omega(F)$.
Case 2. Suppose x_1 and x_2 are adjacent in H_{a_i} for some i. Then $x_1, x_2 \in V(H_{a_i})$. It follows $\phi(x_1) = \phi_i(x_1)$ and $\phi(x_2) = \phi_i(x_2)$. Since ϕ_i preserves adjacency, $\phi(x_1) \cap \phi(x_2) = \phi_i(x_1) \cap \phi_i(x_2) \neq \emptyset$. Thus, $\phi(x_1)$ and $\phi(x_2)$ are adjacent in $\Omega(F)$.
Case 3. Suppose $x_1 = a_i$ and $x_2 \in V(H_{a_i})$. Then $\phi(x_1) = \phi(a_i) = S_i \cup T_i$ and $\phi(x_2) = \phi_i(x_2)$. Since $\phi_i(x_2) \subseteq S_i$, $\phi(x_1) \cap \phi(x_2) \neq \emptyset$. Thus, $\phi(x_1)$ and $\phi(x_2)$ are adjacent in $\Omega(F)$.

Suppose A and B are adjacent in $\Omega(F)$. That is, $A \cap B \neq \emptyset$. The case $A \in F_i$ and $B \in F_j$, where $i, j \neq 0$ and $i \neq j$, is not possible, since $S_i \cap S_j = \emptyset$ in this case. Consider the following cases:
Case 1. Suppose $A, B \in F_i$, for some i. Since ϕ_i is onto, there exists
$x_1, x_2 \in V(H_{a_i})$ such that $\phi_i(x_1) = A$ and $\phi_i(x_2) = B$. Since ϕ_i preserves adjacency, x_1

and x_2 are adjacent in H_{a_i}. It follows that x_1 and x_2 are adjacent in $G \circ H$.
Case 2. Suppose $A = S_i \cup T_i$ and $B = S_j \cup T_j$ for some $i, j = 1, 2, 3, ..., n$, $i \neq j$. Since $A \cap B \neq \emptyset$, $(S_i \cap S_j) \cup (S_i \cap T_j) \cup (T_i \cap S_j) \cup (T_i \cap T_j) \neq \emptyset$. Note that $S_i \cap S_j = \emptyset$, $S_i \cap T_j = \emptyset$, $T_i \cap S_j = \emptyset$. Consequently, $(T_i \cap T_j) \neq \emptyset$. Moreover, $\phi(a_i) = A$ and $\phi(a_j) = B$. Let $t \in T_i \cap T_j$. Then $t \in T_i$ and $t \in T_j$. This implies that $t = (0, r)$ where $a_i \in B_r$ and $t = (0, s)$ where $a_j \in B_s$. Obviously, $r = s$ and $a_i, a_j \in B_r$. It follows that a_i and a_j are adjacent in G. Accordingly, a_i and a_j are adjacent in $G \circ H$.
Case 3. Suppose $A \in F_i$ and $B = S_j \cup T_j$ for some i and j. Suppose $i \neq j$. Then $\phi(a) = \phi_i(a) = A$ for some $a \in V(H_{a_i})$ and $\phi(a_j) = B$. Since $A \cap B \neq \emptyset$, $(A \cap S_j) \cup (A \cap T_j) \neq \emptyset$. Since $A \subseteq S_i$, $A \cap T_j \subseteq S_i \cap T_j = \emptyset$ and $A \cap S_j \subseteq S_i \cap S_j = \emptyset$. This is a contradiction. Thus, $i = j$. Consequently, $a \in V(H_{a_j})$. It follows that a and a_j are adjacent in $G \circ H$. Hence ϕ preserves adjacency.

Therefore, $\Omega(F) \cong G \circ H$.
Accordingly,

$$\begin{aligned}
\omega(G \circ H) &\leq |S| \\
&= \sum_{i=0}^{n} |S_i| \\
&= |S_o| + \sum_{i=1}^{n} |S_i| \\
&= co(G) + \sum_{i=1}^{n} \omega(H) \\
&= co(G) + n \cdot \omega(H) \\
&= co(G) + |V(G)| \cdot \omega(H).
\end{aligned}$$

Therefore, $\omega(G \circ H) \leq co(G) + |V(G)| \cdot \omega(H)$.

□

Corollary 1. Let G be a connected graph and $n \geq 2$. Then $\omega(K_n \circ G) = 1 + n \cdot \omega(G)$.

Proof. By Theorem 3, $\omega(K_n \circ G) \leq co(K_n) + |V(K_n)| \cdot \omega(G)$. By Theorem 2, $co(K_n) = 1$. Thus,

$$\begin{aligned}
\omega(K_n \circ G) &\leq co(K_n) + |V(K_n)| \cdot \omega(G) \\
&= 1 + n \cdot \omega(G).
\end{aligned}$$

Suppose $\omega(K_n \circ G) < 1 + n \cdot \omega(G)$. Let $V(K_n) = \{a_1, a_2, ..., a_n\}$ and for each i, $1 \leq i \leq n$, let G_i be the ith copy of G corresponding to the vertex a_i. Let F be a collection of subsets of $S = \{1, 2, 3, ..., \omega(K_n \circ G)\}$ such that $\Omega(F) \cong K_n \circ G$. Let $\phi : V(K_n \circ G) \to F$ be an isomorphism. For each i, $1 \leq i \leq n$, $\{\phi(x) : x \in V(G_i)\}$ is a set representation for G_i. Thus, $|\cup_{x \in V(G_i)} \phi(x)| \geq \omega(G_i) = \omega(G)$. Note that for each i, j, $i \neq j$, and each $a \in G_i$

and $b \in G_j$, $ab \notin E(K_n \circ G)$. Consequently, $E_i = \cup_{x \in V(G_i)} \phi(x)$ and $E_j = \cup_{x \in V(G_j)} \phi(x)$ are disjoint whenever $i \neq j$. Now,

$$\begin{aligned} |\cup_{i=1}^n E_i| &= \sum_{i=1}^n |E_i| \\ &\geq \sum_{i=1}^n \omega(G) \\ &= n \cdot \omega(G). \end{aligned}$$

It follows that the elements of $S - (\cup_{i=1}^n E_i)$ are used for the set representation of G. Note that

$$\begin{aligned} |S - (\cup_{i=1}^n E_i)| &= |S| - |(\cup_{i=1}^n E_i)| \\ &\leq \omega(K_n \circ G) - n \cdot \omega(G), \text{since we suppose } \omega(K_n \circ G) < 1 + n \cdot \omega(G). \\ &< 1. \end{aligned}$$

That is, $|S - (\cup_{i=1}^n E_i)| = 0$. This implies, $S = \cup_{i=1}^n E_i$. Since a_1 and a_2 are adjacent, $\phi(a_1) \cap \phi(a_2) \neq \emptyset$. Let $t \in \phi(a_1) \cap \phi(a_2)$. Then $t \in \phi(a_1)$ and $t \in \phi(a_2)$. Since $S = \cup_{i=1}^n E_i$, $t \in E_r$ for some r. Thus $t \in \phi(x)$ for $x \in V(G_r)$. Therefore, $\langle \{x, a_1, a_2\} \rangle$ is complete. This is a contradiction.

Therefore, $\omega(K_n \circ G) = 1 + n \cdot \omega(G)$.

□

Corollary 2. Let G be a connected graph and $n \geq 2$. Then $\omega(P_n \circ G) = (n-1) + n \cdot \omega(G)$.

Proof. By Theorem 3, $\omega(P_n \circ G) \leq co(P_n) + |V(P_n)| \cdot \omega(G)$. By Theorem 2, $co(P_n) = n - 1$. Thus,

$$\begin{aligned} \omega(P_n \circ G) &\leq co(P_n) + |V(P_n)| \cdot \omega(G) \\ &= (n-1) + n \cdot \omega(G). \end{aligned}$$

Suppose $\omega(P_n \circ G) < (n-1) + n \cdot \omega(G)$. Let $V(P_n) = \{a_1, a_2, ..., a_n\}$, $E(P_n) = \{a_i a_{i+1} : 1 \leq i \leq n-1\}$ and for each i, $1 \leq i \leq n$, let G_i be the ith copy of G corresponding to the vertex a_i. Let F be a collection of subsets of $S = \{1, 2, 3, ..., \omega(P_n \circ G)\}$ such that $\Omega(F) \cong P_n \circ G$. Let $\phi : V(P_n \circ G) \to F$ be an isomorphism. For each i, $1 \leq i \leq n$, $\{\phi(x) : x \in V(G_i)\}$ is a set representation for G_i. Thus, $|\cup_{x \in V(G_i)} \phi(x)| \geq \omega(G_i) = \omega(G)$. Note that for each i, j, $i \neq j$, and each $a \in G_i$ and $b \in G_j$, $ab \notin E(P_n \circ G)$. Consequently, $E_i = \cup_{x \in V(G_i)} \phi(x)$ and $E_j = \cup_{x \in V(G_j)} \phi(x)$ are disjoint whenever $i \neq j$. Now,

$$\begin{aligned} |\cup_{i=1}^n E_i| &= \sum_{i=1}^n |E_i| \\ &\geq \sum_{i=1}^n \omega(G) \end{aligned}$$

$$= n \cdot \omega(G).$$

It follows that the elements of $S - (\cup_{i=1}^{n} E_i)$ are used for the set representation of G. Note that

$$\begin{aligned} |S - (\cup_{i=1}^{n} E_i)| &= |S| - |(\cup_{i=1}^{n} E_i)| \\ &\leq \omega(P_n \circ G) - n \cdot \omega(G), \text{since we suppose } \omega(P_n \circ G) < (n-1) + n \cdot \omega(G). \\ &< n - 1. \end{aligned}$$

Since a_i and a_{i+1} are adjacent, $\phi(a_i) \cap \phi(a_{i+1}) \neq \emptyset$, for every i, $1 \leq i \leq n-1$. Let $A_i = \phi(a_i) \cap \phi(a_{i+1})$, $1 \leq i \leq n-1$. Since $|S - (\cup_{i=1}^{n} E_i)| < n-1$, there exist i, j with $i < j$, such that $A_i \cap A_j \neq \emptyset$. Let $t \in A_i \cap A_j$. Then $t \in A_i$ and $t \in A_j$. It follows that $t \in \phi(a_i)$ and $t \in \phi(a_{j+1})$. Note that $j \geq i+1$, it follows a_i and a_{j+1} are adjacent. This is a contradiction.

Hence, $\omega(P_n \circ G) = (n-1) + n \cdot \omega(G)$.

□

Corollary 3. Let G be a connected graph. Then

$$\omega(C_n \circ G) = \begin{cases} 1 + 3\omega(G), & \text{if } n = 3 \\ n + n \cdot \omega(G), & \text{if } n \geq 4. \end{cases}$$

Proof. By Theorem 3, $\omega(C_n \circ G) \leq co(C_n) + |V(C_n)| \cdot \omega(G)$. By Theorem 2,

$$co(C_n) = \begin{cases} 1, & if\ n = 3 \\ n, & if\ n \geq 4 \end{cases}$$

The case $n = 3$, follows from Corollary 1 and for $n \geq 4$,

$$\begin{aligned} \omega(C_n \circ G) &\leq co(C_n) + |V(C_n)| \cdot \omega(G) \\ &= n + n \cdot \omega(G). \end{aligned}$$

Suppose $\omega(C_n \circ G) < n + n \cdot \omega(G)$. Let $V(C_n) = \{a_1, a_2, ..., a_n\}$, $E(C_n) = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_1 a_n\}$ and for each i, $1 \leq i \leq n$, let G_i be the ith copy of G corresponding to the vertex a_i. Let F be a collection of subsets of $S = \{1, 2, 3, ..., \omega(C_n \circ G)\}$ such that $\Omega(F) \cong C_n \circ G$. Let $\phi : V(C_n \circ G) \to F$ be an isomorphism. For each i, $1 \leq i \leq n$, $\{\phi(x) : x \in V(G_i)\}$ is a set representation for G_i. Thus, $|\cup_{x \in V(G_i)} \phi(x)| \geq \omega(G_i) = \omega(G)$. Note that for each i, j, $i \neq j$, and each $a \in G_i$ and $b \in G_j$, $ab \notin E(C_n \circ G)$. Consequently, $E_i = \cup_{x \in V(G_i)} \phi(x)$ and $E_j = \cup_{x \in V(G_j)} \phi(x)$ are disjoint whenever $i \neq j$. Now,

$$|\cup_{i=1}^{n} E_i| = \sum_{i=1}^{n} |E_i|$$

$$\geq \sum_{i=1}^{n} \omega(G)$$
$$= n \cdot \omega(G).$$

It follows that the elements of $S - (\cup_{i=1}^{n} E_i)$ are used for the set representation of G. Note that

$$\begin{aligned}|S - (\cup_{i=1}^{n} E_i)| &= |S| - |(\cup_{i=1}^{n} E_i)| \\ &\leq \omega(C_n \circ G) - n \cdot \omega(G), \text{since we suppose } \omega(C_n \circ G) < n + n \cdot \omega(G). \\ &< n.\end{aligned}$$

Since a_i and a_{i+1} are adjacent, $\phi(a_i) \cap \phi(a_{i+1}) \neq \emptyset$, for every i, $1 \leq i \leq n$. Let $A_i = \phi(a_i) \cap \phi(a_{i+1})$, $1 \leq i \leq n$. Since $|S - (\cup_{i=1}^{n} E_i)| < n$, there exist i, j with $i < j$, such that $A_i \cap A_j \neq \emptyset$. Let $t \in A_i \cap A_j$. Then $t \in A_i$ and $t \in A_j$. It follows that $t \in \phi(a_i)$ and $t \in \phi(a_{j+1})$. Note that $j \geq i + 1$, it follows a_i and a_{j+1} are adjacent. This is a contradiction.

Hence, $\omega(C_n \circ G) = n + n \cdot \omega(G)$.

□

Corollary 4. Let $n \geq 3$. Then

$$\omega(Cr_n) = \begin{cases} 4, & \text{if } n = 3 \\ 2n, & \text{if } n \geq 4. \end{cases}$$

Proof. The proof follows from Corollary 3.

□

Acknowledgements

The author would like to thank the peer reviewers of the paper and this research is funded by the Mindanao State University at Naawan.

References

[1] Paul Erdos, A Goodman, and Louis Posa. The representation of a graphing by set intersections. *Canadian Journal of Mathematics*, 18:106–112, 1966.

[2] Frank Harary. *Graph Theory.* Addison-Wesly Publishing Company, Massachusetts, 1972.

[3] Palco J and Paluga R. Intersection number of some graphs. *The Mindanawan Journal of Mathematics*, 3:63–75, 2012.