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Solution of Integral Equations Via Laplace ARA Transform

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Abstract. This research article demonstrates an efficient method for solving partial integro-differential equations. The intention of this research is to establish the solution of some different classes of integral equations, by utilizing the double Laplace ARA transform. We present some definitions and basic concepts related to the double Laplace ARA transform. The results of the examples support the theoretical results and show the accuracy and applicability of the presented approach.

2020 Mathematics Subject Classifications: 44A05, 34A25

Key Words and Phrases: Double Laplace-ARA transform, ARA transform, Laplace transform, Partial integro-differential equations

1. Introduction

Integral transforms play a vital role in solving integral equations and partial integro-differential equations. For this reason, many phenomena in the field of engineering, science, and mathematical physics can be represented by integral equations of different types [6, 8, 9, 12, 15, 16, 26]. Using integral transformations, we can transform integral equations into algebraic or differential equations and get the exact solution of the target integral equations. Developed through the hard work of many scientists and researchers, these techniques are used today to tackle challenging problems in contemporary arithmetic. These transformations enable us to get the exact solutions of the objective equations without the need for linearization or discretization, like Laplace, Fourier, Elzaki, Natural, Sumudu, and ARA transformations [14, 17, 18, 24, 25, 27]. They are used in transforming the partial differential equations into ordinary equations using a simple transformation, or into algebraic equations using a double integral transformation.
The double transformations have also widespread applied to solve partial differential equations with unknown two variable functions, and as a result, double transformations have been considered to be very effective in handling partial differential equations compared to other numerical approaches [3, 7, 11]. In addition, extensions of the double transformation

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have been developed in the relevant literature, such as double Laplace transform, double Shehu transform [10], double Sumudu transform [21, 28], double Elzaki transform [13], double Laplace-Sumudu transform [20], ARA–Sumudu transform [5, 19] and double ARA transform [22, 23].

The ARA transformation is introduced in 2020 [18]. It is defined by the improper integral.

$$\mathcal{G}_n \left[\varphi \left(u\right)\right] = v \int_0^\infty u^{n-1} e^{-uv} \varphi \left(u\right) du, \quad v > 0.$$

This transformation has attracted much attention from researchers due to its ability to generate multiple transformations of index n, and it could also easily overcome the challenges of having singular points in differential equations. Despite all these merits it could be used to solve different kinds of problems [1, 4].

In this research, we introduce a new Laplace and ARA combination, so that we can take advantage of these two powerful transforms. This combination is called the double Laplace-ARA transform (DL-ARAT) [2]. Basic properties and concepts related to DL-ARAT are obtained and proven, also we process the values of some functions by DL-ARAT. To help us in solving integral equations new relations related to the double convolution theorem and partial derivatives are implemented and established. The novelty of this research is evident in these combinations between Laplace and ARA transforms, in which the new DL-ARAT have the advantages of the two transforms, the applicability of ARA in handling some singular points found in the equations and the simplicity of Laplace.

In this work we use the first order ARA transform $\mathcal{G}_1 \left[\varphi(u)\right]$, which we denote by $\mathcal{G} \left[\varphi(u)\right]$ for the sake of simplicity.

The motivation of this work is to present a novel double integral transform, that combines two powerful transforms, Laplace and ARA transforms. The new approach has the merits of the two transforms and can solve different kinds of problems.

The remaining part of the paper is set up as follows. Section 2 defines the basic definitions and properties of the ARA transform and Laplace transform. In Section 3 basic properties and theorems of DL-ARAT are presented and proved, and we apply DL-ARAT to some functions. By applying the integral transform DL-ARAT to solve the second type nonlinear VIE and solving significant examples in Section 4, the effectiveness and efficiency of the proposed method are illustrated. Finally, in Section 5, the conclusion of the work is presented.

2. Preliminaries and Notations

In this part, we will provide the basic definitions and some properties of the Laplace and the ARA transforms that will be needed in later sections.

Definition 1. *[27] The Laplace transform of the function $\varphi(t)$ of $t > 0$ is the function $\Phi\left(s\right) = \mathcal{L}[\varphi(t)]$, defined by*

$$\mathcal{L}\left[\varphi\left(t\right)\right] = \int_0^\infty e^{-st} \varphi\left(t\right) dt, \qquad Re(s) > 0, \tag{1}$$

inverse Laplace transform of $\Phi(s)$ *is given by*

$$\mathcal{L}^{-1}[\Phi(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Phi(s)\,ds = \varphi(t), \quad t>0. \tag{2}$$

Theorem 1. *[27] If the piecewise continuous function* $\varphi(t)$ *and of exponential order* k *on the interval* $0 \le t < \infty$. *Then* $\mathcal{L}[\varphi(t)]$ *exists for* $Re(s) > k$ *and satisfies*

$$|\varphi(t)| \le Me^{kt}, \qquad M>0,$$

where M *is a constant. Then Laplace transform integral converges absolutely for* $Re(s) > k$.

Proof. Using the definition of Laplace transform, we get

$$|\Phi(s)| = \left|\int_0^\infty e^{-st}\varphi(t)dt\right| \le \int_0^\infty e^{-st}|\varphi(t)|dt \le M\int_0^\infty e^{-(s-k)t}dt = \frac{M}{s-k},$$

where $Re(s) > k$.
Thus, Laplace transform integral converges absolutely for $Re(s) > k$.

Definition 2. *[18] The first order ARA integral transform of a continuous function* $\varphi(u)$ *on the interval* $(0,\infty)$ *is introduced as*

$$\mathcal{G}[\varphi(u)](v) = \Phi(v) = v\int_0^\infty e^{-uv}\varphi(u)du, \quad Re(v)>0. \tag{3}$$

The inverse ARA transform is defined by

$$\mathcal{G}^{-1}[\mathcal{G}[\varphi(u)]] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{uv}\Phi(v)\,dv = \varphi(u), \tag{4}$$

where

$$\Phi(v) = \int_0^\infty e^{-uv}\varphi(u)du.$$

Theorem 2. *[18] If* $\varphi(u)$ *is piecewise continuous in* $0 \le u \le k$ *and satisfies*

$$|u\varphi(u)| \le Me^{ku}, \qquad M>0,$$

where M *is a constant, then the ARA transform exists for all* $Re(v) > k$.

Proof. Using the definition of the ARA transform, we have

$$\begin{aligned}|\Phi(n,v)| &= \left|v\int_0^\infty ue^{-uv}\varphi(u)du\right| = \left|v\int_0^\alpha ue^{-uv}\varphi(u)du + v\int_\alpha^\infty ue^{-uv}\varphi(u)du\right| \\ &\le v\left|\int_\alpha^\infty ue^{-uv}\varphi(u)du\right| \le v\int_\alpha^\infty e^{-uv}|u\,\varphi(u)|du \\ &\le v\int_\alpha^\infty e^{-uv}Me^{ku}du = vM\int_\alpha^\infty e^{-(v-k)u}du = \frac{vM}{v-k}e^{-\alpha(v-k)}.\end{aligned}$$

This integral converges for all $Re(v) > k$.
Thus, $\mathcal{G}\left[\varphi\left(u\right)\right]$ exists.

In the table below (Table 1) we introduce the Laplace transform and the ARA transform for some functions and give some basic properties of both transforms, where $\varphi\left(t\right)$ and $\psi(t)$ are continuous functions, $\alpha, \beta \in \mathbb{R}$.

Table 1: Laplace transform and ARA transform for some basic functions.

	Laplace transform	ARA transform
$\alpha f \varphi\left(t\right) + \beta\ \psi(t)$	$\alpha\ \mathcal{L}\left[\varphi\left(t\right)\right] + \beta\ \mathcal{L}\left[\psi(t)\right]$	$\alpha\ \mathcal{G}\left[\varphi\left(t\right)\right] + \beta\ \mathcal{G}\left[\psi(t)\right]$
t^{α}	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \alpha \geq 0$	$\frac{\Gamma(\alpha+1)}{v^{\alpha}},\quad \alpha > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}\ , \alpha \in \mathbb{R}$	$\frac{v}{v-\alpha}$
$\varphi\prime\left(t\right)$	$s\mathcal{L}\left[\varphi\left(t\right)\right] - \varphi\left(0\right)$	$v\mathcal{G}\left[\varphi\left(t\right)\right] - v\varphi(0)$
$\varphi^{(m)}\left(t\right)$	$s^{m}\mathcal{L}\left[\varphi\left(t\right)\right] - \sum_{l=1}^{m} s^{m-l}\varphi^{(l-1)}\left(0\right)$	$v^{m}\mathcal{G}\left[\varphi\left(t\right)\right] - \sum_{l=1}^{m} v^{m-l+1}\varphi^{(l-1)}\left(0\right)$
$\sin\left(\alpha t\right)$	$\frac{\alpha}{s^2+\alpha^2}$	$\frac{\alpha v}{v^2+\alpha^2}$
$\cos\left(\alpha t\right)$	$\frac{s}{s^2+\alpha^2}$	$\frac{v^2}{v^2+\alpha^2}$
$\sinh\left(\alpha t\right)$	$\frac{\alpha}{s^2-\alpha^2}$	$\frac{\alpha v}{v^2-\alpha^2}$
$\cosh\left(\alpha t\right)$	$\frac{s}{s^2-\alpha^2}$	$\frac{v^2}{v^2-\alpha^2},$
$(\varphi * \psi)(t)$	$\mathcal{L}\left[\varphi\left(t\right)\right]\ \mathcal{L}\left[\psi\left(t\right)\right]$	$\frac{\mathcal{G}[\varphi(t)]\ \mathcal{G}[\psi(t)]}{v}$

3. Double Laplace-ARA Transform of First Order (DL-ARAT)

The integral transform DL-ARAT is introduced in this section that combines the Laplace transform and the ARA transform of first order. A fundamental properties and theorems for DL-ARA are presented.

Definition 3. *The DL-ARAT of a continuous function $\varphi(t,u)$ is defined as*

$$\mathcal{L}_t\mathcal{G}_u\left[\varphi\left(t,u\right)\right] = \Phi\left(s,v\right) = v\int_0^{\infty}\int_0^{\infty} e^{-st-uv}\varphi\left(t,u\right)dt\ du,\quad s,v > 0. \tag{5}$$

Clearly that DL-ARAT is a linear

$$\mathcal{L}_t\mathcal{G}_u\left[\alpha\ \varphi\left(t,u\right) + \beta\ \ \psi(t,u)\right] = \alpha\ \mathcal{L}_t\mathcal{G}_u\left[\ \varphi\left(t,u\right)\right] + \beta\ \ \mathcal{L}_t\mathcal{G}_u\left[\ \psi(t,u)\right], \tag{6}$$

where α and β are constants and the functions $\mathcal{L}_t\mathcal{G}_u\left[\varphi\left(t,u\right)\right]\ , \mathcal{L}_t\mathcal{G}_u\left[\psi(t,u)\right]$ are exists. The inverse of the DL-ARAT is provided by

$$\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\Phi\left(s,v\right)\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}ds\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{e^{uv}}{v}\Phi\left(s,v\right)dv = \varphi\left(t,u\right). \tag{7}$$

Properties

(i) Suppose that $\varphi(t,u) = \varphi_1(t)\,\varphi_2(u)$, $t,\ u > 0$. Then

$$\mathcal{L}_t\mathcal{G}_u\left[\varphi(t,u)\right] = \mathcal{L}_t\left[\varphi_1(t)\right]\mathcal{G}_u\left[\varphi_2(u)\right].$$

Proof.

$$\begin{aligned}\mathcal{L}_t\mathcal{G}_u\left[\varphi(t,u)\right] = \mathcal{L}_t\mathcal{G}_u\left[\varphi_1(t)\,\varphi_2(u)\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}\varphi_1(t)\,\varphi_2(u)dt\ du\\ &= \int_0^\infty \varphi_1(t)\,e^{-ts}dt\cdot v\int_0^\infty \varphi_2(u)\,e^{-uv}du = \mathcal{L}_t\left[\varphi_1(t)\right]\mathcal{G}_u\left[\varphi_2(u)\right].\end{aligned}$$

(ii) DL-ARAT of basic functions

- Suppose that $\varphi(t,u) = 1$, $t,\ u > 0$. Then

$$\mathcal{L}_t\mathcal{G}_u\left[1\right] = v\int_0^\infty\int_0^\infty e^{-ts-uv}dt\ du = \int_0^\infty e^{-ts}dt\ v\int_0^\infty e^{-uv}du = \frac{1}{s},$$

where $Re(t) > 0$.

- Suppose that $\varphi(t,u) = t^\alpha u^\beta$, α,β are constants, and $t,u > 0$. Then

$$\begin{aligned}\mathcal{L}_t\mathcal{G}_u\left[t^\alpha\ u^\beta\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}t^\alpha u^\beta dt\ du = \int_0^\infty e^{-ts}t^\alpha dt\ v\int_0^\infty e^{-uv}u^\beta du\\ &= \frac{\Gamma(\alpha+1)\ \Gamma(\beta+1)}{s^{\alpha+1}v^\beta},\quad Re(\alpha) > -1\ , Re(\beta) > -1.\end{aligned}$$

- Suppose that $\varphi(t,u) = e^{\alpha t+\beta u}$, α,β are constants, and $t,u > 0$. Then

$$\begin{aligned}\mathcal{L}_t\mathcal{G}_u\left[e^{\alpha t+\beta u}\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}e^{\alpha t+\beta u}dt\ du\\ &= \int_0^\infty e^{-ts}e^{\alpha t}dt\ v\int_0^\infty e^{-uv}e^{uv}du\\ &= \frac{v}{(s-a)(v-b)}.\end{aligned}$$

Likewise,

$$\mathcal{L}_t\mathcal{G}_u\left[e^{i(\alpha t+\beta u)}\right] = \frac{v}{(s-i\alpha)(v-i\beta)} = \frac{v(sv-\alpha\beta)+iv(s\beta+v\alpha)}{(s^2+\alpha^2)(v^2+\beta^2)}.$$

Consequently,

$$\begin{aligned}\mathcal{L}_t\mathcal{G}_u\left[\sin(\alpha t+\beta u)\right] &= \frac{v(\beta s+\alpha v)}{(s^2+\alpha^2)(sv^2+\beta^2)},\\ \mathcal{L}_t\mathcal{G}_u\left[\cos(\alpha t+\beta u)\right] &= \frac{v(sv-\alpha\beta)}{(s^2+\alpha^2)(v^2+\beta^2)}.\end{aligned}$$

- Suppose that $\varphi(t,u) = \sinh(\alpha t + \beta u)$ or $\varphi(t,u) = \cosh(\alpha t + \beta u)$.
  Recall that

$$\mathcal{L}_t\mathcal{G}_u\left[\sinh(\alpha t + \beta u)\right] = \frac{v(s\beta + v\alpha)}{(s^2-\alpha^2)(v^2-\beta^2)},$$
$$\mathcal{L}_t\mathcal{G}_u\left[\cosh(\alpha t + \beta u)\right] = \frac{v(sv + \alpha\beta)}{(s^2-\alpha^2)(v^2-\beta^2)}.$$

- Suppose that $\varphi(t,u) = J_0\left(c\sqrt{tu}\right)$

$$\begin{aligned}\mathcal{L}_t\mathcal{G}_u\left[J_0\left(c\sqrt{tu}\right)\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}J_0\left(c\sqrt{tu}\right)dt\ du\\ &= \int_0^\infty e^{-ts}J_0\left(c\sqrt{tu}\right)dt\ s\int_0^\infty e^{-uv}du\\ &= v\int_0^\infty e^{-\frac{c^2}{4v}u}e^{-uv}du = \frac{4v}{4sv+c^2}.\end{aligned}$$

- Suppose the function $\varphi(t,u)$ of exponential order α and β as t and u goto ∞. If $\exists N > 0$ such that $\forall t > T$ and $u > U$, we have

$$|\varphi(t,u)| \leq Ne^{\alpha t+\beta u}.$$

  We can write $\varphi(t,u) = O(e^{\alpha t+\beta u})$ as t and u goes to ∞, $s > \alpha$ and $v > \beta$.

Theorem 3. *Suppose that $\varphi(t,u)$ of exponential orders α and β is a continuous function on $[0,T)\times[0,U)$. Then $\mathcal{L}_t\mathcal{G}_u[\varphi(t,u)]$ exists for s and v gave $Re(s) > \alpha$ and $Re(v) > \beta$.*

Proof. By the definition of DL-ARAT, we get

$$\begin{aligned}|\Phi(s,v)| &= \left|v\int_0^\infty\int_0^\infty e^{-ts-uv}\varphi(t,u)dt\ du\right| \leq v\int_0^\infty\int_0^\infty e^{-ts-uv}\left|\varphi(t,u)\right|dt\ du\\ &\leq K\int_0^\infty e^{-(s-\alpha)t}dt\ v\int_0^\infty e^{-(v-\beta)u}du = \frac{Kv}{(u-\alpha)(v-\beta)},\\ &Re(s) > \alpha,\ Re(v) > \beta.\end{aligned}$$

Thus, $\mathcal{L}_t\mathcal{G}_u[\varphi(t,u)]$ exists.

Theorem 4. *Suppose that $\mathcal{L}_t\mathcal{G}_u[\varphi(t,u)]$ and $\mathcal{L}_t\mathcal{G}_u[\psi(t,u)]$ are exists and $\mathcal{L}_t\mathcal{G}_u[\varphi(t,u)] = \Phi(s,v)$, $\mathcal{L}_t\mathcal{G}_u[\psi(t,u)] = \Psi(s,v)$, then the double convolution*

$$\mathcal{L}_t\mathcal{G}_u\left[\varphi(t,u) ** \psi(t,u)\right] = \frac{1}{v}\Phi(s,v)\ \Psi(s,v), \tag{8}$$

where

$$\varphi(t,u) ** \psi(t,u) = \int_0^t\int_0^u \varphi(t-\tau,u-\rho)\psi(\tau,\rho)d\tau\ d\rho. \tag{9}$$

Proof. By the definition of DL-ARAT, we get

$$\begin{aligned}\mathcal{L}_t\mathcal{G}_u\left[\varphi(t,u)**\psi(t,u)\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}\left(\varphi(t,u)**\psi(t,u)\right)dt\ du\\ &= v\int_0^\infty\int_0^\infty e^{-ts-uv}\left(\int_0^t\int_0^u \varphi(t-\tau,u-\rho)\psi(\tau,\rho)d\tau\ d\rho\right)dt\ du.\end{aligned} \tag{10}$$

Equation (10) can be written as

$$\begin{aligned}&\mathcal{L}_t\mathcal{G}_u\left[\varphi(t,u)**\psi(t,u)\right]\\ &\quad= v\int_0^\infty\int_0^\infty e^{-ts-uv}\left(\int_0^\infty\int_0^\infty \varphi(t-\tau,u-\rho)H\left(t-\tau,u-\rho\right)\psi(\tau,\rho)d\tau\ d\rho\right)dt\ du\\ &\quad= \int_0^\infty\int_0^\infty \psi\left(\tau,\rho\right)d\tau\ d\rho\left(v\int_0^\infty\int_0^\infty e^{-s(t+\tau)-v(u+\rho)}\varphi(t-\tau,u-\rho)H\left(t-\tau,u-\rho\right)\right)dt\ du\\ &\quad= \Psi(s,v)\int_0^\infty\int_0^\infty e^{-s\tau-v\rho}\varphi\left(\tau,\rho\right)d\tau\ d\rho = \frac{1}{v}\Psi(s,v)\Phi\left(s,v\right).\end{aligned}$$

where $H\left(t-\tau,u-\rho\right)$ is the Heaviside unit step function.

Theorem 5. *Suppose that $\varphi(t,u)$ is a continuous function and $\mathcal{L}_t\mathcal{G}_u\left[\varphi(t,u)\right]=\Phi\left(s,v\right)$. Then, we have the following properties of derivatives*

(i)
$$\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial\varphi(t,u)}{\partial u}\right] = v\Phi\left(s,v\right) - v\mathcal{L}_t\left[\varphi(t,0)\right],$$

(ii)
$$\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial\varphi(t,u)}{\partial t}\right] = s\Phi\left(s,v\right) - \mathcal{G}_u\left[\varphi(0,u)\right],$$

(iii)
$$\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial^2\varphi(t,u)}{\partial u^2}\right] = v^2\Phi\left(s,v\right) - v^2\mathcal{L}_t\left[\varphi(t,0)\right] - v\mathcal{L}_t\left[\frac{\partial\varphi(t,0)}{\partial u}\right],$$

(iv)
$$\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial^2\varphi(t,u)}{\partial t^2}\right] = s^2\Phi\left(s,v\right) - s\mathcal{G}_u\left[\varphi(0,u)\right] - \mathcal{G}_u\left[\frac{\partial\varphi(0,u)}{\partial u}\right],$$

(v)
$$\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial^2\varphi(t,u)}{\partial t\partial u}\right] = sv\Phi\left(s,v\right) - sv\mathcal{L}_t\left[\varphi(t,0)\right] - v\mathcal{G}_u\left[\varphi(0,u)\right] + v\varphi(0,0).$$

Proof.

(i)

$$\begin{aligned}
\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial\varphi(t,u)}{\partial u}\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}\frac{\partial\varphi(t,u)}{\partial u}dt\ du\\
&= \int_0^\infty e^{-ts}dt\ v\int_0^\infty e^{-uv}\frac{\partial\varphi\left(t,u\right)}{\partial u}du\\
&= \int_0^\infty e^{-ts}\left(v\Phi\left(t,v\right)-v\varphi\left(t,0\right)\right)dt\\
&= v\Phi\left(s,v\right)-v\mathcal{L}_t\left[\varphi(t,0)\right].
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial\varphi(t,u)}{\partial t}\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}\frac{\partial\varphi(t,u)}{\partial t}dt\ du\\
&= v\int_0^\infty e^{-uv}du\ \int_0^\infty e^{-ts}\frac{\partial\varphi(t,u)}{\partial t}dt\\
&= v\int_0^\infty e^{-uv}\left(\Phi(s,u)-\varphi\left(0,u\right)\right)du\\
&= s\Phi\left(s,v\right)-\mathcal{G}_u\left[\varphi(0,u)\right].
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial^2\varphi(t,u)}{\partial u^2}\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}\frac{\partial^2\varphi(t,u)}{\partial u^2}dt\ du\\
&= \int_0^\infty e^{-ts}dt\ \ v\int_0^\infty e^{-uv}\frac{\partial^2\varphi(t,u)}{\partial u^2}du\\
&= \int_0^\infty e^{-ts}\left(v^2\Phi\left(t,v\right)-v^2\varphi(t,0)-v\frac{\partial\varphi(t,0)}{\partial u}\right)dt\\
&= v^2\Phi\left(s,v\right)-v^2\mathcal{L}_t\left[\varphi(t,0)\right]-v\mathcal{L}_t\left[\frac{\partial\varphi(t,0)}{\partial u}\right].
\end{aligned}$$

(iv)

$$\begin{aligned}
\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial^2\varphi(t,u)}{\partial t^2}\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}\frac{\partial^2\varphi(t,u)}{\partial t^2}dt\ du\\
&= v\int_0^\infty e^{-uv}du\ \int_0^\infty e^{-ts}\frac{\partial^2\varphi(t,u)}{\partial t^2}dt\\
&= v\int_0^\infty e^{-uv}\left(s^2\Phi\left(s,u\right)-s\varphi(0,u)-\frac{\partial\varphi(0,u)}{\partial t}\right)du\\
&= s^2\Phi\left(s,v\right)-s\mathcal{G}_u\left[\varphi(0,u)\right]-\mathcal{G}_u\left[\frac{\partial\varphi(0,u)}{\partial t}\right].
\end{aligned}$$

(v)

$$\begin{aligned}\mathcal{L}_t\mathcal{G}_u\left[\frac{\partial^2\varphi(t,u)}{\partial t\partial u}\right] &= v\int_0^\infty\int_0^\infty e^{-ts-uv}\frac{\partial^2\varphi(t,u)}{\partial t\partial u}dt\ du\\ &= v\int_0^\infty e^{-uv}du\ \int_0^\infty e^{-ts}\frac{\partial^2\varphi(t,u)}{\partial t\partial u}dt\\ &= sv\int_0^\infty\int_0^\infty e^{-ts-uv}\frac{\partial\varphi(t,u)}{\partial u}dt\ du - v\int_0^\infty e^{-st}\frac{\partial\varphi(0,u)}{\partial u}du\\ &= sv\Phi\left(s,v\right) - \ v\ \mathcal{G}_u\left[\varphi(0,u)\right] - sv\ \mathcal{L}_t\left[\varphi(t,0)\right] + v\ \varphi(0,0).\end{aligned}$$

4. Applications of DL-ARAT to Solve Volterra Integral Differential Equations

In this part, we apply DL-ARAT to the following classes of Volterra integral equations (VIEs) and Volterra partial integro-differential equations (VPIDEs) first and second order.

4.1. VIEs of two variables

Considering the following VIE

$$\varphi\left(t,u\right) = \omega\left(t,u\right) + a\int_0^t\int_0^u \varphi\left(t-m,u-n\right)\psi\left(m,n\right)dm\ dn, \tag{11}$$

where a is constant, $\omega\left(t,u\right)$ and $\psi\left(t,u\right)$ are two known functions, and $\varphi\left(t,u\right)$ is an unknown function.
Running DARA-ST on equation (1)

$$\mathcal{L}_t\mathcal{G}_u\left[\varphi\left(t,u\right)\right] = \mathcal{L}_t\mathcal{G}_u\left[\omega\left(t,u\right)\right] + \mathcal{L}_t\mathcal{G}_u\left[a\int_0^t\int_0^u \varphi\left(t-m,u-n\right)\psi\left(m,n\right)dm\ dn\right]. \tag{12}$$

According to Theorem 5 and the linearity property (6), equation (12) can be written

$$\Phi\left(s,v\right) = \Omega\left(s,v\right) + a\frac{1}{v}\Phi\left(s,v\right)\Psi\left(s,v\right), \tag{13}$$

where $\Phi\left(s,v\right) = \mathcal{L}_t\mathcal{G}_u\left[\varphi\left(t,u\right)\right]$, $\Omega\left(s,v\right) = \mathcal{L}_t\mathcal{G}_u\left[\omega\left(t,u\right)\right]$ and $\Psi\left(s,v\right) = \mathcal{L}_t\mathcal{G}_u\left[\psi\left(x,u\right)\right]$.
Consequently,

$$\Phi\left(s,v\right) = \frac{v\ \Omega\left(s,v\right)}{v-a\ \Psi\left(s,v\right)}. \tag{14}$$

Using the inverse transform $\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}$, we get the exact solution of (11)

$$\varphi\left(t,u\right) = \mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{v\ \Omega\left(s,v\right)}{v-a\ \Phi\left(s,v\right)}\right]. \tag{15}$$

Now, we give three illustrative problems to above technique.

Problem 1. *Consider the following VIE*

$$\varphi(t,u) = b - a\int_0^t \int_0^u \varphi(m,n)\,dm\ dn, \tag{16}$$

where a and b are constant.

Solution. By implementing DL-ARA in equations (16) and using the linearity property and convolution theorem, we get

$$\Phi(s,v) = \frac{b}{s} - a\frac{1}{vs}\Phi(s,v). \tag{17}$$

As a result,

$$\Phi(s,v) = \frac{b\ v}{sv+a}. \tag{18}$$

Using the inverse transform $\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}$, we get the exact solution of (18)

$$\varphi(t,u) = \mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{b\ v}{sv+a}\right] = b\ J_0\left(2\sqrt{atu}\right).$$

Problem 2. *Consider the following VIE*

$$a^2 u = \int_0^t \int_0^u \varphi(t-m,u-n)\,\varphi(m,n)\,dm\ dn, \tag{19}$$

where a is a constant.

Solution. By implementing DL-ARA in equations (19) and using Theorem 4 on equation (19), we get

$$\frac{a^2}{sv} = \frac{1}{v}\Phi^2(s,v). \tag{20}$$

Thus,

$$\Phi(s,v) = \frac{a}{\sqrt{s}}. \tag{21}$$

Using the inverse transform $\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}$, we obtain the solution exact equation of (19) as following

$$\varphi(t,u) = \mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{a}{\sqrt{s}}\right] = \frac{a}{\sqrt{\pi}}\frac{1}{\sqrt{t}}. \tag{22}$$

Problem 3. *Consider the following VIE*

$$\int_0^t \int_0^u e^{m-n}\varphi(t-m,u-n)dm\ dn = te^{t-u} - te^t. \tag{23}$$

Solution. By implementing DL-ARA in equations (23) and using Theorem 4 on (23), we get

$$\frac{v\ \Phi(s,v)}{(s-1)\,(1+v)}=\frac{v}{(v+1)\,(s-1)^2}-\frac{1}{(s-1)^2}. \tag{24}$$

Thus,

$$\Phi(s,v)=\frac{-1}{v\,(s-1)}. \tag{25}$$

Taking the inverse transform $\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}$, we have the exact solution of (23) as follows

$$\varphi\,(t,u)=\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{-1}{v\,(s-1)}\right]=-u\ e^t. \tag{26}$$

4.2. VPIDEs of first order

Considering the following VPIDE

$$\frac{\partial\varphi\,(t,u)}{\partial t}+\frac{\partial\varphi\,(t,u)}{\partial u}=\omega\,(t,u)+a\int_0^t\int_0^u\varphi\,(t-m,u-n)\,\psi\,(m,n)\,dm\ dn, \tag{27}$$

with the conditions

$$\varphi\,(t,0)=f_0\,(t)\,,\quad \varphi\,(0,u)=h_0\,(u)\,, \tag{28}$$

where a is a constant, $\varphi\,(t,u)$ is an unknown function, $\omega\,(t,u)$ and $\psi\,(t,u)$ are known functions.
Implement DL-ARAT on Equation (27), we achieve

$$s\Phi\,(s,v)-\mathcal{G}_u\,[\varphi\,(0,u)]+s\Phi(s,v)-s\mathcal{L}_t\,[\varphi\,(t,0)]=\Omega\,(s,v)+a\frac{1}{v}\Phi\,(s,v)\,\Psi\,(s,v)\,.$$

Substituting the values of the transformed condition (28)

$$\Phi\,(s,v)=\frac{v\Omega\,(s,v)+vH_0(v)+v^2F_0(s)}{sv+v^2-a\Psi\,(v,s)}, \tag{29}$$

where $F_0\,(s)=\mathcal{L}_t\,[\psi\,(t,0)]$ and $H_0\,(v)=\mathcal{G}_u\,[\psi\,(0,u)]$.
Running the inverse transform $\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}$, we get the exact solution of (27) as follows

$$\varphi\,(t,u)=\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{v\Omega\,(s,v)+vH_0(v)+v^2F_0(s)}{sv+v^2-a\Psi\,(s,v)}\right]. \tag{30}$$

Now, we give illustrative problem to above technique.

Problem 4. *Consider the following VPIDE*

$$\frac{\partial\varphi\,(t,u)}{\partial t}+\frac{\partial\varphi\,(t,u)}{\partial u}=-1+e^t+e^u+e^{t+u}+\int_0^t\int_0^u\varphi\,(t-m,u-n)\,dm\ dn, \tag{31}$$

with the conditions

$$\varphi\,(t,0)=e^t,\quad \varphi\,(0,u)=e^u. \tag{32}$$

Solution. By implementing DL-ARA in equation (32) and the source function $\omega(t,u)$

$$\begin{cases} F_0(s) = \mathcal{L}_t\left[e^t\right] = \frac{1}{s-1}, \\ H_0(v) = \mathcal{G}_u\left[e^u\right] = \frac{v}{v-1}, \\ \omega(t,u) = \mathcal{L}_t\mathcal{G}_u\left[-1+e^t+e^u+e^{t+u}\right] = \frac{v(1-2sv)}{(v-v^2)(-1+s)s}. \end{cases} \tag{33}$$

Now, putting values in (33), into (30), we get the solution of (31) as follows

$$\begin{aligned} \varphi(t,u) &= \mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{v\left(\frac{v(1-2sv)}{(v-v^2)(-1+s)s}\right)}{s\ v+v^2-\frac{1}{s}} + \frac{v\frac{v}{v-1}+v^2\frac{1}{s-1}}{s\ v+v^2-\frac{1}{s}}\right] \\ &= \mathcal{L}_x^{-1}\mathcal{G}_u^{-1}\left[\frac{v}{(-1+v)(-1+s)}\right] = e^{t+u}. \end{aligned} \tag{34}$$

4.3. VPIDE of Second order

Given the following VPIDE

$$\frac{\partial^2\varphi(t,u)}{\partial u^2} - \frac{\partial^2\varphi(t,u)}{\partial t^2} + \varphi(t,u) + \int_0^t\int_0^u \psi(t-\delta,u-\varepsilon)\quad \varphi(\delta,\varepsilon)d\delta\ d\varepsilon = \omega(t,u), \tag{35}$$

with the conditions

$$\begin{aligned} \varphi(t,0) &= f_0(t), \quad \varphi_u(t,0) = f_1(t), \\ \varphi(0,u) &= h_0(u), \quad \varphi_t(0,u) = h_1(u). \end{aligned} \tag{36}$$

Applying DL-ARAT to (36), we have

$$\begin{aligned} v^2\Phi(s,v) - v^2\mathcal{L}_t[\varphi(t,0)] - v\mathcal{L}_t[\varphi_u(t,0)] - \left(s^2\Phi(s,v) - s\mathcal{G}_u[\varphi(0,u)] - \mathcal{G}_u[\varphi_t(0,u)]\right) \\ + \Phi(s,v) + \frac{1}{v}\Phi(s,v)\Psi(s,v) = \Omega(s,v). \end{aligned}$$

After simple calculations, one can obtain

$$\Phi(s,v) = \frac{v^3F_0(s)+v^2F_1(s)-svH_0(v)}{v^3-s^2v+v+\Psi(s,v)} + \frac{-vH_1(v)+v\Omega(s,v)}{v^3-s^2v+v+\Psi(s,v)}, \tag{37}$$

where $F_0(s) = \mathcal{L}_t[\varphi(t,0)]$, $F_1(s) = \mathcal{L}_t[\varphi_u(\S,0)]$, $H_0(v) = \mathcal{G}_u[\varphi(0,u)]$ and $H_1(v) = \mathcal{G}_u[\varphi_t(0,u)]$.
Running the inverse transform $\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}$, we get the solution of (35) as follows

$$\varphi(t,u) = \mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{v^3F_0(s)+v^2F_1(s)}{v^3-s^2v+v+\Phi(s,v)} + \frac{-svH_0(v)-vH_1(v)+v\Omega(s,v)}{v^3-s^2v+v+\Psi(s,v)}\right]. \tag{38}$$

Now, we give illustrative problem to above technique.

Problem 5. *Consider the following VPIDE*

$$\frac{\partial^2\varphi(t,u)}{\partial u^2}-\frac{\partial^2\varphi(t,u)}{\partial t^2}+\varphi(t,u)+\int_0^t\int_0^u e^{t-m+u-n}\varphi(m,n)dm\ dn=e^{t+u}+tue^{t+u},\quad (39)$$

with conditions

$$\begin{aligned}\varphi(t,0)&=e^t, & \varphi_u(t,0)&=e^t,\\ \varphi(0,u)&=e^u, & \varphi_t(0,u)&=e^u.\end{aligned}\quad (40)$$

Solution. By implementing DL-ARA in equation (40) and the functions $\Psi(s,v)$ and $\omega(t,u)$, we achieve

$$\begin{cases}F_0(s)=F_1(s)=\frac{1}{s-1},\\ H_0(u)=H_1(u)=\frac{u}{u-1},\\ \Psi(s,v)=\frac{v}{(v-1)(s-1)},\\ \Omega(s,v)=\frac{v(2-s+v(s-1))}{(-1+s)^2(-1+v)^2}.\end{cases}\quad (41)$$

Now, putting values in (41), into (38), we obtain the solution of equation (39) as follows

$$\begin{aligned}\varphi(t,u)&=\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{v^3\frac{1}{s-1}+v^2\frac{1}{s-1}-sv\frac{v}{v-1}-v\frac{v}{v-1}}{v^3-s^2v+v+\frac{v}{(v-1)(s-1)}}+\frac{v\frac{v(2-s+v(-1+s))}{(v-1)^2(s-1)^2}}{v^3-s^2v+v+\frac{v}{(v-1)(s-1)}}\right]\\&=\mathcal{L}_t^{-1}\mathcal{G}_u^{-1}\left[\frac{v}{(v-1)(s-1)}\right]=e^{t+u}.\end{aligned}$$

5. Conclusion

In this research manuscript, we propose DL-ARAT approach to solve IDEs. Theorems and basic properties of the new DL-ARAT are presented in detail. Two types of integral equations have been discussed: partial integral and PIDEs. The solutions for IDEs are examined and found to best represent the true dynamics of the problem. The method offers a useful way to develop an analytical treatment for these equations. In a future work we will use the proposed scheme to solve other nonlinear equations.

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