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The generalization of integral transforms combined with He's polynomial

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Abstract. Our goal in this paper is to generalize the integral transforms and use it with He's polynomial method to find the solution of the nonlinear partial differential equations. All results of theoretical studies regarding the generalization and its properties are presented. For the He's polynomial method, it is used to solve the nonlinear part of the partial differential equation. It is shown that the importance of my research is the combination of generalization of integral transforms with He's polynomial method allows for exact and approximate solutions configurations to be determined. Furthermore, the generalization of integral transforms has been shown to include most, or even all, of the integral transforms and be applicable to a variety of equations, making it a crucial tool in solving them. Finally, the capability of solutions to be obtained quickly and easily through this combined technique provides an invaluable tool for solving problems.

2020 Mathematics Subject Classifications: 35Q35, 44A05, 41A10, 41A58, 65D25, 65J15, 74H15.

Key Words and Phrases: Generalization of integral transform, He's polynomial method, nonlinear partial differential equations

1. Introduction

Integral transforms have become an important tool in mathematics in recent years and have been extensively studied by many researchers [24, 25, 28]. These transforms have been successfully applied to solve many linear equations, such as ordinary and partial differential equations (PDEs) and integral equations [17, 19, 20]. The utility of integral transforms is their ability to simplify the solution process, making it easier to calculate the solutions in a fraction of time. This has enabled the application of integral transforms to a wide variety of problems, allowing for the production of new solutions and the refinement of existing ones [18].

Furthermore, the development of integral transforms have allowed for the efficient computation of PDEs, opening up new possibilities in the production of exact and approximate solutions of the equations among them, PDEs [5, 23, 32]. Recently, various applications

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of integral transforms have been found in different areas of engineering, mathematics and physics such as image processing, signal analysis and electric [21, 26, 30]. This is due to their properties and ability to transform a function from one domain to another domain while preserving important features of the function. Their use in these areas makes them a valuable tool as they provide efficient and accurate results. Consequently, integral transforms are being used more and more in applied mathematics [13, 15].

It is worth mentioning that the integral transforms can be used in combination with other methods to address the nonlinear parts of equations. This can provide a much more efficient solution, and can be used to analyse many different systems with greater accuracy [7, 29, 32].

One of these methods is He's polynomial, it is a reliable numerical approach for solving nonlinear PDEs. This method is based on the idea that a nonlinear PDE can be approximated by a polynomial in a certain variable. It works by expanding the nonlinear function into a set of truncated series of polynomials [6, 10]. This set of polynomials is then used to approximate the solution of the PDE. He's polynomial method is computationally efficient, since it only requires the evaluation of a few polynomial coefficients rather than a full numerical solution [9]. It has been used in a wide range of problems in fields such as fluid dynamics, astrophysics, and acoustics, and it is a powerful tool for solving nonlinear PDEs [27].

By understanding the underlying principles of integral transforms, it may become possible to develop more efficient and accurate methods for solving equations [1, 4, 8, 14, 16]. In this paper, the integral transforms are generalized, and then combined it with He's polynomial method to solve nonlinear PDEs. As the examples, we apply the combination of generalization (GN) of integral transforms and He's polynomial method to solve the nonlinear PDEs, which include the nonlinear gas dynamic equation, system of coupled nonlinear Burgers' equation and the non-homogeneous gas dynamic equation.

There are some scientific applications of these types of nonlinear equations. For instance, in ideal gas dynamics, several kinds of waves in nonlinear systems are described including discontinuities in contact, shock fronts and rare factions. It's worth mentioning that many researchers have solved the gas dynamics equations [2, 3]. For the coupled Burgers' equation, it falls into the category of integrable equations, such as nonlinear Schrödinger, Korteweg–De Vries, and Bogoyavlensky-Konopelchenko equations [11, 12, 22, 31].

The structure of this article is as follows: In Section (2), a general structure for the GN of the integral transforms is presented. Then, some theorems for the generalized integral transform of functions are proved in Section (3). As main examples, we prove the constant, polynomial, trigonometric, and exponential functions. In Section (4), we examine some useful properties for the GN of integral transform. In Section (5), our mathematical method is illustrated by using He's polynomial for PDEs with applying the GN. In Section (6), based on appropriate GN of integral transforms with He's polynomial method, we present three different examples for nonlinear PDEs: the nonlinear gas dynamic equation, system of coupled nonlinear Burgers' equation and the non-homogeneous gas dynamic equation, the effectiveness of the method is evaluated by deriving the solutions of the corresponding PDEs.

2. A generalization of integral transforms

Here, we present a GN of integral transforms which includes most integral transforms, or even all of them in the class of Laplace transforms (see, e.g., [8, 14, 15, 17]). In the next definition, we define some functions related to the GN, and then formulate the general form of the GN. After that, based on this GN, the transforms of some functions will be found by proving some theorems.

Definition 1. Assume that the function $F(\Psi(\vartheta)\ t),\ t \in [0, \infty)$ is an integrable. Given the positive real functions $h(\vartheta) \neq 0$ and $\sigma(\vartheta)$, the following formula shows the general form of the GN ($GN(\vartheta)$) of $F(\Psi(\vartheta)\ t)$:

$$GN[F(t)] = GN(\vartheta) = h(\vartheta) \int_0^\infty F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t \tag{1}$$

$$= \lim_{\tau \to \infty} h(\vartheta) \int_0^\tau F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t, \tag{2}$$

then, when the limit in (2) is exist, we get,

$$\left| h(\vartheta) \int_0^{\tau'} F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t \right| \leq h(\vartheta) \int_0^{\tau'} \left| F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} \right| t \ \longrightarrow 0, \tag{3}$$

where, $\tau \to \infty$ for all $\tau' > \tau$. Then $GN(F(\Psi(\vartheta)\ t); \vartheta)$ will be convergence.

Definition 2. Let $F(t)$, has an exponential order κ, which means, for any $L > 0$, $\exists\ M > 0$, where κ, L and M are constants, such that:

$$|F(t)| \leq M\ e^{\kappa t}, \quad t > L. \tag{4}$$

3. Results of the preliminary study

In this section, we will establish the theorems needed that can be used to demonstrate the GN of integral transforms for solving PDEs.

Theorem 1. Using the GN of integral transforms in the definition 1, we can get the generalized transforms for bellow functions:

(1.A) When $F(t) = 1$, then

$$GN[1] = h(\vartheta) \int_0^\infty e^{-\sigma(\vartheta) t} t,$$

$$= \frac{-h(\vartheta)}{\sigma(\vartheta)} e^{-\sigma(\vartheta) t} \Big|_0^\infty = \frac{h(\vartheta)}{\sigma(\vartheta)}. \tag{5}$$

(1.B) When $F(t) = t$, then

$$GN[t] = h(\vartheta) \int_0^\infty \psi(\vartheta)t \ e^{-\sigma(\vartheta)t} t,$$

by integrating by parts, we obtain

$$GN[t] = \frac{h(\vartheta)\psi(\vartheta)}{\sigma^2(\vartheta)}. \tag{6}$$

(1.C) When $F(t) = t^n$, then

$$GN[t^n] = \frac{n! \ h(\vartheta)\psi^n(\vartheta)}{\sigma^{n+1}(\vartheta)}. \tag{7}$$

(1.D) When $F(t) = \sin(t)$, then

$$GN[\sin(t)] = h(\vartheta) \int_0^\infty \sin(\psi(\vartheta)t) e^{-\sigma(\vartheta)t} t,$$

using integrating by part twice, we obtain:

$$GN[\sin(t)] = \frac{h(\vartheta)\psi(\vartheta)}{\sigma^2(\vartheta) + \psi^2(\vartheta)}, \tag{8}$$

(1.E) When $F(t) = e^t$, then

$$GN[e^t] = \frac{h(\vartheta)}{\sigma(\vartheta) - \psi(\vartheta)}. \tag{9}$$

Theorem 2. Suppose $F(t)$ is differentiable, if $F(t)$ and it's derivatives $(F'(t), F''(t), \ldots, F^{(n-1)}(t))$ are of exponential order κ and are piecewise continuous on $[0, \infty)$ and the n^{th} derivative $F^{(n)}(t)$ is a piecewise continuous on $[0, \infty)$, then,

(2.A)

$$GN[F'(t)] = -\frac{h(\vartheta)}{\Psi(\vartheta)} \ F(0) + \frac{\sigma(\vartheta)}{\Psi(\vartheta)} \ GN[F(t)], \tag{10}$$

and,

(2.B)

$$GN[F''(t)] = -\frac{h(\vartheta)}{\Psi(\vartheta)} \ F'(0) - \frac{h(\vartheta)\sigma(\vartheta)}{\Psi^2(\vartheta)} \ F(0) + \frac{\sigma^2(\vartheta)}{\Psi^2(\vartheta)} \ GN[F(t)], \tag{11}$$

then, in general,

(2.C)

$$GN[F^n(t)] = \frac{\sigma^n(\vartheta)}{\Psi^n(\vartheta)}\ GN[F(t)] - \sum_{k=0}^{n-1} \frac{h(\vartheta)\ \sigma^{n-k-1}(\vartheta)}{\Psi^{n-k}(\vartheta)}\ F^{(k)}(0). \tag{12}$$

Proof. **(2.A)** To find $GN[F'(t)]$, we can integrate as follows:

$$GN[F'(t)] = h(\vartheta) \int_0^\infty F'(\Psi(\vartheta)t)\ e^{-\sigma(\vartheta)t} t.$$

Using integrating by part, we get,

$$\begin{aligned} GN[F'(t)] =& h(\vartheta) \left[\frac{1}{\Psi(\vartheta)}\ e^{-\sigma(\vartheta)t}\ F(\Psi(\vartheta)t)\Big|_0^\infty - \int_0^\infty \frac{-\sigma(\vartheta)}{\Psi(\vartheta)} e^{-\sigma(\vartheta)t}\ F(\Psi(\vartheta)t)t \right] \\ =& \frac{-h(\vartheta)}{\Psi(\vartheta)} F(0) + \frac{\sigma(\vartheta)}{\Psi(\vartheta)}\ GN[F(t)]. \end{aligned} \tag{13}$$

Proof. **(2.B)** To find $GN[F''(t)]$: Let $\mathcal{F}(t) = F'(t)$, then $\mathcal{F}'(t) = F''(t)$. Since we know,

$$GN[\mathcal{F}'(t)] = \frac{-h(\vartheta)}{\Psi(\vartheta)}\ \mathcal{F}(0) + \frac{\sigma(\vartheta)}{\Psi(\vartheta)}\ GN[\mathcal{F}(t)], \tag{14}$$

substitute $\mathcal{F}'(t) = F''(t)$, and $\mathcal{F}(t) = F'(t)$ in (14), we have

$$GN[F''(t)] = \frac{-h(\vartheta)}{\Psi(\vartheta)} F'(0) + \frac{\sigma(\vartheta)}{\Psi(\vartheta)}\ GN[F'(t)].$$

Using the relation **(2.A)**, we get,

$$GN[F''(t)] = \frac{-h(\vartheta)}{\Psi(\vartheta)} F'(0) + \frac{\sigma(\vartheta)}{\Psi(\vartheta)}\ \left[\frac{-h(\vartheta)}{\Psi(\vartheta)}(\vartheta)\ F(0) + \frac{\sigma(\vartheta)}{\Psi(\vartheta)}\ GN[F(t)] \right],$$

then,

$$GN[F''(t)] = \frac{-h(\vartheta)}{\Psi(\vartheta)} F'(0) - \frac{h(\vartheta)\ \sigma(\vartheta)}{\Psi^2(\vartheta)}\ F(0) + \frac{\sigma^2(\vartheta)}{\Psi^2(\vartheta)}\ GN[F(t)]. \tag{15}$$

It is also possible to prove the general form **(2.C)** in (12) in a similar way.

Theorem 3. Suppose that $h(\vartheta)$ and $\sigma(\vartheta)$ are differentiable and $\sigma'(\vartheta) \neq 0$, then,

(3.A)

$$GN[t\ F(t)] = -\frac{h(\vartheta)}{\sigma'(\vartheta)} \left(\frac{}{\vartheta} \left(\frac{GN[F(t)]}{h(\vartheta)} \right) \right) + \frac{h(\vartheta)}{\sigma'(\vartheta)}\ \int_0^\infty \frac{}{\vartheta} (F(\Psi(\vartheta)t)) e^{-\sigma(\vartheta)t} t. \tag{16}$$

(3.B)

$$GN[t^2\ F(t)] = \frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F(t)]}{h(\vartheta)}\right)\right)\right)\right)$$
$$-\frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\int_0^\infty \frac{}{\vartheta}(F(\Psi(\vartheta)t))\ e^{-\sigma(\vartheta)\ t}t\right)\right)\right)$$
$$+\frac{h(\vartheta)}{\sigma'(\vartheta)}\int_0^\infty t\ \frac{}{\vartheta}(F(\Psi(\vartheta)t))\ e^{-\sigma(\vartheta)\ t}t. \tag{17}$$

(3.C) In general,

$$GN[t^n\ F(t)] = (-1)^n\frac{h(\vartheta)}{\sigma'(\vartheta)}\underbrace{\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\cdots\right.\right.\right.\right.}_{n-1\ \text{times}}\left(\frac{}{\vartheta}\left(\frac{GN[F(t)]}{h(\vartheta)}\right)\right)\left.\left.\left.\left.\right)\right)\right)\right)$$
$$+\frac{h(\vartheta)}{\sigma'(\vartheta)}\sum_{k=0}^{n-1}(-1)^k\underbrace{\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\cdots\right.\right.\right.\right.}_{k\ \text{times}}$$
$$\times\left(\int_0^\infty (t)^{n-k-1}\frac{}{\vartheta}(F(\Psi(\vartheta)t))\ e^{-\sigma(\vartheta)\ t}t\right)\left.\left.\left.\left.\right)\right)\right)\right). \tag{18}$$

(3.D)

$$GN[t\ F'(t)] = -\frac{h(\vartheta)}{\sigma'(\vartheta)}\ \frac{}{\vartheta}\left(\frac{GN[F'(t)]}{h(\vartheta)}\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)}\ \int_0^\infty \frac{}{\vartheta}(F'(\Psi(\vartheta)\ t))e^{-\sigma(\vartheta)\ t}t. \tag{19}$$

(3.E)

$$GN[t\ F''(t)] = -\frac{h(\vartheta)}{\sigma'(\vartheta)}\ \frac{}{\vartheta}\left(\frac{GN[F'(t)]}{h(\vartheta)}\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)}\ \int_0^\infty \frac{}{\vartheta}(F''(\Psi(\vartheta)\ t))e^{-\sigma(\vartheta)t}t. \tag{20}$$

(3.F) In general,

$$GN[t\ F^{(n)}(t)] = -\frac{h(\vartheta)}{\sigma'(\vartheta)}\frac{}{\vartheta}\left(\frac{GN[F^{(n)}(t)]}{h(\vartheta)}\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)}\int_0^\infty \frac{}{\vartheta}(F^{(n)}(\Psi(\vartheta)t))e^{-\sigma(\vartheta)t}t. \tag{21}$$

Proof. **(3.A)**
By using the definition of the GN in equation (1),

$$GN[F(t)] = h(\vartheta)\int_0^\infty F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t}t,$$
$$\frac{GN[F(t)]}{h(\vartheta)} = \int_0^\infty F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t}t,$$

derive both side with respect to ϑ, gives,

$$\begin{aligned}
\frac{}{\vartheta}\left(\frac{GN[F(t)]}{h(\vartheta)}\right) &= \int_0^\infty \frac{}{\vartheta}\left(F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t}\right) t \\
&= \int_0^\infty \left(-t\ F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t}\sigma'(\vartheta) + \frac{F(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t}\right) t \\
&= -\frac{\sigma'(\vartheta)}{h(\vartheta)}\ h(\vartheta) \int_0^\infty t\ F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t + \int_0^\infty \frac{F(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t} t \\
&= -\frac{\sigma'(\vartheta)}{h(\vartheta)}\ GN[t\ F(t)] + \int_0^\infty \frac{F(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t} t,
\end{aligned}$$

then,

$$GN[t\ F(t)] = -\frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F(t)]}{h(\vartheta)}\right)\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)}\int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t.$$

Proof. **(3.B)**
Using the formula in **(3.A)** and the definition of the GN in equation (1)

$$GN[t\ F(t)] = -\frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F(t)]}{h(\vartheta)}\right)\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)}\int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t,$$

$$h(\vartheta)\int_0^\infty F(\Psi(\vartheta)\ t)\ t\ e^{-\sigma(\vartheta)\ t} t = -\frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F(t)]}{h(\vartheta)}\right)\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)}\int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t,$$

derive both side with respect to ϑ, gives,

$$\int_0^\infty F(\Psi(\vartheta)\ t)\ t^2(-\sigma'(\vartheta))\ e^{-\sigma(\vartheta)\ t} t + \int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ t\ e^{-\sigma(\vartheta)\ t} t =$$

$$-\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F(\Psi(\vartheta)\ t)]}{h(\vartheta)}\right)\right)\right) + \frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ t\ e^{-\sigma(\vartheta)\ t} t\right)\right),$$

so,

$$-\frac{\sigma(\vartheta)}{h(\vartheta)}GN[t^2F(t)] + \int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ t\ e^{-\sigma(\vartheta)\ t} t =$$

$$-\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F(\Psi(\vartheta)\ t)]}{h(\vartheta)}\right)\right)\right) + \frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ t\ e^{-\sigma(\vartheta)\ t} t\right)\right),$$

then,

$$GN[t^2F(t)] = \frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F(\Psi(\vartheta)\ t)]}{h(\vartheta)}\right)\right)\right)\right) -$$

$$\frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t\right)\right)\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\int_0^\infty \frac{}{\vartheta}F(\Psi(\vartheta)\ t)\ t\ e^{-\sigma(\vartheta)\ t} t\right)$$

We can prove the general case **(3.C)** of theorem (3) in a similar way.

Proof. **(3.D)**
Since,

$$
\begin{aligned}
GN[F'(t)] =& h(\vartheta) \int_0^\infty F'(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t, \\
\frac{GN[F'(t)]}{h(\vartheta)} =& \int_0^\infty F'(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t,
\end{aligned}
$$

derive both side with respect to ϑ, gives,

$$
\begin{aligned}
\frac{}{\vartheta}\left(\frac{GN[F'(t)]}{h(\vartheta)}\right) &= \int_0^\infty \frac{}{\vartheta}\left(F'(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t}\right) t \\
&= \int_0^\infty \left(-t\ F'(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t}\sigma'(\vartheta) + \frac{F'(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t}\right) t \\
&= -\frac{\sigma'(\vartheta)}{h(\vartheta)}\ h(\vartheta) \int_0^\infty t\ F'(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t} t + \int_0^\infty \frac{F'(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t} t \\
&= -\frac{\sigma'(\vartheta)}{h(\vartheta)}\ GN[t\ F'(t)] + \int_0^\infty \frac{F'(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t} t,
\end{aligned}
$$

then,

$$
GN[t\ F'(t)] = -\frac{h(\vartheta)}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F'(t)]}{h(\vartheta)}\right)\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)} \int_0^\infty \frac{F'(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t} t. \tag{22}
$$

We can prove the forms **(3.E)** and **(3.F)** of theorem (3) in similar way.

Theorem 4. Given these functions $h(\vartheta)$, $\sigma(\vartheta)$ and $F(t)$ are differentiable ($\sigma'(\vartheta) \neq 0$), then we can prove

$$
\begin{aligned}
GN[t^2\ F^{(n)}(t)] =& \frac{h(\vartheta)}{\sigma'(\vartheta)}\ \left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F^{(n)}]}{h(\vartheta)}\right)\right)\right)\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)} \\
& \times \left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\int_0^\infty \frac{F^{(n)}(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t} t\right)\right)\right) \\
& + \frac{h(\vartheta)}{\sigma'(\vartheta)} \int_0^\infty t\ \frac{F^{(n)}(\Psi(\vartheta)\ t)}{\vartheta}\ e^{-\sigma(\vartheta)\ t} t.
\end{aligned} \tag{23}
$$

Proof. Recall the general form **(3.F)** of theorem (3),

$$
\begin{aligned}
GN[t\ F^{(n)}(t)] &= h(\vartheta) \int_0^\infty t\ F^{(n)}(\Psi(\vartheta t)) e^{-\sigma(\vartheta) t} t \\
&= -\frac{h}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F^{(n)}(t)]}{h(\vartheta)}\right)\right) + \frac{h(\vartheta)}{\sigma'(\vartheta)} \int_0^\infty \frac{}{\vartheta}(F^{(n)}(\Psi(\vartheta)t)) e^{-\sigma(\vartheta)t} t,
\end{aligned} \tag{24}
$$

derive both sides of (24) with respect to ϑ, to obtain,

$$\int_0^\infty t^2\ F^{(n)}(\Psi(\vartheta)\ t)\ e^{-\sigma(\vartheta)\ t}(-\sigma')t + \int_0^\infty t\ \frac{}{\vartheta}(F^{(n)}(\Psi(\vartheta)\ t))\ e^{-\sigma(\vartheta)\ t}t =$$
$$-\frac{}{\vartheta}\left(\frac{h}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F^{(n)}(t)]}{1}\right)\right)\right) + \frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\int_0^\infty \frac{}{\vartheta}(F^{(n)}(\Psi(\vartheta)t))e^{-\sigma(\vartheta)t}t\right)\right), \tag{25}$$

Thus, simplifying more, we prove theorem (4) as follows:

$$\frac{\sigma'(\vartheta)}{h(\vartheta)}GN[t^2\ F^{(n)}(t)] = \frac{}{\vartheta}\left(\frac{h}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{GN[F^{(n)}(t)]}{1}\right)\right)\right)$$
$$-\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\int_0^\infty \frac{}{\vartheta}(F^{(n)}(\Psi(\vartheta)\ t))\ e^{-\sigma(\vartheta)t}t\right)\right). \tag{26}$$

Theorem 5. Let $h(\vartheta)$ and $\sigma(\vartheta)$ and $F(\Psi(\vartheta)\ t))$ are differentiable ($\sigma'(\vartheta) \neq 0$), then,

(5.A)

$$GN[t^n\ F'(t)] = (-1)^n \frac{h(\vartheta)}{\sigma'(\vartheta)}\Bigg[\underbrace{\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\cdots\right.\right.\right.}_{n-1 \text{ times}}\ \left(\frac{}{\vartheta}\left(\frac{GN[F'(t)]}{h(\vartheta)}\right)\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg]$$
$$+\frac{h(\vartheta)}{\sigma'(\vartheta)}\sum_{k=0}^{n-1}(-1)^k \Bigg[\underbrace{\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\cdots\right.\right.\right.}_{k \text{ times}}$$
$$\times\ \left(\int_0^\infty (t)^{n-k-1}\frac{}{\vartheta}(F'(\Psi(\vartheta)t))\ e^{-\sigma(\vartheta)\ t}t\right)\Bigg)\Bigg)\Bigg)\Bigg]. \tag{27}$$

(5.B)

$$GN[t^n\ F''(t)] = (-1)^n \frac{h(\vartheta)}{\sigma'(\vartheta)}\Bigg[\underbrace{\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\cdots\right.\right.\right.}_{n-1 \text{ times}}\ \left(\frac{}{\vartheta}\left(\frac{GN[F''(t)]}{h(\vartheta)}\right)\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg]$$
$$+\frac{h(\vartheta)}{\sigma'(\vartheta)}\sum_{k=0}^{n-1}(-1)^k \Bigg[\underbrace{\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\cdots\right.\right.\right.}_{k \text{ times}}$$
$$\times\ \left(\int_0^\infty (t)^{n-k-1}\frac{}{\vartheta}(F''(\Psi(\vartheta)t))\ e^{-\sigma(\vartheta)\ t}t\right)\Bigg)\Bigg)\Bigg)\Bigg]. \tag{28}$$

(5.C)

$$
\begin{aligned}
&GN[t^n\ F^{(m)}(t)] = \\
&(-1)^n \frac{h(\vartheta)}{\sigma'(\vartheta)} \underbrace{\left[\frac{-}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{-}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\cdots\right.\right.\right.\right.}_{n-1 \text{ times}} \left(\frac{-}{\vartheta}\left(\frac{GN[F^{(m)}(t)]}{h(\vartheta)}\right)\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg] \\
&+ \frac{h(\vartheta)}{\sigma'(\vartheta)} \sum_{k=0}^{n-1} (-1)^k \underbrace{\left[\frac{-}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\left(\frac{-}{\vartheta}\left(\frac{1}{\sigma'(\vartheta)}\cdots\right.\right.\right.\right.}_{k \text{ times}} \\
&\times \left(\int_0^\infty (t)^{n-k-1} \frac{-}{\vartheta}(F^{(m)}(\Psi(\vartheta)t))\ e^{-\sigma(\vartheta)\ t} t\right)\Bigg)\Bigg)\Bigg)\Bigg]. \qquad (29)
\end{aligned}
$$

where, in the same way as the theorems (2-4), this theorem is proven.

4. Some properties of the generalization of integral transforms

Here, we study some properties for the GN of integral transforms.

Theorem 6. (**Linearity property**) Let $S(t)$ and $P(t)$ be functions defined for $t \geq 0$, then,

$$GN\{\eta\ S(t) + \omega\ P(t)\} = \eta\ GN\{S(t)\} + \omega\ GN\{P(t)\}, \qquad (30)$$

here, η and ω are scalers.

Proof. Depending on the definition of the GN of integral transforms, we obtain:

$$
\begin{aligned}
GN\{\eta\ S(t) + \omega\ P(t)\} &= h(\vartheta) \int_0^\infty [\eta\ S(\Psi(\vartheta)t) + \omega\ P(\Psi(\vartheta)t)]\ e^{-\sigma(\vartheta)t} t \\
&= \eta\ h(\vartheta) \int_0^\infty S(\Psi(\vartheta)t) e^{-\sigma(\vartheta)t} t + \omega\ h(\vartheta) \int_0^\infty P(\Psi(\vartheta)t) e^{-\sigma(\vartheta)t} t \\
&= \eta\ GN\{S(t)\} + \omega\ GN\{P(t)\}. \qquad (31)
\end{aligned}
$$

Theorem 7. (**Convolution property**) If $S(t)$ and $P(t)$ are of exponential order and piecewise continuous on $[0, 1)$, then,

$$GN\{S(t) * P(t)\} = GN_1\{S(t)\}\ GN_2\{P(t)\}. \qquad (32)$$

Proof. Applying the general transform GN on both functions $S(t)$ and $P(t)$, we obtain

$$GN_1\{S(t)\} = h(\vartheta) \int_0^\infty S(\Psi(\vartheta)\epsilon) e^{-\sigma(\vartheta)\epsilon} \epsilon,$$

$$
\begin{aligned}
GN_2\{P(t)\} =& h(\vartheta)\int_0^\infty P(\Psi(\vartheta)\nu)e^{-\sigma(\vartheta)\nu}\nu,\\
GN_1\{S(t)\}GN_2\{P(t)\} =& \left(h(\vartheta)\int_0^\infty S(\Psi(\vartheta)\epsilon)e^{-\sigma(\vartheta)\epsilon}\epsilon\right)\times\left(h(\vartheta)\int_0^\infty P(\Psi(\vartheta)\nu)e^{-\sigma(\vartheta)\nu}\nu\right)\\
=& h^2(\vartheta)\left(\int_0^\infty\int_0^\infty S(\Psi(\vartheta)\epsilon)\ P(\Psi(\vartheta)\nu)e^{-\sigma(\vartheta)(\epsilon+\nu)}\epsilon\ \nu\right)\\
=& h^2(\vartheta)\int_0^\infty S(\Psi(\vartheta)\nu)\nu\int_0^\infty\ P(\Psi(\vartheta)\epsilon)e^{-\sigma(\vartheta)(\epsilon+\nu)}\epsilon. \qquad (33)
\end{aligned}
$$

Let,

$$t = \epsilon+\nu, \qquad t=\epsilon, \qquad (34)$$

substitute (34) into (33), leads to

$$
\begin{aligned}
GN_1\{S(t)\}\ GN_2\{P(t)\} =& h^2(\vartheta)\int_0^\infty S(\Psi(\vartheta)\nu)\nu\int_0^\infty e^{-\sigma(\vartheta)t}\ P(\Psi(\vartheta)(t-\nu))t\\
=& h^2(\vartheta)\int_0^\infty e^{-\sigma(\vartheta)t}t\int_0^t\ S(\Psi(\vartheta)\nu)\ P(\Psi(\vartheta)(t-\nu))\nu\\
=& h(\vartheta)\int_0^\infty e^{-\sigma(\vartheta)t}\left(h(\vartheta)\int_0^t S(\Psi(\vartheta)\nu)P(\Psi(\vartheta)(t-\nu))\nu\right)t.
\end{aligned}
$$

So,

$$GN_1\{S(t)\}\ GN_2\{P(t)\} = GN\{S(t)*P(t)\}. \qquad (35)$$

Theorem 8. **(Commutativity property)** The convolution between two functions is commutative.

Proof. Let $S(t)$ and $P(t)$, then

$$S(t)*P(t) = h(\vartheta)\int_0^\infty e^{-\sigma(\vartheta)\nu}S(\Psi(\vartheta)\nu)P(\Psi(\vartheta)(t-\nu))\nu. \qquad (36)$$

Suppose,

$$t-\nu=\omega, \qquad -\nu=\omega, \qquad (37)$$

substitute the relation in (37) into (36), gives,

$$
\begin{aligned}
S(t)*P(t) =& -h(\vartheta)\int_t^{-\infty} e^{-\sigma(\vartheta)(t-\omega)}S(\Psi(\vartheta)(t-\omega))P(\Psi(\vartheta)\omega)\omega,\\
=& h(\vartheta)\int_0^\infty e^{-\sigma(\vartheta)\omega}P(\Psi(\vartheta)\omega)\ S(\Psi(\vartheta)(t-\omega))\omega.
\end{aligned}
$$

So,

$$S(t)*P(t) = P(t)*S(t). \qquad (38)$$

5. Mathematical Method: The generalization of integral transforms with He's polynomial for the PDEs

As a starting point, we consider nonlinear PDEs in the following equation:

$$\dot{\mathcal{L}}[g(x,t)] + R[g(x,t)] + \mathcal{G}(x,t) = 0, \tag{39}$$

here, $\dot{\mathcal{L}}$ is an invertible operator of first order, R consists of linear as well as nonlinear functions, $g(x,0)$ is the initial condition (IC) for (39) and $g(x,0)$ and $\mathcal{G}(x,t)$ are both known functions.
First, the GN to (39) can be applied as:

$$GN[\dot{\mathcal{L}}[g(x,t)]] + GN[R[g(x,t)]] + GN[\mathcal{G}(x,t)] = 0. \tag{40}$$

Now, the IC of (39) and theorem (2) will be used as follows,

$$-\frac{h(\vartheta)}{\psi(\vartheta)}g(x,0) + \frac{\sigma(\vartheta)}{\psi(\vartheta)}GN[g(x,t)] + GN[R[g(x,t)]] + GN[\mathcal{G}(x,t)] = 0,$$

$$\frac{\sigma(\vartheta)}{\psi(\vartheta)}GN[g(x,t)] = \frac{h(\vartheta)}{\psi(\vartheta)}g(x,0) - \left(GN[R[g(x,t)]] + GN[\mathcal{G}(x,t)]\right),$$

$$GN[g(x,t)] = \frac{\psi(\vartheta)}{\sigma(\vartheta)}\left(\frac{h(\vartheta)}{\psi(\vartheta)}g(x,0) - \left(GN[R[g(x,t)]] + GN[\mathcal{G}(x,t)]\right)\right),$$

$$GN[g(x,t)] = \frac{h(\vartheta)}{\sigma(\vartheta)}g(x,0) - \frac{\psi(\vartheta)}{\sigma(\vartheta)}\left(GN[R[g(x,t)]] + GN[\mathcal{G}(x,t)]\right). \tag{41}$$

Taking the inverse of the GN of integral transforms (GN^{-1}) to (41), we can find $g(x,t)$, this yields,

$$g(x,t) = g(x,0) - GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN[R[g(x,t)] + \mathcal{G}(x,t)]\right]. \tag{42}$$

So, the solution can be represented by an infinite series:

$$g(x,t) = \sum_{n=0}^{\infty} g_n(x,t), \tag{43}$$

and by using He's polynomial, we will deal with the nonlinear parts

$$R[g(x,t)] = \sum_{n=0}^{\infty} \mathcal{H}_n, \tag{44}$$

where, $\mathcal{H}_n$ is defined as fallow

$$\mathcal{H}_n(g_0,\ldots,g_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[R\left(\sum_{n=0}^{\infty} p^n g_n(x,t)\right)\right]_{p=0}, \tag{45}$$

here, $n = 0, 1, 2, \ldots$, substitute (43) and (44) into (42), gives:

$$\sum_{n=0}^{\infty} g_n(x,t) = g(x,0) - GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)} GN\left[\sum_{n=0}^{\infty} \mathcal{H}_n^m + \sum_{n=0}^{\infty} g_n(x,t) + \mathcal{G}(x,t)\right]\right], \tag{46}$$

where, m represents the number of nonlinear terms in (39). Compering both sides of (46), we obtain,

$$\begin{aligned} g_0(x,t) =& g(x,0), \\ g_1(x,t) =& - GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)} GN\left[\mathcal{H}_0^m + g_0 + \mathcal{G}(x,t)\right]\right], \\ g_2(x,t) =& - GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)} GN\left[\mathcal{H}_1^m + g_1 + \mathcal{G}(x,t)\right]\right], \end{aligned}$$

then, the general form will be as,

$$g_{n+1}(x,t) = -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)} GN\left[\mathcal{H}_n^m + g_n + \mathcal{G}(x,t)\right]\right], \tag{47}$$

here, $n \geq 0$. This calculation results a series expression and the convergence of this expression leads to an approximate solution.

6. Examples: Numerical results for nonlinear PDEs

Here, we apply our mathematical method to three different examples of PDEs.

Example 1. Let us consider the nonlinear equation for gas dynamics,

$$\begin{aligned} g_t(x,t) =& - g(x,t)\ g_x(x,t) + g(x,t)(1 - g(x,t)), \\ g(x,0) =& e^{-x}. \end{aligned} \tag{48}$$

Here, $\mathcal{G}(x,t) = 0$, first, we apply the GN integral transforms of (48):

$$\begin{aligned} -\frac{h(\vartheta)}{\psi(\vartheta)} g(x,0) + \frac{\sigma(\vartheta)}{\psi(\vartheta)} GN[g(x,t)] &= GN[-gg_x + g - g^2(x,t)], \\ \frac{\sigma(\vartheta)}{\psi(\vartheta)} GN[g(x,t)] &= \frac{h(\vartheta)}{\psi(\vartheta)} g(x,0) + GN[-gg_x + g - g^2(x,t)], \end{aligned}$$

multiply both side by $\frac{\psi(\vartheta)}{\sigma(\vartheta)}$, gives

$$GN[g(x,t)] = \frac{h(\vartheta)}{\sigma(\vartheta)}\ \mathrm{e}^{-x} + \frac{\psi(\vartheta)}{\sigma(\vartheta)} GN[-gg_x + g - g^2(x,t)], \tag{49}$$

take GN^{-1} to (49), then we obtain:

$$g(x,t) = GN^{-1}\left[\frac{h(\vartheta)}{\sigma(\vartheta)}\ \mathrm{e}^{-x} + \frac{\psi(\vartheta)}{\sigma(\vartheta)} GN[-gg_x + g - g^2(x,t)]\right],$$

$$g(x,t) = e^{-x} + GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN[-gg_x + g - g^2(x,t)]\right]. \tag{50}$$

Now, we deal with the nonlinear parts $g\ g_x$ and g^2 by using He's polynomial. So, using the general form (47) will give,

$$g_{n+1} = -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\mathcal{H}_n^1 - g_n + \mathcal{H}_n^2\right]\right], \quad n \geq 0, \tag{51}$$

where, $\mathcal{H}^1 = gg'$ and $\mathcal{H}^2 = g^2$, and $g_0 = e^{-x}$, using the formula (45), we have

$$\mathcal{H}_0^1 = g_0 g_0' = -e^{-2x}, \tag{52}$$

$$\mathcal{H}_0^2 = g_0^2 = e^{-2x}. \tag{53}$$

So,

$$\begin{aligned} g_1(x,t) &= -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\mathcal{H}_0^1 - g_0 + \mathcal{H}_0^2\right]\right] \\ &= -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[-e^{-2x} - e^{-x} + e^{-2x}\right]\right] \\ &= GN^{-1}\left[\frac{\psi(\vartheta)e^{-x}}{\sigma(\vartheta)}GN\left[1\right]\right] \\ &= GN^{-1}\left[\frac{\psi(\vartheta)e^{-x}}{\sigma(\vartheta)} \times \frac{h(\vartheta)}{\sigma(\vartheta)}\right], \end{aligned}$$

then,

$$g_1(x,t) = t\ e^{-x}. \tag{54}$$

The next step, we need,

$$\mathcal{H}_1^1 = g_0 g_1' + g_1 g_0' = -2te^{-2x}, \tag{55}$$

$$\mathcal{H}_1^2 = 2g_0 g_1 = 2te^{-2x}. \tag{56}$$

So, the second iteration is:

$$\begin{aligned} g_2(x,t) &= -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\mathcal{H}_1^1 - g_1 + \mathcal{H}_1^2\right]\right] \\ &= -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[-2te^{-2x} - te^{-x} + 2te^{-2x}\right]\right] \\ &= GN^{-1}\left[\frac{\psi(\vartheta)e^{-x}}{\sigma(\vartheta)}GN\left[t\right]\right] \\ &= GN^{-1}\left[\frac{\psi(\vartheta)e^{-x}}{\sigma(\vartheta)} \times \frac{h(\vartheta)\psi(\vartheta)}{\sigma^2(\vartheta)}\right] \\ &= e^{-x}GN^{-1}\left[\frac{h(\vartheta)\psi^2(\vartheta)}{\sigma^3(\vartheta)}\right], \end{aligned}$$

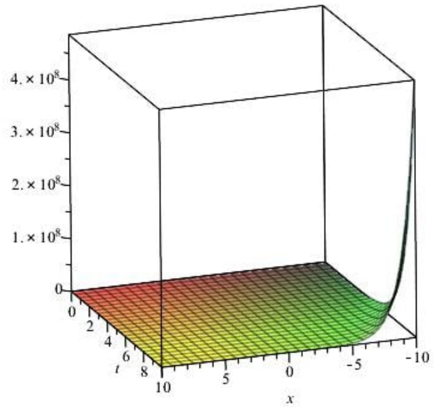

Figure 1: The plot of the solution (59)

then,

$$g_2(x,t) = \frac{1}{2}t^2\ e^{-x}, \tag{57}$$

and so on,

$$g(x,t) = \lim_{n\to\infty}\sum_{k=0}^{n} g_k(x,t) = e^{-x} + te^{-x} + \frac{1}{2}t^2e^{-x} + \dots. \tag{58}$$

In such a way, the series solution (58) of (48) represents Taylor's expansion of the function $g(x,t)$

$$g(x,t) = e^{t-x}, \tag{59}$$

in the variable t (see Figure (1)).

Example 2. Meditate the system of nonlinear Burgers' equation,

$$\begin{aligned} &g_t(x,t) - g_{xx}(x,t) - 2\ g(x,t)\ g_x(x,t) + (g(x,t)v(x,t))_x = 0,\\ &v_t(x,t) - v_{xx}(x,t) - 2\ v(x,t)\ v_x(x,t) + (g(x,t)v(x,t))_x = 0,\\ &g(x,0) = \sin(x) = v(x,0). \end{aligned} \tag{60}$$

Here, $\mathcal{G}(x,t) = 0$. First apply the GN of integral transforms of (60):

$$\begin{aligned} &-\frac{h(\vartheta)}{\psi(\vartheta)}g(x,0) + \frac{\sigma(\vartheta)}{\psi(\vartheta)}GN[g(x,t)] = -GN[-g_{xx}(x,t) - 2g(x,t)g_x(x,t) + (g(x,t)v(x,t))_x],\\ &-\frac{h(\vartheta)}{\psi(\vartheta)}g(x,0) + \frac{\sigma(\vartheta)}{\psi(\vartheta)}GN[g(x,t)] = -GN[-v_{xx}(x,t) - 2v(x,t)v_x(x,t) + (g(x,t)v(x,t))_x], \end{aligned} \tag{61}$$

apply GN^{-1} to (61), then we obtain:

$$\begin{aligned} g(x,t) =& g(x,0) - GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[-g_{xx}(x,t) - 2\ g(x,t)\ g_x(x,t) + (g(x,t)v(x,t))_x\right]\right], \\ v(x,t) =& v(x,0) - GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[-v_{xx}(x,t) - 2\ v(x,t)\ v_x(x,t) + (g(x,t)v(x,t))_x\right]\right]. \end{aligned} \tag{62}$$

Now, we deal with the nonlinear parts $g\ g_x$, $v\ v_x$ and $(g\ v)_x$ by using He's polynomial and the general formula (47), as follows:

$$\begin{aligned} g_{n+1} =& -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[-g_{n,xx} - 2H_n^1 + \mathcal{H}_n^2\right]\right], \\ v_{n+1} =& -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[-v_{n,xx} - 2H_n^3 + \mathcal{H}_n^2\right]\right], \end{aligned} \tag{63}$$

here, $n \geq 0$, $\mathcal{H}^1 = gg'$, $\mathcal{H}^3 = vv'$ and $\mathcal{H}^2 = (gv)_x$, and since $g_0 = v_0 = \sin(x)$, $H_n^2 = (gv)_x = g_x^2$ or v_x^2, then using the formula (45) yields,

$$\begin{aligned} \mathcal{H}_0^1 =& g_0 g_0' = \sin(x)\cos(x), \qquad g_0(x,t) = g(x,0) = \sin(x), \\ \mathcal{H}_0^3 =& v_0 v_0' = \sin(x)\cos(x), \qquad v_0(x,t) = v(x,0) = \sin(x), \\ \mathcal{H}_0^2 =& (g_0^2)_x = 2g_0 g_0' = 2\sin(x)\cos(x), \end{aligned}$$

$$\begin{aligned} g_1 =& -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[-g_{0,xx} - 2H_0^1 + \mathcal{H}_0^2\right]\right], \\ v_0 =& -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[-v_{0,xx} - 2H_0^3 + \mathcal{H}_0^2\right]\right], \end{aligned}$$

$$\begin{aligned} g_1 &= -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\sin(x) - 2\sin(x)\cos(x) + 2\sin(x)\cos(x)\right]\right], \\ v_1 &= -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\sin(x) - 2\sin(x)\cos(x) + 2\sin(x)\cos(x)\right]\right], \end{aligned}$$

$$\begin{aligned} g_1 =& -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}\sin(x)\frac{h(\vartheta)}{\sigma(\vartheta)}\right], \\ v_1 =& -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}\sin(x)\frac{h(\vartheta)}{\sigma(\vartheta)}\right], \end{aligned}$$

$$\begin{aligned} g_1 =& -\sin(x)t, \\ v_1 =& -\sin(x)t. \end{aligned}$$

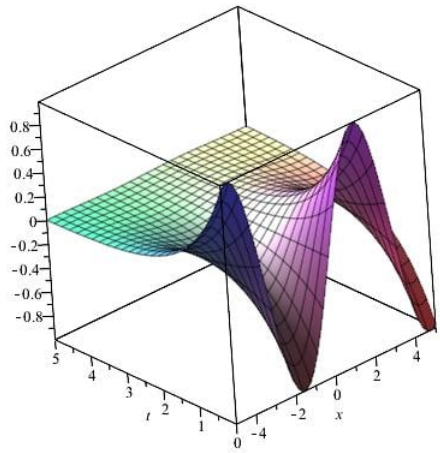

Figure 2: The plot of the solution (65)

So, in the second iteration, we need to find the second He's polynomial,

$$\begin{aligned} \mathcal{H}_1^1 =& g_0 g_1' + g_0' + g_1 = 2\sin(x)\cos(x)t, \\ \mathcal{H}_1^3 =& v_0 v_1' + v_0' + v_1 = 2\sin(x)\cos(x)t, \\ \mathcal{H}_1^2 =& (2g_0 g_1)_x = 4\sin(x)\cos(x)t, \end{aligned} \tag{64}$$

then, substitute (64) into the system (63), yields,

$$\begin{aligned} g_2 =& \frac{1}{2}\sin(x)t^2, \\ v_2 =& \frac{1}{2}\sin(x)t^2, \end{aligned}$$

and so on, then, the Taylor's expansion of $g(x,t)$ and $v(x,t)$ give the following solutions (see Figure (2)):

$$\begin{aligned} g(x,t) =& \lim_{n\to\infty}\sum_{k=0}^{n} g_k = \sin(x)e^{-t}, \\ v(x,t) =& \lim_{n\to\infty}\sum_{k=0}^{n} v_k = \sin(x)e^{-t}. \end{aligned} \tag{65}$$

Example 3. The non-homogeneous gas dynamic equation is given below,

$$\begin{aligned} g_t(x,t) =& -g(x,t)\, g_x(x,t) + g(x,t)(1-g(x,t)) - \mathrm{e}^{t-x}, \\ g(x,0) =& 1-\mathrm{e}^{-x}. \end{aligned} \tag{66}$$

First take general integral transform of (66) as:

$$-\frac{h(\vartheta)}{\psi(\vartheta)}g(x,0) + \frac{\sigma(\vartheta)}{\psi(\vartheta)}GN[g(x,t)] = GN[-gg_x + g - g^2(x,t)] - GN[\mathrm{e}^{t-x}],$$

$$\frac{\sigma(\vartheta)}{\psi(\vartheta)}GN[g(x,t)] = \frac{h(\vartheta)}{\psi(\vartheta)}g(x,0) - \mathrm{e}^{-x}\frac{h(\vartheta)}{\sigma(\vartheta)-\psi(\vartheta)} + GN[-gg_x + g - g^2(x,t)],$$

then, multiply both side by $\frac{\psi(\vartheta)}{\sigma(\vartheta)}$, gives,

$$GN[g(x,t)] = \frac{h(\vartheta)}{\sigma(\vartheta)}(1-\mathrm{e}^{-x}) - \mathrm{e}^{-x}\frac{\psi(\vartheta)}{\sigma(\vartheta)}\frac{h(\vartheta)}{\sigma(\vartheta)-\psi(\vartheta)} + \frac{\psi(\vartheta)}{\sigma(\vartheta)}GN[-gg_x + g - g^2(x,t)], \tag{67}$$

take GN^{-1} to (67), we obtain:

$$\begin{aligned} g(x,t) = GN^{-1}\Bigg[\frac{\psi(\vartheta)}{\sigma(\vartheta)}(1-\mathrm{e}^{-x}) - \mathrm{e}^{-x}\frac{\psi(\vartheta)}{\sigma(\vartheta)}\frac{h(\vartheta)}{\sigma(\vartheta)-\psi(\vartheta)} \\ + \frac{\psi(\vartheta)}{\sigma(\vartheta)}GN[-gg_x + g - g^2(x,t)]]\Bigg], \\ g(x,t) = 1 - \mathrm{e}^{-x} - \mathrm{e}^{-x}(\mathrm{e}^{t}-1) + GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN[-gg_x + g - g^2(x,t)]\right]. \end{aligned} \tag{68}$$

After that, we will deal with the nonlinear parts $g\ g_x$ and g^2 by using He's polynomial. So, using the general form (47) will be as,

$$g_{n+1} = -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\mathcal{H}_n^1 - g_n + \mathcal{H}_n^2\right]\right],\ n \geq 0, \tag{69}$$

where, $\mathcal{H}^1 = gg'$ and $\mathcal{H}^2 = g^2$, and $g_0(x,t) = 1 - \mathrm{e}^{t-x}$, using the formula (45), we have

$$\mathcal{H}_0^1 = g_0 g_0' = \mathrm{e}^{t-x} - \mathrm{e}^{2(t-x)}, \tag{70}$$

$$\mathcal{H}_0^2 = g_0^2 = 1 - 2\mathrm{e}^{t-x} + \mathrm{e}^{2(t-x)}, \tag{71}$$

so,

$$\begin{aligned} g_1(x,t) =& - GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\mathcal{H}_0^1 - g_0 + \mathcal{H}_0^2\right]\right] \\ =& - GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\mathrm{e}^{t-x} - \mathrm{e}^{2(t-x)} - 1 + \mathrm{e}^{t-x} + 1 - 2\mathrm{e}^{t-x} + \mathrm{e}^{2(t-x)}\right]\right] \\ =& GN^{-1}\left[\frac{\psi(\vartheta)e^{-x}}{\sigma(\vartheta)}GN\left[1\right]\right] \\ =& 0, \end{aligned}$$

then,

$$g_1(x,t) = 0. \tag{72}$$

The next step, we need

$$\mathcal{H}_1^1 = g_0 g_1' + g_1 g_0' = 0, \tag{73}$$

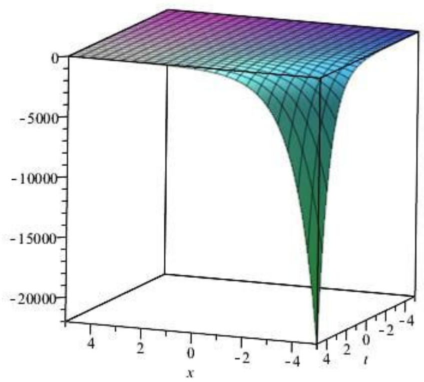

Figure 3: The plot of the exact solution (76)

$$\mathcal{H}_1^2 = 2g_0g_1 = 0. \tag{74}$$

So,

$$\begin{aligned} g_2(x,t) =& -GN^{-1}\left[\frac{\psi(\vartheta)}{\sigma(\vartheta)}GN\left[\mathcal{H}_1^1 - g_1 + \mathcal{H}_1^2\right]\right] \\ =& 0, \end{aligned}$$

and so on. Thus, the series solution $g(x,t)$

$$\begin{aligned} g(x,t) =& \lim_{n\to\infty}\sum_{k=0}^{\infty} g_k(x,t) \\ =& 1 - \mathrm{e}^{t-x} + 0 + \dots, \end{aligned} \tag{75}$$

gives the exact solution of (66) (see Figure (3)):

$$g(x,t) = 1 - e^{t-x}. \tag{76}$$

7. Conclusion

We briefly summarized the output of the current paper. First of all, the GN of integral transforms has been developed by formulating a new form of the generalized integral transform. The development started with introducing some definitions about the functions which are related to the GN. Then, formulating the theorems of the GN, in which we proved that the GN of integral transforms of some functions and the derivatives of unknown functions, which were used in the equations, has been fulfilled. These transforms have been used as an essential tool in determining the solutions to equations. Additionally, the development of the GN of integral transform involved studying some useful properties. In a second study, a mathematical method has been exhibited by using our newly developed GN together with He's polynomial method to solve nonlinear PDEs where the He's polynomial method evaluates nonlinear terms. For this combination of our GN and He's polynomial method, a convergent series has been obtained. Consequently, a variety of equations have been successfully solved using this combination, including PDEs in three types of equations. The nonlinear gas dynamic equation and the system of coupled nonlinear Burgers' equation have been solved in the first and second examples, respectively, which determined approximate solution (see Figure (1) and (2)), whereas in the third example, the non-homogeneous gas dynamic equation has been solved to obtain an exact solution (see Figure (3)). Finally, the advantages of our method are its efficiency and accuracy as it led to more accurate results and more efficient calculations, as well as the fact that it can be used to solve any order of nonlinear equations. This makes it an invaluable tool for finding the solutions of a various problems in mathematical physics.

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