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Computational Experience with Modified Coefficients Conjugate Gradient for Image Restoration

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Abstract. The Conjugate Gradient Method is a numerical optimization technique that finds the optimal solution by focusing on the coefficient conjugate. This paper presents a new coefficients conjugate gradient method for removing impulse noise from images, which is based on a quadratic function and is proven to be globally convergent. Results show that it is an effective method for image restoration.

2020 Mathematics Subject Classifications: 65K10, 49M37, 90C06
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1. Introduction

This work presents a class of iterative methods for optimization problems where the objective function is an edge-preserving regularization (EPR) functional. The goal is to find the optimal solution to the problem. Impulse noise can be identified and filtered using an adaptive median filter (AMF) equation. The true image is denoted by X and the index set of X is denoted by $A = 1, 2, 3, ..., M \times 1, 2, 3, ..., N$. The sets $N \subset A$ and $P_{i,j}$ represent the indices of noise pixels detected in the first phase and the four closest neighbors of the pixel at position $(i, j) \in A$, respectively. Given a pixel value $y_{i,j}$ at position $(i, j), u_{i,j} = [u_{i,j}]_{(i,j)\in N}$ is a vector of length c that is ordered lexicographically. the number of elements in a set N, represented by the letter c. Here c is the number of elements of N. A two-step process for recovering noise pixels. The first step involves minimizing a functional, and the second step involves minimizing a different functional to recover the noise pixels.

$$f_a(u) = \sum_{(i,j)\in N} \left[|u_{i,j} - y_{i,j}| + \frac{\beta}{2}(2 \times S^1_{i,j} + S^2_{i,j}) \right] \tag{1}$$

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where, β is the regularization parameter, and: $S^1_{i,j} = 2\sum_{(m,n)\in P_{i,j}\cap N^c} \varphi_\alpha(u_{i,j} - y_{m,n})$, $S^2_{i,j} = \sum_{(m,n)\in P_{i,j}N} \varphi_\alpha(u_{i,j} - y_{m,n})$ and the $\varphi_\alpha = \sqrt{a + x^2}, \alpha > 0$ edge-preserving potential function is an example of a potential function that preserves edges in an image. It is used to reduce noise and enhance the edges of an image, making it easier to identify features. Optimization methods can be used to minimize a smooth edge-preserving regularization (EPR) functional. It is noted that the non-smooth data-fitting term is not needed in the second phase, where only noisy pixels are restored:

$$f_a(u) = \sum_{(i,j)\in N} \left[(2 \times S^1_{i,j} + S^2_{i,j}\right] \tag{2}$$

Referring to [14] of additional information.

Conjugate gradient method is commonly used in image restoration, where it is used to minimize a given objective function:

$$Minf(x), u \in R^n \tag{3}$$

where a function is continuously differentiable. The Conjugate Gradient Method works by taking a starting point and then iteratively improving the solution:

$$u_{k+1} = u_k + \alpha_k d_k \tag{4}$$

until it converges to the optimal solution, a search direction d_k and exact step size α_k can be used to optimize the quadratic case. These two elements are essential for achieving the best possible results:

$$\alpha_k = \frac{-g_k^T d_k}{d_k^T Q d_k}. \tag{5}$$

In order to solve for general nonlinear functions, an iterative procedure must be used [11]. The Wolfe conditions are a set of criteria used to determine the step length in a given work:

$$f(u_k + \alpha_k d_k) \leq f(u_k) + \delta \alpha_k g_k^T d_k \tag{6}$$

$$d_k^T g(u_k + a_k d_k) \geq \sigma d_k^T g_k \tag{7}$$

where $0 < \delta < \sigma < 1$. Additional information. The following is the search direction equation:

$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{8}$$

where the coefficient β_k is selected in such a way that and must satisfy the conjugacy property. Fletcher and Reeves (FR) [2] and Dai and Yuan (DY) [1] are two well-known β_k that are thought to be the most effective approaches, as stated by:

$$\beta_k^{FR} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}, \beta_k^{DY} = \frac{g_{k+1}^T g_{k+1}}{d_k^T y_k} \tag{9}$$

Alternatively, by other equations (see, for example, [3],[10], [12], [13]. Some computer-generated imagery conjugate gradient approaches are numerically efficient, while others are conceptually efficient. All of these strategies are equivalent if the objective function is strictly convex quadratic. When applied to generic non-quadratic functions, they behave differently, as shown in [4]. The conjugate gradient approach was later extended to larger unconstrained optimization problems in formula [2]. Conjugate gradient approaches are now recognized as being particularly useful for dealing with large-scale unconstrained optimization problems since they do not require the storing of matrices. Many conjugate gradient techniques have lately been proposed that offer both strong global convergence properties and good numerical performance. These approaches are based on the conjugacy requirement of multiple conjugate gradient parameters see, [7], [9], [8], [6]. Previous conjugate gradient algorithms made advantage of the conjugacy requirement:

$$d_{k+1}^T Q d_k = 0. \tag{10}$$

this is essential for the mathematical experiment and the convergence analysis [8].

Based on the quadratic function, a novel conjugate gradient method was developed, rationalized, and studied. Numerical studies demonstrated that the suggested technique outperformed other conjugate gradient methods.

2. Modification of Coefficients Conjugate Gradient

We utilizes the quadratic model to derivation the new coefficients conjugate gradient:

$$f_{k+1} = f_k + s_k^T g_k + \frac{1}{2} s_k^T Q(u_k) s_k \tag{11}$$

where $Q(u_k)$ is the Hessian matrix. Taking the derivative of (11) on both sides for s_k, we get:

$$\nabla f_{k+1} = g_k + Q(u_k) s_k \tag{12}$$

By combining 5 and 12 in 11, we get:

$$s_k^T Q(u_k) s_k = f_k - f_{k+1} + \frac{1}{2} s_k^T y_k \tag{13}$$

We may express the equation 13 using 3 as follows:

$$s_k^T Q(x_k) s_k = 2 \frac{(g_k^T s_k)^2}{(s_k^T y_k + 2(f_k - f_{k+1}))}. \tag{14}$$

The conjugacy condition, yields:

$$d_{k+1}^T Q s_k = 0 \tag{15}$$

From 14 through 15, and after some algebra, we get:

$$\beta_k = \frac{2 \frac{(g_k^T s_k)^2}{s_k^T y_k (s_k^T y_k + 2(f_k - f_{k+1}))} g_{k+1}^T y_k}{d_k^T y_k} \tag{16}$$

If exact line search is utilized in 16, then β_k is such that:

$$\beta_k^{BE1} = \frac{2\frac{(g_k^T s_k)^2}{s_k^T y_k (s_k^T y_k + 2(f_k - f_{k+1}))} \|g_{k+1}\|^2}{d_k^T y_k} \tag{17}$$

and

$$\beta_k^{BE2} = \frac{2\frac{(g_k^T s_k)^2}{s_k^T g_k (2(f_k - f_{k+1}) - s_k^T g_k)} \|g_{k+1}\|^2}{d_k^T g_k} \tag{18}$$

By the above modification, it is not difficult to see that:

$$\beta_k^{BE3} = \frac{2\frac{(g_k^T s_k)^2}{a_k \|g_k\|^2 (a_k \|g_k\|^2 + 2(f_k - f_{k+1}))} \|g_{k+1}\|^2}{\|g_k\|^2} \tag{19}$$

from 17, 18 and 19 we define new formula denote by BE1, BE2 and BE3. Inspired by the above direction, we refer to the proposed algorithm as:

Algorithm (2.1): ***(BE1,BE2 and BE3)***.

Stage 1. Choose the first starting location u_1. Set $d_1 = -g_1$.
Stage 2. if $\|g_k\| = 0$ stop.
Stage 3. State Determine if α_k meets the requirements 6 and 7.
Stage 4. Compute β_k as given in formula (2.17-2.19).
Stage 5. Create the search direction as $d_{k+1} = -g_{k+1} + \beta_k d_k$.
Stage 6. Set $k = k + 1$ and go to Stage 2.

3. Convergence analysis

To give the convergence result, the following assumptions are given. Assumptions:

In an open convex set Ψ that contains the level set $\Psi = x \in R^n : f(x) \leq f(x_0)$, where x_0 is given, f is differentiable and its gradient is Lipschitz continuous, namely, there exists a constant $L > 0$ such that:

$$\|g(z) - g(u)\| = L \|z - u\|, \forall z, u \in R^n. \tag{20}$$

With a constant ¥ ≥ 0 such that $\|\nabla f(x)\| =$ ¥ , see [9].

The following important result was obtained by Zoutendijk [16].

Lemma 1. *Any iteration method in which a_k is obtained by the Wolfe line search and the Assumptions holds. Then:*

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty. \tag{21}$$

Theorem 1. *Let the direction d_{k+1} be yielded by the new Algorithms. Then, we obtained:*

$$d_{k+1}^T g_{k+1} < 0 \text{ and } d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k. \tag{22}$$

Proof.
Using 8, it holds that:

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \beta_k d_k^T g_{k+1} \tag{23}$$

On 17 and after some algebra another is propose as:

$$\|g_{k+1}\|^2 = \frac{\beta_k^{BE1} d_k^T y_k}{2\frac{(g_k^T s_k)^2}{s_k^T y_k (s_k^T y_k + 2(f_k - f_{k+1}))}} \tag{24}$$

Applying 24 on 23 will lead to the following:

$$d_{k+1}^T g_{k+1} = \beta_k^{BE1}[d_k^T g_{k+1} - d_k^T y_k] = \beta_k^{BE1} d_k^T g_k < 0 \tag{25}$$

This inequality implies 22 is satisfied for $k+1$.

Theorem 2. *Suppose that Assumptions hold. Let $\{u_k\}$ be generated by new Algorithm. Then one has:*

$$\lim_{k\to\infty} inf|g_k| = 0. \tag{26}$$

Proof. Similar to the proof done in Hassan [6].

4. Computational Experience

After that, we do numerical experiments to evaluate Algorithm (2.1) and compare the performance of the FR technique to salt-and-pepper impulse noise reduction. Table 1 shows the techniques used to evaluate photographs: placeLena, Home, Cameraman, and Elaine. To assess restoration success qualitatively, we use the PSNR (peak signal to noise ratio):

$$PSNR = 10log_{10}\frac{255^2}{\frac{1}{MN}\sum_{i,j}(u_{i,j}^r - u_{i,j}^*)^2} \tag{27}$$

where $u_{i,j}^r$ and $u_{i,j}^*$ represent the restored and original image pixel values.
Both approaches' stopping criteria are:

$$\frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} \le 10^{-4} \; and \; \|f(u_k)\| = 10^{-4}(1 + |f(u_k)|) \tag{28}$$

For more details see [5],[15].
Table (1) displays the PSNR, total number of iterates (NI), and function evaluation (NF).

Table 1: Numerical results of FR, BE1, BE2 and BE3 algorithms.

Image	Noise level r (%)	FR-Method			BE1-Method			BE2-Method			BE3-Method		
		NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)
Le	50	82	153	30.5529	33	70	30.6475	49	61	30.4516	61	65	30.5955
	70	81	155	27.4824	49	98	27.0736	67	81	27.4784	74	80	27.3493
	90	108	211	22.8583	87	173	22.8978	69	84	22.8632	71	102	22.7517
Ho	50	52	53	30.6845	26	52	34.6636	38	47	35.1291	41	44	34.4129
	70	63	116	31.2564	39	77	31.1108	33	39	31.0401	54	58	31.0637
	90	111	214	25.287	80	159	24.8605	63	74	24.8948	71	78	24.847
El	50	35	36	33.9129	26	52	33.9227	32	38	33.8945	36	37	33.9154
	70	38	39	31.864	32	61	31.8809	34	40	31.7373	41	42	31.7657
	90	65	114	28.2019	52	101	28.1536	53	60	28.1151	52	54	28.3089
c512	50	59	87	35.5359	29	60	35.3234	40	51	35.3997	36	48	35.4794
	70	78	142	30.6259	48	97	30.787	52	62	30.664	46	55	30.6654
	90	121	236	24.3962	72	145	24.0228	63	75	24.920	70	77	24.8121

Table (1), shows that the suggested algorithms beat the FR technique in terms of iterations and function evaluations, as well as peak signal to noise ratio.

Lena image

Figure 1: Comparing images results of algorithms: (a) denoised images with 70% salt-and-pepper noise, (b) recovered images through FR, (c), (d) and (e) restored images using BE1, BE2 and BE3 of 256×256.

House image

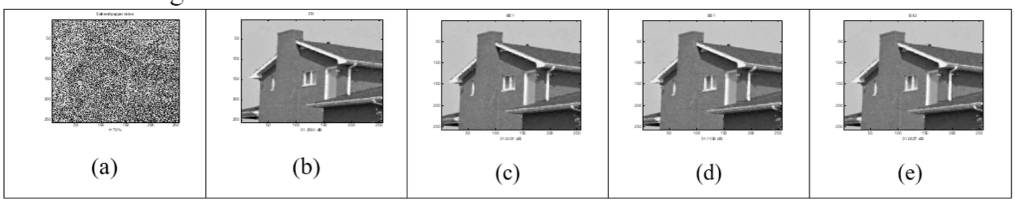

Figure 2: Comparing images results of algorithms: (a) denoised images with 70% salt-and-pepper noise, (b) recovered images through FR, (c), (d) and (e) restored images using BE1, BE2 and BE3 of 256×256.

Elaine image

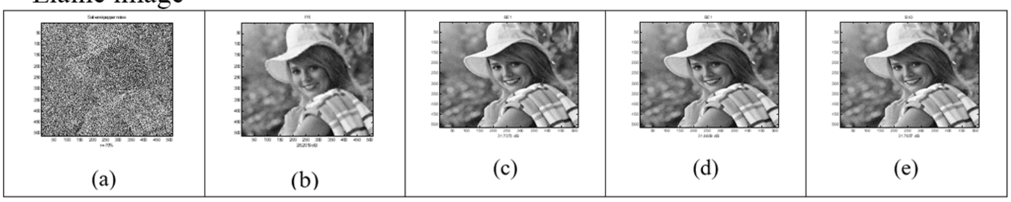

Figure 3: Comparing images results of algorithms: (a) denoised images with 70% salt-and-pepper noise, (b) recovered images through FR, (c), (d) and (e) restored images using BE1, BE2 and BE3 of 256×256.

Cameraman image

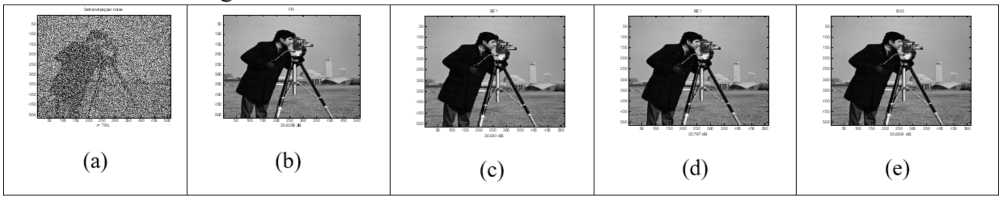

Figure 4: Comparing images results of algorithms: (a) denoised images with 70% salt-and-pepper noise, (b) recovered images through FR, (c), (d) and (e) restored images using BE1, BE2 and BE3 of 256×256.

5. Conclusions

In this study, we introduce a parameter conjugate gradient strategy based on the quadratic function for picture restoration problems, and we analyze its global convergence under various mild circumstances. Our solution is promising and practical, as demonstrated by numerical tests for restricted picture restoration challenges.

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