EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 3, 2023, 1913-1939
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Localization in the Category $COMP(G_r(A-Mod))$ of Complex associated to the Category $G_r(A-Mod)$ of Graded left $A-$modules over a Graded Ring

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Abstract. The main results of this paper are:
If $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded duo-ring, S_H is a part formed of regulars homogeneous elements of A, $\overline{S}_H$ is the homogeneous multiplicatively closed subset of A generated by S_H, then:

(i) The relation $C_H(-) : G_r(\overline{S}_H^{-1}A - Mod) \longrightarrow COMP(G_r(\overline{S}_H^{-1}A - Mod))$ which that for all graded left $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}M$ of $G_r(\overline{S}_H^{-1}A - Mod)$ we correspond the associate complex sequence $(\overline{S}_H^{-1}M)_*$ to a graded $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}M$ and for all graded morphism of graded left $\overline{S}_H^{-1}A-$modules $\overline{S}_H^{-1}f : \overline{S}_H^{-1}M \longrightarrow \overline{S}_H^{-1}N$ of degree k we correspond the associate complex chain $(\overline{S}_H^{-1}f)_*^k$ to a morphism of graded left $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}f : \overline{S}_H^{-1}M \longrightarrow \overline{S}_H^{-1}N$ is additively exact covariant functor.

(ii) The relation $(C_H \circ \overline{S}_H^{-1})(-) : G_r(A - Mod) \longrightarrow COMP(G_r(\overline{S}_H^{-1}A - Mod))$ which that for all graded left $A-$module M of $G_r(A - Mod)$ we correspond the associate complex sequence $(C_H \circ \overline{S}_H^{-1})(M) = (\overline{S}_H^{-1}M)_*$ to a graded $A-$module M and for all graded morphism of graded left $A-$modules $f : M \longrightarrow N$ of degree k we correspond the associate complex chain $(C_H \circ \overline{S}_H^{-1})(f) = (\overline{S}_H^{-1}f)_*^k$ to a morphism of graded left $A-$module $f : M \longrightarrow N$ is additively exact covariant functor.

(iii) For all $n \in \mathbb{Z}$ fixed and for all $M \in G_r(A - Mod)$ we have:

$$\overline{S}_H^{-1}((H_n \circ C)(M)) \cong H_n(C_H \circ \overline{S}_H^{-1})(M)).$$

2020 Mathematics Subject Classifications: 13A02, 16W50, 18C40, 18G35, 13D45
Key Words and Phrases: Duo-ring, graded ring, graded module, multiplicatively closed subset of duo-ring generated by regular homogeneous elements, Category, sequence complex,Complex chain and homology functor

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DOI: https://doi.org/10.29020/nybg.ejpam.v16i3.4753

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1. Introduction

In this article A is supposed unitary graded ring and all left $A-$module is a unitary.
In this article, we study the localization in the category $COMP(G_r(A-Mod))$ of complexes of graded left $A-$modules so for this gaol we used the localization in the category $G_r(A-Mod)$, of graded left $A-$modules, the functor $S_H^{-1}: G_r(A-Mod) \longrightarrow G_r(S_H^{-1}A-Mod)$ with S_H is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of a graded ring A and the functor $H_n : COMP(G_r(A-Mod)) \longrightarrow G_r(A-Mod)$.
This work finds its roots in particular as regards the functor $S^{-1}: A-Mod \longrightarrow S^{-1}A-Mod$ in [8], and as regards the graduation of graded module of fractions in [2] and [1].
This article is presented as follows:
In the second section we present a reminder containing the definitions and background results of graded rings and modules and homological algebra extracted in [10], [11],[4] and [9].
In section 3, the following results have been shown, among others:
If $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded duo-ring, S_H is a part formed of regular homogeneous elements of A, $\overline{S}_H$ is the homogeneous multiplicatively closed subset of A generated by S_H, then we have:

(i)
$$\overline{S}_H^{-1}(f) : \overline{S}_H^{-1}M \longrightarrow \overline{S}_H^{-1}N$$
$$\frac{m}{s} \longmapsto \overline{S}_H^{-1}(f)(\frac{m}{s}) = \frac{f(m)}{s}$$
is a graded morphism of degree $k \in \mathbb{Z}$ of graded left $\overline{S}_H^{-1}A-$module;

(ii) The relation $\overline{S}_H^{-1}(-) : G_r(A-Mod) \longrightarrow G_r(\overline{S}_H^{-1}A-Mod)$ which that for any graded left $A-$module M we made to correspond $\overline{S}_H^{-1}(M)$ and for all graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules $f : M \longrightarrow N$ we correspond $\overline{S}_H^{-1}(f)$ of degree $k \in \mathbb{Z}$ is a exact additively covariant functor;

(iii) Furthermore let P a prime ideal of A and S_H is a part formed of regular homogeneous elements of $A\backslash P$ and $\overline{S}_{P_H}$ is the homogeneous multiplicatively closed subset of A generated by S_H, then the relation $\overline{S}_{P_H}^{-1}(-) : G_r(A-Mod) \longrightarrow \overline{S}_{P_H}^{-1}A-Mod$ which that for any graded left $A-$module M we correspond $\overline{S}_{P_H}^{-1}(M)$ and for all graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules $f : M \longrightarrow N$ we correspond $\overline{S}_{P_H}^{-1}(f)$ of degree $k \in \mathbb{Z}$ is a exact additively covariant functor.

In the last section the following results among others have been shown: if $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded duo-ring, S_H is a part formed of regular homogeneous elements of A, $\overline{S}_H$ is the homogeneous multiplicatively closed subset of A generated by S_H, then we have the following results:

(i) The following complex sequence :

$$\overline{S}_H^{-1}(M_*) : \cdots \longrightarrow \overline{S}_H^{-1}(M(n+1)) \overset{\overline{S}_H^{-1}(d_{n+1})}{\longrightarrow} \overline{S}_H^{-1}(M(n)) \overset{\overline{S}_H^{-1}(d_n)}{\longrightarrow} \overline{S}_H^{-1}(M(n-1)) \longrightarrow \cdots$$

with

$$d_n : M(n) \longrightarrow M(n-1)$$
$$x = y + z \longmapsto y$$

with $(y, z) \in M_n \times M(n+1)$;

(ii) The following complex chain :

$$\begin{array}{ccccccccc}
\overline{S}_H^{-1}(M_*) : \cdots & \longrightarrow & \overline{S}_H^{-1}(M(n+1)) & \overset{\overline{S}_H^{-1}(d_{n+1})}{\longrightarrow} & \overline{S}_H^{-1}(M(n)) & \overset{\overline{S}_H^{-1}(d_n)}{\longrightarrow} & \overline{S}_H^{-1}(M(n-1)) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \overline{S}_H^{-1}(f^k_*)} & & \downarrow{\scriptstyle \overline{S}_H^{-1}(f^k(n+1))} & & \downarrow{\scriptstyle \overline{S}_H^{-1}(f^k(n))} & & \downarrow{\scriptstyle \overline{S}_H^{-1}(f^k(n-1))} & & \\
\overline{S}_H^{-1}(N_*) : \cdots & \longrightarrow & \overline{S}_H^{-1}(N(n+1)) & \overset{\overline{S}_H^{-1}(d'_{n+1+k})}{\longrightarrow} & \overline{S}_H^{-1}(N(n)) & \overset{\overline{S}_H^{-1}(d'_{n+k})}{\longrightarrow} & \overline{S}_H^{-1}(N(n-1)) & \longrightarrow & \cdots
\end{array}$$

(iii) The relation $C_H(-) : G_r(\overline{S}_H^{-1}A - Mod) \longrightarrow COMP(G_r(\overline{S}_H^{-1}A - Mod))$ which that for all graded left $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}M$ of $G_r(\overline{S}_H^{-1}A - Mod)$ we correspond the associate complex sequence $(\overline{S}_H^{-1}M)_*$ to a graded $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}M$ and for all graded morphism of graded left $\overline{S}_H^{-1}A-$modules $\overline{S}_H^{-1}f : \overline{S}_H^{-1}M \longrightarrow \overline{S}_H^{-1}N$ of degree k we correspond the associate complex chain $(\overline{S}_H^{-1}f)^k_*$ to a morphism of graded left $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}f : \overline{S}_H^{-1}M \longrightarrow \overline{S}_H^{-1}N$ is additively exact covariant functor.

(iv) The relation $(C_H \circ \overline{S}_H^{-1})(-) : G_r(A - Mod) \longrightarrow COMP(G_r(\overline{S}_H^{-1}A - Mod))$ which that for all graded left $A-$module M of $G_r(A - Mod)$ we correspond the associate complex sequence $(C_H \circ \overline{S}_H^{-1})(M) = (\overline{S}_H^{-1}M)_*$ to a graded $A-$module M and for all graded morphism of graded left $A-$modules $f : M \longrightarrow N$ of degree k we correspond the associate complex chain $(C_H \circ \overline{S}_H^{-1})(f) = (\overline{S}_H^{-1}f)^k_*$ to a morphism of graded left $A-$module $f : M \longrightarrow N$ is additively exact covariant functor.

(v) We have the composed functor $\mathcal{H}_n = H_n \circ C$, $\mathcal{H}_n : G_r(A - Mod) \longrightarrow G_r(A - Mod)$. With $C() : G_r(A - Mod) \longrightarrow COMP(G_r(A - Mod))$ and $H_n : COMP(G_r(A - Mod)) \longrightarrow G_r(A - Mod)$.

(vi) For all $n \in \mathbb{Z}$ fixed and for all $M \in G_r(A - Mod)$ we have:

$$\overline{S}_H^{-1}((H_n \circ C)(M)) \cong H_n(C_H \circ \overline{S}_H^{-1})(M)).$$

2. Reminder and preliminary results

Definition 1. *Let A be a ring, then we say that A is a graded ring if there exists a suite $(A_n)_{n\in\mathbb{Z}}$ of additive subgroups of A such that*

(i) $A = \bigoplus_{n\in\mathbb{Z}} A_n$;

(ii) $A_n \cdot A_m \subset A_{n+m}, \forall\ n,\ m \in \mathbb{Z}$.

Definition 2. *Let A be a graded ring, and x be a non-zero element of A. then we say that x is homogeneous of degree n, if there exist n such that $x \in A_n$ and we note $\deg(x) = n$.*

In all that follows, A and M are supposed unitary.

Definition 3. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring and M be a left $A-$module, we say that M is a graded left $A-$module if there exists a suite $(M_n)_{n\in\mathbb{Z}}$ of sub-groups of M such that:*

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$;

(ii) $A_n \cdot M_d \subset M_{n+d}, \forall\ n,\ d \in \mathbb{Z}$.

Definition 4. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded left $A-$module and N is a sub-module of M, then we say that N is a graded sub-module of M, if $\forall x \in N$ such that $x = \sum_{n\in\mathbb{Z}} x_n$, then $x_n \in N$, $\forall n \in \mathbb{Z}$.*

Proposition 1. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is graded left $A-$module, then for all $n \in \mathbb{Z}$ fixed, we have*

$$M(n) = \bigoplus_{k\geq n} M_k$$

is a graded sub-module of M and we have the descendant sequence:

$$\cdots M(n+2) \subset M(n+1) \subset M(n) \subset \cdots.$$

Proof. For all $n \in \mathbb{Z}$ fixed, $M(n) = \bigoplus_{k\geq n} M_k$ is a sub-group of M and

$$A_s \cdot M(n)_k = A_s \cdot M_{n+k} \subset M_{n+k+s} = M_{n+(k+s)} = M(n)_{k+s}.$$

In the other hand, it suffices to remark that

$$M(n) = \bigoplus_{k\geq n} M_k = M_n \bigoplus M(n+1).$$

Hence $M(n+1) \subset M(n)$. Thus

$$\cdots M(n+2) \subset M(n+1) \subset M(n) \subset \cdots.$$

Definition 5. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and* $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *two graded left* $A-$*modules and* $f : M \longrightarrow N$ *is a morphism of left* $A-$*modules, then we say that* f *is a graded morphism of degree* $k \in \mathbb{Z}$ *if for any* $m \in M_s$ *then* $f(m) \in N_{s+k}$.

Theorem 1. *Let* A *be a graded ring, then the following information:*

(i) The class of objects are the graded left $A-$*modules;*

(ii) The class of morphisms are the graded morphisms of degree $k \in \mathbb{Z}$.

constitute a category called the category of graded left $A-$*module and it is denoted by* $G_r(A - Mod)$.

Proof. See [3]

Definition 6. *A complex sequence* $(C, d) : \ldots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \overset{d_{n-1}}{\to} \ldots$ *is a sequence of morphisms of* $A-$ *modules satisfying* $d_n \circ d_{n+1} = 0$, *for all* $n \in \mathbb{Z}$.

Definition 7. *A complex chain* $f : (C, d) \to (C', d')$ *is a sequence of homomorphisms* $(f_n : C_n \longrightarrow C'_n)_{n\in\mathbb{Z}}$ *of* $A-$ *modules making the following diagram commute:*

$$\begin{array}{ccccccccc} (C,d): \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\ \downarrow f & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ (C',d'): \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \longrightarrow & \cdots \end{array}$$

i.e $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$, *for all* $n \in \mathbb{Z}$.

Proposition 2. *We called the category of complexes of* $A-$*modules and we denote* $COMP$, *the category whose:*

(i) The objects are the sequences complex;

(ii) The morphisms are the complex chains.

Proof. See [3]

Proposition 3. *We called functor homology* H_n *the functor* $H_n : COMP \longrightarrow Ab$ *defined by:*

(i) For all objet (C, d) *of* $COMP$, $H_n((C, d)) = \ker d_n / Imd_{n+1}$

(ii) For all chain $f : (C, d) \to (C', d')$ *of* $COMP$

$$H_n(f) : H_n((C, d)) \longrightarrow H_n((C', d'))$$

$$\overline{z_n} \longmapsto \overline{f_n(z_n)}$$

Proof. See [3]

Theorem 2. *Let*

$$(0) \longrightarrow ((M,d)) \xrightarrow{f} ((N,d^{'})) \xrightarrow{g} ((L,d^{''})) \longrightarrow (0)$$

be a short exact complex sequence, then for all $n \in \mathbb{Z}$ there exist a morphism of left $A-$module

$$\delta_n : H_n((L,d^{'}) \longrightarrow H_{n-1}((M,d))$$

called connecting morphism such that the following long exact sequence is exact

$$\cdots \longrightarrow H_n((M,d)) \stackrel{H_n(f)}{\longrightarrow} H_n((N,d^{'})) \stackrel{H_n(g)}{\longrightarrow} H_n((L,d^{''})) \xrightarrow{\delta_n} H_{n-1}(M,d) \stackrel{H_{n-1}(f)}{\longrightarrow}$$
$$H_{n-1}(N,d^{'}) \longrightarrow \cdots$$

Proof. See [3]

Definition 8. *Let A be a ring, we say that A is duo ring if every left ideal of A is two-sided, and any right ideal of A is two-sided.*

Proposition 4. *Let A be a ring, then A is a duo-ring if, and only if, $\forall a \in A, aA = Aa$.*

Proof. See [6].

Proposition 5. *Let A be a duo-ring then, the set of all regular elements of A is a multiplicatively closed subset of A verifies the conditions Ore.*

Proof. See [6].

Proposition 6. *Let A be a duo-ring and S be a nonempty subset formed of regular elements of A, then there exists a multiplicatively closed subset of A satisfying the left conditions of Ore containing S.*

Proof. It suffices to note that the set of all regular elements of A is a multiplicatively closed subset satisfying the conditions Ore and containing S.

Definition 9. *Let A be a duo-ring and S be a nonempty subset formed of regular elements of A, then the smaller multiplicatively closed subset of A satisfying the conditions of Ore containing S is called the multiplicatively closed subset of A satisfying the left conditions of Ore generated by S and denoted by $\overline{S}$.*

Proposition 7. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring and S_H be a nonempty subset formed of regular homogeneous elements of A, then there exists a homogeneous multiplicatively closed subset of A satisfying the left conditions of Ore containing S, and denoted by $\overline{S}_H$.*

Proof. Put $\overline{S}_H$ the the smaller multiplicatively closed subset of A satisfying the conditions Ore containing S, $\overline{S}_H$ exist because the set of regular elements of A is a multiplicatively closed subset of A satisfying the conditions Ore containing S. Then it is enough to proof that $\overline{S}_H$ is homogeneous. We have the elements of $\overline{S}_H$ are of the form $\prod_i s_i$, $s_i \in S$ which $\prod_i s_i$ is homogeneous.

Corollary 1. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring then the set of all regular homogeneous of A is multiplicatively closed subset satisfying the left conditions of Ore.*

Proof. Put S the set of all regular homogeneous of A then $\overline{S}_H = S$.

Proposition 8. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring, P is a prime ideal of A and S_{P_H} is the set formed of homogeneous regular elements of $A\backslash P$, then $\overline{S}_{P_H} \subset (A\backslash P)$.*

Proof. The set of regular elements of $A\backslash P$ is a multiplicatively closed subset satisfying the conditions of Ore, (see [5] and [7]) and containing $\overline{S}_{P_H}$, then $\overline{S}_{P_H} \subset (A\backslash P)$.

Corollary 2. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring and P is a prime ideal of A, then the set of regular homogeneous of $A\backslash P$ is a multiplicatively closed subset satisfying the conditions of Ore.*

Proof. Put S the set of all regular homogeneous of $A\backslash P$, then $\overline{S}_{P_H} = S$.

3. Functor Graduation $\overline{S}_H^{-1}$ and Functorization of graded modules

Theorem 3. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ be a two graded left $A-$modules and S is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of a graded ring A.*
Let $f : M \longrightarrow N$ be graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules, then:

$$S^{-1}(f) : S^{-1}M \longrightarrow S^{-1}N$$

$$\frac{m}{s} \quad \longmapsto \quad S^{-1}(f)(\frac{m}{s}) = \frac{f(m)}{s}$$

is a graded morphism of degree $k \in \mathbb{Z}$ of graded left $S^{-1}A-$module.

Proof. Since [1], $S^{-1}(f)$ is a morphism of left $S^{-1}A-$module.
Show that $S^{-1}(f)$ is graded morphism of degree $k \in \mathbb{Z}$, let $m \in M$ homogeneous such that $\frac{m}{s} \in S^{-1}M$ is of degree d, then $d = \deg(\frac{m}{s}) = \deg(m) - \deg(s)$, on the other hand

$$\deg(S^{-1}(f)(\frac{m}{s})) \quad = \quad \deg(\frac{f(m)}{s})$$

$$\begin{aligned} &= \deg(f(m)) - \deg(s) \\ &= \deg(f(m)) - \deg(s) \\ &= (\deg(m) + k) - \deg(s) \\ &= d + k \end{aligned}$$

because f is graded of degree $k \in \mathbb{Z}$, thus $S^{-1}(f)$ has degree k, hence $S^{-1}(f)$ is graded morphism of degree k of graded left $S^{-1}A-$module.

Proposition 9. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded duo-ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and* $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *be a two graded left* $A-$*modules and* S_H *be a part formed of regulars homogeneous elements of* A.
Let $f : M \longrightarrow N$ *be graded morphism of degree* $k \in \mathbb{Z}$ *of graded left* $A-$*modules, then:*

$$\overline{S}_H^{-1}(f) : \overline{S}_H^{-1}M \longrightarrow \overline{S}_H^{-1}N$$

$$\frac{m}{s} \quad \longmapsto \quad \overline{S}_H^{-1}(f)(\frac{m}{s}) = \frac{f(m)}{s}$$

is a graded morphism of degree $k \in \mathbb{Z}$ *of graded left* $\overline{S}_H^{-1}A-$*module.*

Proof. Since the proposition 6, $\overline{S}_H$ is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A and from 9, $\overline{S}_H^{-1}(f)$ is graded morphism of degree k of graded left $\overline{S}_H^{-1}A-$module.

Proposition 10. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$, $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *and* $L = \bigoplus_{n\in\mathbb{Z}} L_n$ *be three graded left* $A-$*modules and* S *be a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of* A, *then for every short exact sequences of a graded morphisms of degree* $k \in \mathbb{Z}$ *of a graded left* $A-$*module*

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\phi} L \longrightarrow 0,$$

we have the following short exact sequences of a graded morphisms of degree $k \in \mathbb{Z}$ *of a graded left* $S^{-1}A-$*modules:*

$$0 \longrightarrow S^{-1}M \xrightarrow{S^{-1}(\varphi)} S^{-1}N \xrightarrow{S^{-1}(\phi)} S^{-1}L \longrightarrow 0.$$

Proof. Since the theorem 3.4 of [8], if $0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\phi} L \longrightarrow 0$ is a short exact sequences of a morphisms of degree $k \in \mathbb{Z}$ of a left $A-$modules, then

$$0 \longrightarrow S^{-1}M \xrightarrow{S^{-1}(\varphi)} S^{-1}N \xrightarrow{S^{-1}(\phi)} S^{-1}L \longrightarrow 0$$

is a short exact sequences of a morphisms of degree $k \in \mathbb{Z}$ of a left $S^{-1}A-$modules, and as S is a set formed of no null homogeneous elements of A and $S^{-1}(-)$ preserve degree, then we have

$$0 \longrightarrow S^{-1}M \overset{S^{-1}(\varphi)}{\longrightarrow} S^{-1}N \overset{S^{-1}(\phi)}{\longrightarrow} S^{-1}L \longrightarrow 0$$

is a short exact sequences of a graded morphisms of degree $k \in \mathbb{Z}$ of a graded left $S^{-1}A-$modules.

Corollary 3. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$, $N = \bigoplus_{n\in\mathbb{Z}} N_n$ and $L = \bigoplus_{n\in\mathbb{Z}} L_n$ be three graded left $A-$modules and S_H be part formed of regulars homogeneous elements of A, then for every short exact sequences of a graded morphisms of degree $k \in \mathbb{Z}$ of a graded left $A-$module*

$$0 \longrightarrow M \overset{\varphi}{\longrightarrow} N \overset{\phi}{\longrightarrow} L \longrightarrow 0$$

we have the following short exact sequences of a graded morphisms of degree $k \in \mathbb{Z}$ of a graded left $\overline{S}_H^{-1}A-$modules:

$$0 \longrightarrow \overline{S}_H^{-1}M \overset{\overline{S}_H^{-1}(\varphi)}{\longrightarrow} \overline{S}_H^{-1}N \overset{\overline{S}_H^{-1}(\phi)}{\longrightarrow} \overline{S}_H^{-1}L \longrightarrow 0$$

Proof. Since the proposition 6, $\overline{S}_H$ is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A.

Corollary 4. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$, $N = \bigoplus_{n\in\mathbb{Z}} N_n$ and $L = \bigoplus_{n\in\mathbb{Z}} L_n$ be three graded left $A-$modules and S_H the set of all regular homogeneous of A, then for every short exact sequences of a graded morphisms of degree $k \in \mathbb{Z}$ of a graded left $A-$module*

$$0 \longrightarrow M \overset{\varphi}{\longrightarrow} N \overset{\phi}{\longrightarrow} L \longrightarrow 0$$

we have the following short exact sequences of a graded morphisms of degree $k \in \mathbb{Z}$ of a graded left $S_H^{-1}A-$modules:

$$0 \longrightarrow S_H^{-1}M \overset{S_H^{-1}(\varphi)}{\longrightarrow} S_H^{-1}N \overset{S_H^{-1}(\phi)}{\longrightarrow} S_H^{-1}L \longrightarrow 0$$

Proof. Similarly to the proof of the corollary precedent 3 with $\overline{S}_H = S_H$

Theorem 4. *Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a graded ring and S be a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A, then the relation $S^{-1}(-) : G_r(A - Mod) \longrightarrow G_r(S^{-1}A - Mod)$ which that for any graded left $A-$module M we correspond $S^{-1}(M)$ and for all graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules $f : M \longrightarrow N$ we correspond $S^{-1}(f)$ of degree $k \in \mathbb{Z}$ is a exact additively covariant functor.*

Proof. Let $f : M \longrightarrow N$ be a graded morphism of degree $k \in \mathbb{Z}$ of a graded left $A-$modules, then

$$\begin{aligned} S^{-1}(f) : S^{-1}M &\longrightarrow S^{-1}N \\ \frac{m}{s} &\longmapsto \frac{f(m)}{s} \end{aligned}$$

is a morphism of degree $k \in \mathbb{Z}$ of left $S^{-1}A-$modules.
So

(i) Let $M \in G_r(A - Mod)$, then $S^{-1}M$ is a graded left $S^{-1}A-$module, thus $S^{-1}M \in G_r(S^{-1}A - Mod)$.

(ii) Let $f : M \longrightarrow N$ be a graded morphism of the graded left $A-$modules, then

$$S^{-1}(g \circ f) : S^{-1}M \longrightarrow S^{-1}N$$

$$\begin{aligned} S^{-1}(g \circ f)(\frac{m}{s}) &= \frac{(g \circ f)(m)}{s} \\ &= \frac{g(f(m))}{s} \\ &= g\left(\frac{f(m)}{s}\right) \\ &= S^{-1}(g)(\frac{f(m)}{s}) \\ &= S^{-1}(g) \circ S^{-1}(f)(\frac{m}{s}) \end{aligned}$$

Thus $\forall \ \frac{m}{s} \in S^{-1}M$, $S^{-1}(g \circ f) = S^{-1}(g) \circ S^{-1}(f)$.

$$\begin{aligned} S^{-1}(1_M) : S^{-1}M &\longrightarrow S^{-1}M \\ \frac{m}{s} &\longmapsto \frac{1_M(m)}{s} = \frac{m}{s} = 1_{S^{-1}M}(\frac{m}{s}) \end{aligned}$$

so $\forall \frac{m}{s} \in S^{-1}M$ we have $S^{-1}(1_M) = 1_{S^{-1}M}$,
so $S^{-1}(-) : G_r(A - Mod) \longrightarrow G_r(S^{-1}A - Mod)$ is a covariant functor.

Furthermore $\deg(\frac{m}{s}) = \deg(m) - \deg(s)$ or f is graded of degree $k \in \mathbb{Z}$, then $\deg(m) + k = \deg(f(m))$ so

$$\begin{aligned} \deg(S^{-1}(f)(\frac{m}{s})) &= \deg(\frac{f(m)}{s}) = \deg(f(m)) - \deg(s) \\ &= (\deg(m) + k) - \deg(s) \\ &= \deg(\frac{m}{s}) + k. \end{aligned}$$

Thus $S^{-1}(-)$ is additively exact covariant functor. Or $S^{-1}(-)$ is exact then additively exact covariant functor.

Proposition 11. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring and S_H be the part formed of all regulars homogeneous elements of A, then the relation $\overline{S}_H^{-1}(-) : G_r(A - Mod) \longrightarrow G_r(\overline{S}_H^{-1}A - Mod)$ which that for any graded left $A-$module M we correspond $\overline{S}_H^{-1}(M)$ and for all graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules $f : M \longrightarrow N$ we correspond $\overline{S}_H^{-1}(f)$ of degree $k \in \mathbb{Z}$ is a exact additively covariant functor.*

Proof.
Similarly to the proof of the theorem precedent 4.

Corollary 5. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring and S_H the set of all regular homogeneous of A, then the relation $S_H^{-1}(-) : G_r(A - Mod) \longrightarrow S_H^{-1}A - Mod$ which that for any graded left $A-$module M we correspond $S_H^{-1}(M)$ and for all graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules $f : M \longrightarrow N$ we correspond $S_H^{-1}(f)$ is additively exact covariant functor.*

Proof. S_H is the set of regular homogeneous of A then S_H is homogeneous multiplicatively closed subset so $\overline{S}_H = S_H$ then according to proposition precedent 11.

Proposition 12. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring , P be a prime ideal of A and S_{P_H} be a set formed of homogeneous regular elements of $A\backslash P$, then the relation $\overline{S}_{P_H}^{-1}(-) : G_r(A - Mod) \longrightarrow G_r(\overline{S}_{P_H}^{-1}A - Mod)$ which that for any graded left $A-$module M we correspond $\overline{S}_{P_H}^{-1}(M)$ and for all graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules $f : M \longrightarrow N$ we correspond $\overline{S}_{P_H}^{-1}(f)$ of degree $k \in \mathbb{Z}$ is additively exact covariant functor.*

Proof. Since the proposition 8 $\overline{S}_{P_H}$homogeneous multiplicatively closed subset, so $\overline{S}_{P_H}^{-1}(-) : G_r(A - Mod) \longrightarrow G_r(\overline{S}_{P_H}^{-1}A - Mod)$ is a covariant functor indeed. Let M, N two graded left $A-$modules and $f : M \longrightarrow N$ is a graded morphism of degree $k \in \mathbb{Z}$, then

$$\overline{S}_{P_H}^{-1}(-)(f) : \overline{S}_{P_H}^{-1}M \longrightarrow \overline{S}_{P_H}^{-1}N$$

$$\frac{m}{s} \longmapsto \frac{f(m)}{s}$$

is a graded morphism of degree $k \in \mathbb{Z}$ of $\overline{S}_{P_H}^{-1}A-$modules and for any graded left $A-$module M, $S_{P_H}^{-1}(-)(M) = \overline{S}_{P_H}^{-1}M$ is a graded left $\overline{S}_{P_H}^{-1}A-$module, so $S_{P_H}^{-1}(-)$ is a functor covariant for the category $G_r(A-Mod)$ to the category $G_r(\overline{S}_{P_H}^{-1}A-Mod)$.
Furthermore $\overline{S}_{P_H}^{-1}(-)$ is of degree $k \in \mathbb{Z}$, indeed let $(s,m) \in S \times M$ such that $\deg(\frac{m}{s}) = \deg(m) - \deg(s) = d_1$ so $\overline{S}_{P_H}^{-1}(-)(f)(\frac{m}{s}) = \frac{f(m)}{s}$, and

$$\begin{aligned}
\deg(\overline{S}_{P_H}^{-1}(f)(\frac{m}{s})) &= \deg(\frac{f(m)}{s}) \\
&= \deg(f(m)) - \deg(s) \\
&= (\deg(m) + k) - \deg(s) \\
&= \deg(\frac{m}{s}) + k \\
&= d_1 + k.
\end{aligned}$$

Thus $\overline{S}_{P_H}^{-1}(-)$ is additively exact covariant functor, since $\overline{S}_{P_H}^{-1}(-)$ preserve the exactness.

Definition 10. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded duo-ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a left graded $A-$module, P is a prime ideal of A and S_H be the set of homogeneous regular elements of $A\backslash P$ then:*

(i) $S_{P_H}^{-1}A$ is called homogeneous localized to A in P. and denoted by A_{PH};

(ii) $S_{P_H}^{-1}M$ is called homogeneous localized to M in P. and denoted by M_{PH}.

Corollary 6. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring , P be a prime ideal of A and S_{P_H} be the set of all homogeneous regular elements of $A\backslash P$, then the relation $S_{P_H}^{-1}(-) : G_r(A-Mod) \longrightarrow A_{PH} - Mod$ which that for any graded left $A-$module M we correspond M_{PH} and for all graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules $f : M \longrightarrow N$ we correspond $S_{P_H}^{-1}(f)$ of degree $k \in \mathbb{Z}$ is additively exact covariant functor.*

Proof. It is enough to note that $\overline{S_{P_H}} = S_{P_H}$ since the corollary 2.

4. Localization of complex in $COMP(G_r(A-Mod))$ over a duo-ring

Proposition 13. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ are two graded left $A-$module $f : M \longrightarrow N$ is a graded morphism of degree $k \in \mathbb{Z}$ of a graded left $A-$modules, then for all $n \in \mathbb{Z}$*

$$f^k(n) : M(n) \longrightarrow N(n)$$

$$m \quad \longmapsto \quad f^k(n)(m) = f(m)$$

is graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules.

Proof. We have $f : M \longrightarrow N$ is graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules, and $M(n)$ is a sub-module of graded left $A-$module M then let $m \in M(n)$, so

$$m = \sum_{i\in\mathbb{Z}} m_{i+n} \Longrightarrow f^k(n)(m) = f(m) = f(\sum_{i\in\mathbb{Z}} m_{i+n}) = \sum_{i\in\mathbb{Z}} f(m_{i+n})$$

or $f(m_{i+n}) \in N_{i+n+k} = (N(n))_{i+k}$ thus f is graded morphism of degree $k \in \mathbb{Z}$ of a graded left $A-$modules.

Corollary 7. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ are two graded left $A-$module $f : M \longrightarrow N$ is a graded morphism of degree $k \in \mathbb{Z}$ of a graded left $A-$modules, then $f : M \longrightarrow N(k)$ is graded morphism of graded left $A-$modules.*

Proof. We have $f : M \longrightarrow N$ is graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules, and $N(k)$ is a sub-module of graded left $A-$module N then let $m \in M$, so

$$m = \sum_{i\in\mathbb{Z}} m_i \Longrightarrow f(m) = f(\sum_{i\in\mathbb{Z}} m_i) = \sum_{i\in\mathbb{Z}} f(m_i).$$

Or $f(m_i) \in N_{i+k} = (N(k))_i$ thus f is graded morphism of a graded left $A-$modules.

Theorem 5. *Let A be a graded ring and $G_r(A - Mod)$ the category of a graded left $A-$modules, then for all $n \in \mathbb{Z}$ the relation $(-)(n) : G_r(A - Mod) \longrightarrow G_r(A - Mod)$ which that for any $M \in G_r(A - Mod)$ we made to correspond $M(n)$ and for all graded morphism of degree $k \in \mathbb{Z}$ of a graded left $A-$modules $f : M \longrightarrow N$ we correspond $f^k(n)$ is a additively exact covariant functor.*

Proof. Let $f : M \longrightarrow N$ be a graded morphism of degree $k \in \mathbb{Z}$ of a graded left $A-$modules, we denote by $(-)(n)(f) = f^k(n)$ the morphism of left $A-$modules of $M(n)$ to $N(n)$ thus $(-)(n)(M) = M(n)$ is in $A - Mod$, furthermore $M(n)$ and $N(n)$ are both graded left $A-$module then $M(n), N(n) \in G_r(A - Mod)$. Thus $(-)(n) : M(n) \longrightarrow N(n)$ has a sense.

(i) Let $f : M \longrightarrow N$ is graded morphism of degree $k \in \mathbb{Z}$ of graded left $A-$modules, then

$$(-)(n)(f) : M(n) \longrightarrow N(n)$$
$$f^k(n) : M(n) \longrightarrow N(n)$$
$$m \longmapsto f^k(n)(m) = f(m)$$

is a graded morphism of a graded left $A-$modules. Furthermore

$$(-)(n)(g \circ f)(m) = (g \circ f)^k(n)(m) = (g \circ f)(m) = g[f(m)] = g[f^k(n)(m)]$$

$$= g^k(n)[f^k(n)(m)] = (g^k(n) \circ f^k(n))(m) = (-)(n)(g) \circ (-)(n)(f)(m).$$

So

$$(-)(n)(g \circ f)(m) = (-)(n)(g) \circ (-)(n)(f)(m) \ \forall \ \ m \in M(n).$$

Thus

$$(-)(n)(g \circ f) = (-)(n)(g) \circ (-)(n)(f).$$

On the other hand

$$(-)(n)(1_{M(n)}) : M(n) \longrightarrow M(n)$$

$$1_M(n) : M(n) \longrightarrow M(n)$$

$$m \mapsto 1_{M(n)}(n)(m) = 1_{M(n)}(m) = m = 1_{(-)(n)(M)}(m),$$

so $(-)(n)(1_{M(n)}) = 1_{(-)(n)(M(n))}$, $\forall m \in M(n)$, so $(-)(n)$ is a functor of $G_r(A - Mod)$ to $G_r(A - Mod)$.
Thus $(-)(n) : G_r(A - Mod) \longrightarrow G_r(A - Mod)$ is a functor covariant.
Let $m \in M$ be homogeneous of degree d, then $(-)(n)(f)(m) = f^k(n)(m) = f(m)$ is of degree $k + n$ thus $(-)(n)$ is a additively exact covariant functor of degree $k \in \mathbb{Z}$.

Proposition 14. *Let* $A = \bigoplus\limits_{n\in\mathbb{Z}} A_n$ *be a graded ring and* $M = \bigoplus\limits_{n\in\mathbb{Z}} M_n$ *be a graded left* $A-$*module, then we have the following associate complex sequence* M_* *of a graded* $A-$*module* $M = \bigoplus\limits_{n\in\mathbb{Z}} M_n$ *:*

$$M_* : \cdots \to \ M(n+1) \overset{d_{n+1}}{\to} M(n) \overset{d_n}{\to} M(n-1) \to \cdots$$

with $M(n) = \bigoplus\limits_{k\in\mathbb{Z}} M_{n+k}$ *and*

$$d_n : M(n) \longrightarrow M(n-1)$$

$$x = y + z \longmapsto y$$

with $(y, z) \in M_n \times M(n + 1)$*.*

Proof. We have $M(n) = \bigoplus\limits_{k\in\mathbb{Z}} M_{n+k} = \bigoplus\limits_{k\geq n} M_k = M_n \bigoplus M_{n+1}$ and

$$M(n-1) = M_{n-1} \bigoplus M(n) = M_{n-1} \bigoplus M_n \bigoplus M(n+1).$$

Let $x \in M(n)$, then it is exist a unique $(y, z) \in M_n \times M(n + 1)$ such that $x = y + z$.
Put

$$d_n : M(n) \longrightarrow M(n-1)$$

$$x = y + z \longmapsto y,$$

so $Im(d_n) = M_n$; On the other hand

$$d_{n-1} : M(n-1) \longrightarrow M(n-2)$$

$$w = u + v \longmapsto v$$

with $(u, v) \in M_{n-1} \times M(n)$, so $ker(d_{n-1}) = M(n)$ so $Im(d_n) \subset \ker(d_{n-1})$, so

$$d_{n-1} \circ d_n = 0$$

,thus

$$M_* : \cdots \to \ M(n+1) \overset{d_{n+1}}{\to} M(n) \overset{d_n}{\to} M(n-1) \to \cdots$$

is a complex sequence.

Proposition 15. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$, $N = \bigoplus_{n\in\mathbb{Z}} N_n$ are two graded left $A-$modules and $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n\in\mathbb{Z}} N_n$ is a graded morphism of degree $k \in \mathbb{Z}$ of a graded $A-$modules, then we have the following associate complex f_*^k of graded morphism $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n\in\mathbb{Z}} N_n$ of a graded $A-$modules :*

$$\begin{array}{ccccccccc} M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & M(n-1) & \longrightarrow & \cdots. \\ \downarrow f_*^k & & \downarrow f^k(n+1) & & \downarrow f^k(n) & & \downarrow f^k(n-1) & & \\ N_* : \cdots & \longrightarrow & N(n+1) & \xrightarrow{d'_{n+1+k}} & N(n) & \xrightarrow{d'_{n+k}} & N(n-1) & \longrightarrow & \cdots \end{array}$$

Proof. Prove that for all $n \in \mathbb{Z}$,

$$f^k(n) \circ d_{n+1} = d'_{n+1+k} \circ f^k(n+1).$$

Let $x \in M(n+1)$, then there exist the unique couple $(y, z) \in M_{n+1} \times M(n+2)$ such that $x = y + z$, so

$$(f^k(n) \circ d_{n+1})(x) = f^k(n)[d_{n+1}(x)] = f[d_{n+1}(x)] = f[y] = f(y),$$

and
$(d'_{n+1+k} \circ f^k(n+1))(x) = d'_{n+1+k}[f^k(n+1)(x)] = d'_{n+1+k}[f(x)] = d'_{n+1+k}[f(y+z)] = d'_{n+1+k}[f(y) + f(z)] = f(y)$,
because $f(y) \in N_{n+1+k}$ and $f(z) \in N(n+2+k)$,

$$\Longrightarrow (f^k(n) \circ d_{n+1})(x) = (d'_{n+1+k} \circ f^k(n+1))(x), \quad \forall\ x\ \in M(n+1),$$

so

$$f^k(n) \circ d_{n+1} = d'_{n+1+k} \circ f^k(n+1),$$

thus f_*^k is a complex chain.

Theorem 6. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring, then the following information:*

(i) The objets are the associate complex sequences of a graded left $A-$*modules;*

(ii) The morphisms are the associate complex chains of a graded morphism of a graded left $A-$*modules.*

formed a category called the category of associate complex of a graded left $A-$*modules and denoted by* $COMP(G_r(A-Mod))$.

Proof. Let M_* and N_* two objets of $COMP(G_r(A-Mod))$,then:

(i) $Hom_{COMP(G_r(A-Mod))}(M_*, N_*) = \{$ the set of associate complex chains f_*^k, of M_* to $N_*\}$;

(ii) The morphisms are the associate complex chains of a graded morphism of degrees k of a graded left $A-$modules. then we have :

(a) $\forall\ f_*^k \in Hom_{COMP(G_r(A-Mod))}(M_*, N_*)$; $\forall\ g_*^r \in Hom_{COMP(G_r(A-Mod))}(N_*, P_*)$; $\forall\ \ h_*^s \in Hom_{COMP(G_r(A-Mod))}(P_*, Q_*)$ on a :

$$\begin{array}{ccccccc}
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \ldots \\
\downarrow f_*^k & & \downarrow f^k(n+1) & & \downarrow f^k(n) & & \\
N_* : \cdots & \longrightarrow & N(n+1) & \xrightarrow{d'_{n+1+k}} & N(n) & \xrightarrow{d'_{n+k}} & \ldots \\
\downarrow g_*^r & & \downarrow g^r(n+1) & & \downarrow g^r(n) & & \\
P_* : \cdots & \longrightarrow & P(n+1) & \xrightarrow{d''_{n+1+k+r}} & P(n) & \xrightarrow{d''_{n+k+r}} & \ldots \\
\downarrow h_*^s & & \downarrow h^s(n+1) & & \downarrow h^s(n) & & \\
Q_* : \cdots & \longrightarrow & Q(n+1) & \xrightarrow{d'''_{n+1+k+r+s}} & Q(n) & \xrightarrow{d'''_{n+k+r+s}} & \ldots
\end{array}$$

So $(h_*^s \circ g_*^r) \circ f_*^k = h_*^s \circ (g_*^r \circ f_*^k)$;

(b) Let M_* the object of $COMP(G_r(A-Mod))$, we have:

$$1_{M_*} : M_* \longrightarrow M_*$$

$$\begin{array}{ccccccc}
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \ldots \\
\downarrow 1_{M_*} & & \downarrow 1(n+1) & & \downarrow 1(n) & & \\
M_* : \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \ldots
\end{array}$$

1_{M_*} verified $f_* \circ 1_{M_*} = f_* \quad \forall\ \ f_* \in Hom_{COMP(G_r(A-Mod))}(M_*, N_*)$.
Furthermore $1_{M_*} \circ g_* = g_* \quad \forall\ \ g_* \in Hom_{COMP(G_r(A-Mod))}(N_*, M_*)$.

Thus $COMP(G_r(A-Mod))$ is a category.

Proposition 16. *Let $A=\bigoplus_{n\in\mathbb{Z}}A_n$ be a graded ring, $M=\bigoplus_{n\in\mathbb{Z}}M_n$ and $N=\bigoplus_{n\in\mathbb{Z}}N_n$ are two graded left $A-$modules, $f:M\longrightarrow N$ is a graded morphism of degree k and S be a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A, then we have:*

(i) The following complex sequence:

$$S^{-1}(M_*):\cdots\longrightarrow S^{-1}(M(n+1))\overset{S^{-1}(d_{n+1})}{\longrightarrow}S^{-1}(M(n))\overset{S^{-1}(d_n)}{\longrightarrow}S^{-1}(M(n-1))\longrightarrow\cdots$$

(ii) The following complex chain:

$$\begin{array}{ccccccccc} S^{-1}(M_*):\cdots & \longrightarrow & S^{-1}(M(n+1)) & \overset{S^{-1}(d_{n+1})}{\longrightarrow} & S^{-1}(M(n)) & \overset{S^{-1}(d_n)}{\longrightarrow} & S^{-1}(M(n-1)) & \longrightarrow & \cdots \\ \downarrow S^{-1}(f^k_*) & & \downarrow S^{-1}(f^k(n+1)) & & \downarrow S^{-1}(f^k(n)) & & \downarrow S^{-1}(f^k(n-1)) & & \\ S^{-1}(N_*):\cdots & \longrightarrow & S^{-1}(N(n+1)) & \overset{S^{-1}(d'_{n+1+k})}{\longrightarrow} & S^{-1}(N(n)) & \overset{S^{-1}(d'_{n+k})}{\longrightarrow} & S^{-1}(N(n-1)) & \longrightarrow & \cdots \end{array}$$

Proof. As for all $n\in\mathbb{Z}$, M_* and N_* are two complex sequences of graded left $A-$module, then $S^{-1}(M_*)$ and $S^{-1}(N_*)$ are two complex sequences of a graded left $S^{-1}A-$module.
Prove that for all $n\in\mathbb{Z}$,

$$S^{-1}(f^k(n))\circ S^{-1}(d_{n+1})=S^{-1}(d'_{n+1+k})\circ S^{-1}(f^k(n+1)).$$

Let $\frac{x}{s}\in S^{-1}(M(n+1))$, then it is exist a unique couple $(\frac{y}{t},\frac{z}{r})\in S^{-1}M_{n+1}\times S^{-1}M(n+2)$ such that $\frac{x}{s}=\frac{y}{t}+\frac{z}{r}$, so

$$(S^{-1}f^k(n)\circ S^{-1}d_{n+1})(\frac{x}{s})=S^{-1}f^k(n)[S^{-1}d_{n+1}(\frac{x}{s})]=S^{-1}f^k(n)[\frac{y}{t}]=S^{-1}f[\frac{y}{t}]=\frac{f(y)}{t},$$

and

$$\begin{aligned}(S^{-1}d'_{n+1+k}\circ S^{-1}f^k(n+1))(\frac{x}{s}) &= S^{-1}d'_{n+1+k}[S^{-1}f^k(n+1)(\frac{y}{t}+\frac{z}{r})]\\ &= S^{-1}d'_{n+1+k}[S^{-1}f(\frac{y}{t}+\frac{z}{r})]\\ &= S^{-1}d'_{n+1+k}[S^{-1}f(\frac{y}{t})+S^{-1}f(\frac{z}{r})]\\ &= S^{-1}f(\frac{y}{t})\\ &= \frac{f(y)}{t}\end{aligned}$$

because $S^{-1}f(\frac{y}{t}) \in S^{-1}N_{n+1+k}$ and $S^{-1}f(\frac{z}{r}) \in S^{-1}N(n+2+k)$

$$\Longrightarrow (S^{-1}d'_{n+1+k} \circ S^{-1}f^k(n+1))(\frac{x}{s}) = (S^{-1}f^k(n) \circ S^{-1}d_{n+1})(\frac{x}{s}) \quad \forall \ \frac{x}{s} \in S^{-1}M(n+1)$$

so

$$(S^{-1}d'_{n+1+k} \circ S^{-1}f^k(n+1))) = (S^{-1}f^k(n) \circ S^{-1}d_{n+1})$$

thus $S^{-1}(f^k_*)$ is a complex chain.

Corollary 8. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ are two graded left $A-$modules, $f : M \longrightarrow N$ is a graded morphism of degree k and S_H be a part formed of regulars homogeneous elements of A, then we have:*

(i) The following complex sequence:

$$\overline{S}_H^{-1}(M_*) : \cdots \longrightarrow \overline{S}_H^{-1}(M(n+1)) \overset{\overline{S}_H^{-1}(d_{n+1})}{\longrightarrow} \overline{S}_H^{-1}(M(n)) \overset{\overline{S}_H^{-1}(d_n)}{\longrightarrow} \overline{S}_H^{-1}(M(n-1)) \longrightarrow \cdots$$

(ii) The following complex chain:

$$\begin{array}{ccccccccc} \overline{S}_H^{-1}(M_*) : \cdots & \longrightarrow & \overline{S}_H^{-1}(M(n+1)) & \overset{\overline{S}_H^{-1}(d_{n+1})}{\longrightarrow} & \overline{S}_H^{-1}(M(n)) & \overset{\overline{S}_H^{-1}(d_n)}{\longrightarrow} & \overline{S}_H^{-1}(M(n-1)) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \overline{S}_H^{-1}(f^k_*)} & & \downarrow{\scriptstyle \overline{S}_H^{-1}(f^k(n+1))} & & \downarrow{\scriptstyle \overline{S}_H^{-1}(f^k(n))} & & \downarrow{\scriptstyle \overline{S}_H^{-1}(f^k(n-1))} & & \\ \overline{S}_H^{-1}(N_*) : \cdots & \longrightarrow & \overline{S}_H^{-1}(N(n+1)) & \overset{\overline{S}_H^{-1}(d'_{n+1+k})}{\longrightarrow} & \overline{S}_H^{-1}(N(n)) & \overset{\overline{S}_H^{-1}(d'_{n+k})}{\longrightarrow} & \overline{S}_H^{-1}(N(n-1)) & \longrightarrow & \cdots \end{array}$$

Proof. Since the proposition 6 $\overline{S}_H$ is multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A, and the rest is similarly to the proof of the proposition 16.

Proposition 17. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ two graded left $A-$modules, $f : M \longrightarrow N$ is graded morphism of degree $k \in \mathbb{Z}$ and S be a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A, then we have:*

(i) The following complex sequence:

$$B^* : \cdots \longrightarrow S^{-1}A \otimes (M(n+1)) \overset{S^{-1}A\otimes(d_{n+1})}{\longrightarrow} S^{-1}A \otimes (M(n)) \overset{S^{-1}A\otimes(d_n)}{\longrightarrow}$$
$$S^{-1}A \otimes (M(n-1)) \longrightarrow \cdots$$

(ii) The following complex chain:

$$\begin{array}{ccccccccc} B^* : \cdots & \longrightarrow & S^{-1}A \otimes (M(n+1)) & \overset{S^{-1}A\otimes(d_{n+1})}{\longrightarrow} & S^{-1}A \otimes (M(n)) & \overset{S^{-1}A\otimes(d_n)}{\longrightarrow} & S^{-1}A \otimes (M(n-1)) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle S^{-1}A\otimes f^k_*} & & \downarrow{\scriptstyle S^{-1}A\otimes(f^k(n+1))} & & \downarrow{\scriptstyle S^{-1}A\otimes(f^k(n))} & & \downarrow{\scriptstyle S^{-1}A\otimes(f^k(n-1))} & & \\ D_* : \cdots & \longrightarrow & S^{-1}A \otimes (N(n+1)) & \overset{S^{-1}A\otimes(d'_{n+1+k})}{\longrightarrow} & S^{-1}A \otimes N(n)) & \overset{S^{-1}A\otimes(d'_{n+k})}{\longrightarrow} & S^{-1}A \otimes (N(n-1)) & \longrightarrow & \cdots \end{array}$$

With $B^* = S^{-1}A \otimes (M_*)$ *and* $D_* = S^{-1}A \bigotimes_A (N_*)$.

Proof. We have the functor $S^{-1}()$ and the functor $S^{-1}A \bigotimes_A ()$ are isomorphs. On the other hand it suffices to prove that the following diagram is commutative

$$\begin{array}{ccccccc}
B^* : \cdots & \longrightarrow & S^{-1}A \bigotimes M(n+1) & \xrightarrow{S^{-1}A \bigotimes d_{n+1}} & S^{-1}A \bigotimes M(n) & \xrightarrow{S^{-1}A \bigotimes d_n} & \cdots \\
\downarrow \gamma & & \downarrow \gamma_{n+1} & & \downarrow \gamma_n & & \\
S^{-1}(M_*) : \cdots & \longrightarrow & S^{-1}M(n+1) & \xrightarrow{S^{-1}d_{n+1}} & S^{-1}M(n) & \xrightarrow{S^{-1}d_n} & \cdots \\
\downarrow S^{-1}f^k_* & & \downarrow S^{-1}f^k(n+1) & & \downarrow S^{-1}f^k(n) & & \\
(S^{-1}(N_*) : \cdots & \longrightarrow & S^{-1}N(n+1) & \xrightarrow{S^{-1}d'_{n+1+k}} & S^{-1}N(n) & \xrightarrow{S^{-1}d'_{n+k}} & \cdots \\
\downarrow \lambda & & \downarrow \lambda_{n+1} & & \downarrow \lambda_n & & \\
D_* : \cdots & \longrightarrow & S^{-1}A \bigotimes N(n+1) & \xrightarrow{S^{-1}A \bigotimes d'_{n+1+k}} & S^{-1}A \bigotimes N(n) & \xrightarrow{S^{-1}A \bigotimes d'_{n+k}} & \cdots
\end{array}$$

i.e. prove that for all $n \in \mathbb{Z}$ we have

$$\lambda_n \circ S^{-1}f^k(n) \circ \gamma_n \circ S^{-1}A \bigotimes d_{n+1} = S^{-1}A \bigotimes d'_{n+1+k} \circ \lambda_{n+1} \circ S^{-1}f^k(n+1) \circ \gamma_{n+1}$$

or for all $n \in \mathbb{Z}$, we have $\lambda_n \circ S^{-1}f^k(n) \circ \gamma_n = 1_{S^{-1}A} \bigotimes f^k(n)$.
Let $\frac{1}{s} \otimes m \in S^{-1}A \bigotimes\limits_A M(n+1)$, then it is exist an unique couple $(x, y) \in M_{n+1} \times M(n+2)$ such that $m = x + y$ so

$$\lambda_n \circ S^{-1}f^k(n) \circ \gamma_n \circ S^{-1}A \bigotimes d_{n+1}[\frac{1}{s} \otimes m] = \lambda_n \circ S^{-1}f^k(n) \circ \gamma_n \circ S^{-1}A \bigotimes d_{n+1}[\frac{1}{s} \otimes (x+y)] =$$

$$\lambda_n \circ S^{-1}f^k(n) \circ \gamma_n[\frac{1}{s} \otimes x] = 1_{S^{-1}A} \bigotimes f^k(n)[\frac{1}{s} \otimes x] = \frac{1}{s} \otimes f(x).$$

On the other hand we have

$$S^{-1}A \bigotimes d'_{n+1+k} \circ \lambda_{n+1} \circ S^{-1}f^k(n+1) \circ \gamma_{n+1}[\frac{1}{s} \otimes m] =$$

$$S^{-1}A \bigotimes d'_{n+1+k} \circ \lambda_{n+1} \circ S^{-1}f^k(n) \circ \gamma_{n+1}[\frac{1}{s} \otimes (x+y)] =$$

$$S^{-1}A \bigotimes d'_{n+1+k}[\frac{1}{s} \otimes f(x+y)] = \frac{1}{s} \otimes f(x)$$

thus $S^{-1}A \otimes f^k_*$ is a complex the chain.

Corollary 9. *Let* $A = \bigoplus\limits_{n \in \mathbb{Z}} A_n$ *be a graded duo-ring,* $M = \bigoplus\limits_{n \in \mathbb{Z}} M_n$ *and* $N = \bigoplus\limits_{n \in \mathbb{Z}} N_n$ *two graded left* $A-$*modules,* $f : M \longrightarrow N$ *is graded morphism of degree* $k \in \mathbb{Z}$ *and* S_H *be a part formed of regulars homogeneous elements of* A*, then we have :*

(i) The following complex sequence:

$$B^* : \cdots \longrightarrow \overline{S}_H^{-1}A \otimes (M(n+1)) \overset{\overline{S}_H^{-1}A\otimes(d_{n+1})}{\longrightarrow} \overline{S}_H^{-1}A \otimes (M(n)) \overset{\overline{S}_H^{-1}A\otimes(d_n)}{\longrightarrow} \overline{S}_H^{-1}A \otimes (M(n-1)) \longrightarrow \cdots$$

(ii) The following complex chain:

$$\begin{array}{ccccccccc} B^* : \cdots & \longrightarrow & \overline{S}_H^{-1}A \otimes (M(n+1)) & \xrightarrow{\overline{S}_H^{-1}A\otimes(d_{n+1})} & \overline{S}_H^{-1}A \otimes (M(n)) & \xrightarrow{\overline{S}_H^{-1}A\otimes(d_n)} & \overline{S}_H^{-1}A \otimes (M(n-1)) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \overline{S}_H^{-1}A\otimes f_*^k} & & \downarrow{\scriptstyle \overline{S}_H^{-1}A\otimes(f^k(n+1))} & & \downarrow{\scriptstyle \overline{S}_H^{-1}A\otimes(f^k(n))} & & \downarrow{\scriptstyle \overline{S}_H^{-1}A\otimes(f^k(n-1))} & & \\ D_* : \cdots & \longrightarrow & \overline{S}_H^{-1}A \otimes (N(n+1)) & \xrightarrow{\overline{S}_H^{-1}A\otimes(d'_{n+1+k})} & \overline{S}_H^{-1}A \otimes N(n)) & \xrightarrow{\overline{S}_H^{-1}A\otimes(d'_{n+k})} & \overline{S}_H^{-1}A \otimes (N(n-1)) & \longrightarrow & \cdots \end{array}$$

With $B^* = \overline{S}_H^{-1}A \otimes (M_*)$ *and* $D_* = \overline{S}_H^{-1}A \bigotimes_A (N_*)$.

Proof.

it is sufficient to note that $\overline{S}_H$ is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A.

Corollary 10. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded duo-ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and* $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *two graded left* $A-$*modules,* $f : M \longrightarrow N$ *is graded morphism of degree* k *and* S_H *be the set of all regulars homogeneous elements of* A*, then we have:*

(i) The following complex sequence:

$$B^* : \cdots \longrightarrow S^{-1}A\otimes(M(n+1)) \overset{S^{-1}A\otimes(d_{n+1})}{\longrightarrow} S^{-1}A\otimes(M(n)) \overset{S^{-1}A\otimes(d_n)}{\longrightarrow} S^{-1}A\otimes(M(n-1)) \longrightarrow \cdots$$

(ii) The following complex chain:

$$\begin{array}{ccccccccc} B^* : \cdots & \longrightarrow & S_H^{-1}A \otimes (M(n+1)) & \xrightarrow{S_H^{-1}A\otimes(d_{n+1})} & S_H^{-1}A \otimes (M(n)) & \xrightarrow{S_H^{-1}A\otimes(d_n)} & S_H^{-1}A \otimes (M(n-1)) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle S_H^{-1}A\otimes f_*^k} & & \downarrow{\scriptstyle S_H^{-1}A\otimes(f^k(n+1))} & & \downarrow{\scriptstyle S_H^{-1}A\otimes(f^k(n))} & & \downarrow{\scriptstyle S_H^{-1}A\otimes(f^k(n-1))} & & \\ D_* : \cdots & \longrightarrow & S_H^{-1}A \otimes (N(n+1)) & \xrightarrow{S_H^{-1}A\otimes(d'_{n+1+k})} & S_H^{-1}A \otimes N(n)) & \xrightarrow{S_H^{-1}A\otimes(d'_{n+k})} & S_H^{-1}A \otimes (N(n-1)) & \longrightarrow & \cdots \end{array}$$

With $B^* = S_H^{-1}A \otimes (M_*)$ *and* $D_* = S_H^{-1}A \bigotimes_A (N_*)$.

Proof. it is sufficient to note that $S_H = \overline{S}_H$.

Theorem 7. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring and* $G_r(A-Mod)$ *the category of graded left* $A-$*modules, then the relation* $C(-) : G_r(A-Mod) \longrightarrow COMP(G_r(A-Mod))$ *which that for all graded left* $A-$*module* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *of* $G_r(A-Mod)$ *we correspond the associate* *complex sequence* M_* *to a graded* $A-$*module* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and for all graded morphism of graded left* $A-$*modules* $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n\in\mathbb{Z}} N_n$ *of degree* k *we correspond the associate complex chain* f_*^k *to a morphism of graded left* $A-$*module* $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n\in\mathbb{Z}} N_n$ *is exact additively covariant functor.*

Proof. Let M, N two graded left $A-$modules and $f : M \longrightarrow N$ graded morphism of graded $A-$modules, we note that $C(M) = M_*$ (respectively $C(N) = N_*$) the associate complex sequence M_* (respectively N_*) to a graded $A-$module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ (respectively to a graded $A-$module $N = \bigoplus_{n\in\mathbb{Z}} N_n$) so $M_*, N_* \in COMP(G_r(A - Mod))$.
So $C(f) : M_* \longrightarrow N_*$ has a sense.

(i) Let $M \in G_r(A - Mod)$ then $C(M) = M_*$ is the associate complex sequence to a graded $A-$module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ then $M_* \in COMP(G_r(A - Mod))$.

(ii) Let $f : M \longrightarrow N$ graded morphism of degree k of graded $A-$modules then :

$$C(f) = f_*^k : M_* \longrightarrow N_*$$

the associate complex chain to a graded morphism of degree k of graded left $A-$module. Furthermore

$$C(g \circ f) = (g \circ f)_*^k = g[f]_*^k = g[f_*^k]_*^k = g_*^k \circ f_*^k = C(g) \circ C(f).$$

On other hand

$$C(1_{M(n)}) : M(n)_* \longrightarrow M(n)_*$$

$$1_{M(n)_*} = 1_{C(M)}$$

Thus $C()$ is a covariant functor of $G_r(A - Mod)$ to $COMP(G_r(A - Mod))$.
Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

be the short exact sequence of graded left $A-$modules then we make the functor $C()$ then we have

$$
\begin{array}{ccccccc}
0: \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
M_*: \cdots & \longrightarrow & M(n+1) & \xrightarrow{d_{n+1}} & M(n) & \xrightarrow{d_n} & \ldots \\
\downarrow{\scriptstyle f^k_*} & & {\scriptstyle f^k(n+1)}\downarrow & & \downarrow{\scriptstyle f^k(n)} & & \\
N_*: \cdots & \longrightarrow & N(n+1) & \xrightarrow{d'_{n+1+k}} & N(n) & \xrightarrow{d'_{n+k}} & \ldots \\
\downarrow{\scriptstyle g^r_*} & & \downarrow{\scriptstyle g^r(n+1)} & & \downarrow{\scriptstyle g^r(n)} & & \\
L_*: \cdots & \longrightarrow & L(n+1) & \xrightarrow{d''_{n+1+k+r}} & L(n) & \xrightarrow{d''_{n+k+r}} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
0: \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}
$$

is a short exact complex chain associate to short exact sequence of graded left $A-$modules then

$$0 \longrightarrow M_* \xrightarrow{f^k_*} N_* \xrightarrow{g^k_*} L_* \longrightarrow 0$$

is exact complex chain. Thus $C()$ is exact additively covariant functor of $G_r(A-Mod)$ to $COMP(G_r(A-Mod))$.

Theorem 8. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, S is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A and $G_r(S^{-1}A - Mod)$ the category of graded left $S^{-1}A-$modules, then the relation $C_H(-) : G_r(S^{-1}A - Mod) \longrightarrow COMP(G_r(S^{-1}A - Mod))$ which that for all graded left $S^{-1}A-$module $S^{-1}M$ of $G_r(S^{-1}A - Mod)$ we correspond the associate complex sequence $(S^{-1}M)_*$ to a graded $S^{-1}A-$module $S^{-1}M$ and for all graded morphism of graded left $S^{-1}A-$modules $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$ of degree k we correspond the associate complex chain $(S^{-1}f)^k_*$ to a morphism of graded left $S^{-1}A-$module $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$ is additively exact covariant functor.*

Proof. Similarly to the proof of theorem precedent 7

Theorem 9. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring, S_H is a part formed of regulars homogeneous elements of A and $G_r(\overline{S}_H^{-1}A - Mod)$ the category of graded left $\overline{S}_H^{-1}A-$modules, then the relation $C_H(-) : G_r(\overline{S}_H^{-1}A - Mod) \longrightarrow COMP(G_r(\overline{S}_H^{-1}A - Mod))$ which that for all graded left $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}M$ of $G_r(\overline{S}_H^{-1}A - Mod)$ we correspond the associate complex sequence $(\overline{S}_H^{-1}M)_*$ to a graded $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}M$ and for all graded morphism of graded left $\overline{S}_H^{-1}A-$modules $\overline{S}_H^{-1}f : \overline{S}_H^{-1}M \longrightarrow \overline{S}_H^{-1}N$ of degree k we correspond the associate complex chain $(\overline{S}_H^{-1}f)^k_*$ to a morphism of graded left $\overline{S}_H^{-1}A-$module $\overline{S}_H^{-1}f : \overline{S}_H^{-1}M \longrightarrow \overline{S}_H^{-1}N$ is additively exact covariant functor.*

Proof. Similarly to the proof of theorem precedent 7

Theorem 10. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, S is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A and $G_r(S^{-1}A-Mod)$ the category of graded left $S^{-1}A-$modules, then the relation $(C_H \circ S^{-1})(-) : G_r(A-Mod) \longrightarrow COMP(G_r(S^{-1}A - Mod))$ which that for all graded left $A-$module M of $G_r(A-Mod)$ we correspond the associate complex sequence $(C_H \circ S^{-1})(M) = (S^{-1}M)_*$ to a graded $A-$module M and for all graded morphism of graded left $A-$modules $f : M \longrightarrow N$ of degree k we correspond the associate complex chain $(C_H \circ S^{-1})(f) = (S^{-1}f)^k_*$ to a morphism of graded left $A-$module $f : M \longrightarrow N$ is additively exact covariant functor.*

Proof. Similarly to the proof of theorem precedent 7

Theorem 11. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded duo-ring, S_H is a part formed of regulars homogeneous elements of A and $G_r(\overline{S}_H^{-1}A-Mod)$ the category of graded left $\overline{S}_H^{-1}A-$modules, then the relation $(C_H \circ \overline{S}_H^{-1})(-) : G_r(A-Mod) \longrightarrow COMP(G_r(\overline{S}_H^{-1}A-Mod))$ which that for all graded left $A-$module M of $G_r(A-Mod)$ we correspond the associate complex sequence $(C_H \circ \overline{S}_H^{-1})(M) = (\overline{S}_H^{-1}M)_*$ to a graded $A-$module M and for all graded morphism of graded left $A-$modules $f : M \longrightarrow N$ of degree k we correspond the associate complex chain $(C_H \circ \overline{S}_H^{-1})(f) = (\overline{S}_H^{-1}f)^k_*$ to a morphism of graded left $A-$module $f : M \longrightarrow N$ is additively exact covariant functor.*

Proof. Similarly to the proof of theorem precedent7

Lemma 1. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and a graded left $A-$module then for all $n \in \mathbb{Z}$ $H_n(M_*) \cong M(n + 2)$.*

Proof. Let $M_* : \cdots \to M(n+1) \overset{d_{n+1}}{\to} M(n) \overset{d_n}{\to} M(n-1) \to \cdots$ the complex sequence, then $\ker(d_n) = M(n + 1)$ and $Im(d_n) = M_n$ so

$$H_n(M_*) = \ker(d_n)/Im(d_{n+1}) = M(n+1)/M_{n+1} = (M_{n+1} \oplus M(n+2))/M_{n+1} \cong M(n+2).$$

Theorem 12. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, we have the induced functor of $H_n : COMP(G_r(A - Mod)) \longrightarrow G_r(A - Mod)$ which that for all associate complex sequence M_* to a graded $A-$module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ we correspond $H_n(M_*) = M(n + 2)$ and for all associate complex chain f^k_* to a morphism of graded left $A-$module $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n\in\mathbb{Z}} N_n$ we correspond $H_n(f_*) = f^k(n + 2)$, is a covariant functor.*

Theorem 13. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring, For all* $n \in \mathbb{Z}$ *and for all short exact sequence*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

of a graded left $A-$*modules of graded morphism of degree* $k \in \mathbb{Z}$ *we have the following long exact sequence*

$$\cdots \longrightarrow M(n+2) \overset{f^k(n+2)}{\longrightarrow} N(n+2) \overset{g^k(n+2)}{\longrightarrow} L(n+2) \overset{\delta_n}{\longrightarrow} M(n+1) \overset{f^k(n+1)}{\longrightarrow} N(n+1) \longrightarrow \cdots$$

of a graded left $A-$*modules of graded morphism of degree* $k \in \mathbb{Z}$.
Furthermore, if

S is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A*, we have the following longs exacts sequences of a graded left* $S^{-1}A-$*modules*

$$\cdots S^{-1}M(n+2) \overset{S^{-1}f^k(n+2)}{\longrightarrow} S^{-1}N(n+2) \overset{S^{-1}g^k(n+2)}{\longrightarrow} S^{-1}L(n+2) \overset{S^{-1}\delta_n}{\longrightarrow} S^{-1}M(n+1)\cdots$$

$$\cdots S^{-1}\otimes M(n+2) \overset{S^{-1}\otimes f^k(n+2)}{\longrightarrow} S^{-1}\otimes N(n+2) \overset{S^{-1}\otimes g^k(n+2)}{\longrightarrow} S^{-1}\otimes L(n+2) \overset{S^{-1}\otimes\delta_n}{\longrightarrow} S^{-1}\otimes M(n+1)\cdots$$

Proof. Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

be the short exact sequence of graded left $A-$modules then we make the functor $C()$ to the short exact sequence of graded $A-$modules then we have

$$0 \longrightarrow M_* \xrightarrow{f_*^k} N_* \xrightarrow{g_*^k} L_* \longrightarrow 0$$

the associate short exact complex to a of graded $A-$modules or since precedent theorem 12, it exist a morphism of left $A-$module $H_n(L_*) \xrightarrow{\delta_n} H_{n-1}(M_*)$ such that we have the following long exact sequence of graded left $A-$modules

$$\cdots \longrightarrow H_n(M_*) \overset{H_n(f_*^k)}{\longrightarrow} H_n(N_*) \overset{H_n(g_*^k)}{\longrightarrow} H_n(L_*) \overset{\delta_n}{\longrightarrow} H_{n-1}(M_*) \overset{H_{n-1}(f_*^k)}{\longrightarrow} H_{n-1}(N_*) \longrightarrow \cdots$$

i.e

$$\cdots \longrightarrow M(n+2) \overset{f^k(n+2)}{\longrightarrow} N(n+2) \overset{g^k(n+2)}{\longrightarrow} L(n+2) \overset{\delta_n}{\longrightarrow} M(n+1) \overset{f^k(n+1)}{\longrightarrow} N^k(n+1) \longrightarrow \cdots$$

We have the functor $S^{-1}()$ is exact then we have

$$\cdots S^{-1}M(n+2) \overset{S^{-1}f^k(n+2)}{\longrightarrow} S^{-1}N(n+2) \overset{S^{-1}g^k(n+2)}{\longrightarrow} S^{-1}L(n+2) \overset{S^{-1}\delta_n}{\longrightarrow} S^{-1}M(n+1)\cdots$$

Or the functor $S^{-1}()$ and the functor $S^{-1}A\bigotimes_A$ are isomorph then we have also

$$\cdots S^{-1}\otimes M(n+2) \overset{S^{-1}\otimes f^k(n+2)}{\longrightarrow} S^{-1}\otimes N(n+2) \overset{S^{-1}\otimes g^k(n+2)}{\longrightarrow} S^{-1}\otimes L(n+2) \overset{S^{-1}\otimes\delta_n}{\longrightarrow} S^{-1}\otimes M(n+1)\cdots$$

Proposition 18. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring, For all* $n \in \mathbb{Z}$ *and for all short exact sequence*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

of a graded left $A-$*modules of graded morphism of degree* $k \in \mathbb{Z}$ *we have the following long exact sequence*

$$\cdots \longrightarrow M(n+2) \overset{f^k(n+2)}{\longrightarrow} N(n+2) \overset{g^k(n+2)}{\longrightarrow} L(n+2) \overset{\delta_n}{\longrightarrow} M(n+1) \overset{f^k(n+1)}{\longrightarrow} N(n+1) \longrightarrow \cdots$$

of a graded left $A-$*modules of graded morphism of degree* $k \in \mathbb{Z}$.
Furthermore, if S *is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of* A*, we have the following longs exacts sequences of a graded left* $S^{-1}A-$*modules*

$$\cdots S^{-1}M(n+2) \overset{S^{-1}f^k(n+2)}{\longrightarrow} S^{-1}N(n+2) \overset{S^{-1}g^k(n+2)}{\longrightarrow} S^{-1}L(n+2) \overset{S^{-1}\delta_n}{\longrightarrow} S^{-1}M(n+1) \cdots$$

$$\cdots S^{-1}\otimes M(n+2) \overset{S^{-1}\otimes f^k(n+2)}{\longrightarrow} S^{-1}\otimes N(n+2) \overset{S^{-1}\otimes g^k(n+2)}{\longrightarrow} S^{-1}\otimes L(n+2) \overset{S^{-1}\otimes\delta_n}{\longrightarrow} S^{-1}\otimes M(n+1) \cdots$$

Proof. Similarly to the proof of theorem precedent13

Corollary 11. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded duo-ring, For all* $n \in \mathbb{Z}$ *and for all short exact sequence*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

of a graded left $A-$*modules we have the following long exact sequence*

$$\cdots \longrightarrow M(n+2) \overset{f^k(n+2)}{\longrightarrow} N(n+2) \overset{g^k(n+2)}{\longrightarrow} L(n+2) \overset{\delta_n}{\longrightarrow} M(n+1) \overset{f^k(n+1)}{\longrightarrow} N(n+1) \longrightarrow \cdots$$

of a graded left $A-$*modules.*
Furthermore, if S_H *is the part of regulars homogeneous elements of* A*, we have the following longs exacts sequences of a graded left* $\overline{S}_H^{-1}A-$*modules*

$$\cdots \overline{S}_H^{-1}M(n+2) \overset{\overline{S}_H^{-1}f^k(n+2)}{\longrightarrow} \overline{S}_H^{-1}N(n+2) \overset{\overline{S}_H^{-1}g^k(n+2)}{\longrightarrow} \overline{S}_H^{-1}L(n+2) \overset{\overline{S}_H^{-1}\delta_n}{\longrightarrow} \overline{S}_H^{-1}M(n+1) \cdots$$

$$\cdots \overline{S}_H^{-1}\otimes M(n+2) \overset{\overline{S}_H^{-1}\otimes f^k(n+2)}{\longrightarrow} \overline{S}_H^{-1}\otimes N(n+2) \overset{\overline{S}_H^{-1}\otimes g^k(n+2)}{\longrightarrow} \overline{S}_H^{-1}\otimes L(n+2) \overset{\overline{S}_H^{-1}\otimes\delta_n}{\longrightarrow} \overline{S}_H^{-1}\otimes M(n+1) \cdots$$

Proof. it is sufficient to note that $\overline{S}_H$ is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A and according to proposition 18

Proposition 19. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded duo-ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *a graded left* $A-$*module and* S_H *be the part formed of regulars homogeneous elements of* A*, then for all* $n \in \mathbb{Z}$

$$\overline{S}_H^{-1}(H_n(M_*)) \cong \overline{S}_H^{-1}(A) \bigotimes M(n+2).$$

Moreover $\overline{S}_H^{-1}(H_n(M_*)) \cong H_n((\overline{S}_H^{-1}(M)_*)$.

Proof. We have $H_n(M_*) \cong M(n+2)$ and as $\overline{S}_H$ is a multiplicatively closed subset satisfying the left conditions of Ore formed of homogeneous elements of A then $\overline{S}_H^{-1}(H_n(M_*)) \cong \overline{S}_H^{-1}(M(n+2))$ or $\overline{S}_H^{-1}A \bigotimes M(n) \cong \overline{S}_H^{-1}M(n)$ so

$$\overline{S}_H^{-1}(H_n(M_*)) \cong \overline{S}_H^{-1}A \bigotimes (M(n+2)).$$

On other hand $H_n((\overline{S}_H^{-1}(M))_*) \cong (\overline{S}_H^{-1}(M))(n+2) \cong \overline{S}_H^{-1}(M(n+2)) \cong \overline{S}_H^{-1}(H_n(M_*))$ Thus

$$\overline{S}_H^{-1}(H_n(M_*)) \cong H_n((\overline{S}_H^{-1}(M))_*).$$

Corollary 12. *Let* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded duo-ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and be a graded left* $A-$*modules and* S_H *be the set of all regular homogeneous of* A*, then for all* $n \in \mathbb{Z}$

$$S_H^{-1}(H_n(M_*)) \cong S_H^{-1}(A) \bigotimes M(n+2).$$

Moreover $S_H^{-1}(H_n(M_*)) \cong H_n((S_H^{-1}(M)_*)$.

Proof. it is sufficient to note that $S_H = \overline{S}_H$.

5. Conclusion

In this article, we study the localization in the category $COMP(G_r(A - Mod))$ and we used the localization in the category $G_r(A - Mod)$, and we proof that for all $n \in \mathbb{Z}$ fixed and for all $M \in G_r(A - Mod)$ we have:

$$\overline{S}_H^{-1}((H_n \circ C)(M)) \cong H_n(C_H \circ \overline{S}_H^{-1})(M)).$$

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