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Nonabelian Case of Hopf Galois Structures on Nonnormal Extensions of Degree *pqw*

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Abstract. We look at Hopf Galois structures with square free pqw degree on separable field extensions (nonnormal) L/K. Where E/K is the normal closure of L/K, the group permutation of degree pqw is $G = Gal(E/K)$. We study details of the nonabelian case, where $J_l = \langle \sigma, [\tau, \alpha^l] \rangle$ is a nonabelian regular subgroup of $Hol(N)$ for $1 \leq l \leq w - 1$. We first find the group permutation G, and then the Hopf Galois structures for each G. In this case, there exists four G such that the Hopf Galois structures are admissible within the field extensions L/K.

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1. Introduction

Chase and Sweedler [7] proposed the Hopf Galois theory by investigating inseparable field extensions. Their work marks the start of a slew of new problems about separable field extensions (SFEs). In [14] Greither and Pareigis showed that an SFE can generate a large number of Hopf Galois structures (HGSs), and HGSs can be used by the group theoretic in issues.

If the field extension L/K is normal and separable with degree n, then the Galois extension L/K is classical Galois. Let its Galois group be $G = Gal(L/K)$. The group algebra $K[G]$ then operates on L/K, yielding at least one HGS. On the other hand, there could be a slew of more HGSs on L/K. We have L as Hopf algebras $L \otimes_K H \cong L[N]$ for every group N of order n if the K Hopf algebra H generates one of these HGSs on L/K. We have the type of HGS by the isomorphism type of the group N. The group G determinates the different types of HGS as well as the number of each type.

Consider L/K to be an SFE (presumably nonnormal) of degree n in general. Let the normal closure of L/K is F/K, while the Galois groups of F/K and F/L are $G = Gal(F/K)$

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and $G^{'} = Gal(F/L)$, respectively. In each type the number of HGSs is determined by the group G and its subgroup $G^{'}$. The primary finding of Greither and Pareigis [14] is that HGSs on L/K are congruent to order n groups and act transitively as a permutation group (PG) on the space of left coset $X = G/G^{'}$.

Many authors have studied HGSs since Greither and Pareigis' efforts on the subject. The majority of them are interested in Galois extensions on various forms of SFEs; see [11, 16, 19]. Other authors, such as [9, 12, 13] deal with nonnormal extensions. The study of HGSs on Galois extensions has been more well-known in recent years due to a link between studying HGSs and the solutions of the Yang Baxter equation as set theoretic (skew braces and braces), for more information, see [2, 18].

Byott shows in [4] that there exists a CG of order pq and a nonabelian group of the same order such that a Galois extension of degree pq allows the number of HGSs, based on the condition $p \equiv 1 \pmod{q}$ where p and q are distinct primes. A Galois extension with kinds of groups admits the cyclic and nonabelian HGSs. Furthermore, when the degree is $2pq$ with odd primes p, q and $p = 2q + 1$ (p is safe prime and q is Sophie Germain prime), there is research in numerous resources [5, 10, 17]. The HGSs on L/K of type N with an arbitrary square free degree n are described in [1]. The HGSs were enumerated by dividing the order n into two groups, G and N, with $G = Gal(L/K)$.

Byott and Lyons has shown in their paper [6] that the conclusions of [1] may extend to nonnormal but SFEs L/K of square free degree $n = pq$ ($p = 2q + 1$ is a safe prime and $q \geq 3$ is a Sophie Germain prime). There is at least one cyclic and nonabelian HGS for the PGs admitted by the corresponding field extensions L/K. The issue in [6] then becomes whether the same behaviour applies for square free degrees n in general.

The primary purpose of this research is to answer the question and extend the approach in [6] for $n = pqw$, where a Sophie Germain prime $w \geq 3$, a safe prime $p = 2w+1$, and p, q, w are square free primes. We start with the potential group J_l of order pqw in this work, and then look for PGs that are released by HGSs of type J_l. Where J_l is the nonabelian group. We then enumerate all HGSs of cyclic type on J_l -extension and identify all isomorphism types of PGs of degree pqw.

Now we can show the first of our main results.

Theorem 1. *The total number of isomorphism types admits nonabelian HGSs is* $2 + (r + 1) + \sigma_0(s)$ *of PGs of degree pqw. The nonabelain of order pqw is the regular group.*

The second of our main results shows the total isomorphism types that realise by cyclic and nonabelian group.

Theorem 2. *A HGS of cyclic type can realise isomorphism types in total* $12(r + i + 1)[\sigma_0(s) + \sigma_1(j) + \sigma_0(s)\sigma_1(j)] + 2 + (r + 1) + \sigma_0(s)$ *of PGs G of degree pqw of both cases regular groups (where the Galois extensions have 1 HGS for the cyclic group and* $(p - 1)(q - 1) + 1, 1, r + 1,$ *and* $\sigma_0(s)$ *HGSs for the nonabelian group of the cyclic type).*

2. Materials and Methods

In this section, we review the fundamental facts and concepts related to HGSs, as well as the relationship between them and PGs. Using the method provided in [3], we demonstrate how to count HGSs. We recommend [[8], Chapter 2] to the reader for further information on counting HGSs.

A PG is defined as a finite group G with a one to one homomorphism ρ from G into the PG of a finite set X ($\rho : G \to Perm(X)$). let $y \in X, h \in G$ then we express $\rho(h)(x) = h.y$. The degree of G is the order of X. If there is a unique $h \in G$ (respectively, some $h \in G$) with $h.y = x$ for each $x, y \in X$, then G is regular (respectively, transitive) on X. We assume all PGs to be transitive groups in our study.

We define the subgroup $G_y = \{h \in G : h.y = y\}$ as the stabilizer of $y \in X$, so the stabilizer of $h.y$ is referred to be as hG_yh^{-1}. The core $\cap_{h\in G} hG_yh^{-1}$ of G_y in G is simple as a result of the fact that X affects transitively by G and G embedded in $Perm(X)$. Additionally, by the left multiplication action $\mu : G \to Perm(G/G_y)$, G acts as a PG on $G/G_y = \{hG_y : h \in G\}$ the set of left cosets, where $\mu(h)(h^{'}G_y) = (hh^{'})G_y$. As a result, the left translation on $G/G^{'}$ acts up to isomorphism on the abstract group G as a PG of degree n, with the subgroup $G^{'}$ having a trivial core with index n. We utilize the automorphism definition.

If $Aut(G, G^{'})$ is defined as

$$Aut(G, G^{'}) = \{\phi \in Aut(G) : \phi(G^{'}) = G^{'}\}.$$

Thus it is clear that $Aut(G, G^{'})$ forms a PG of automorphisms ϕ of G such that ϕ fixes the left coset $1_G G^{'}$ of $G/G^{'}$ (1_G is the identity of G).

Let we have a finite SFE L/K of degree n with a fixed algebraic closure F as normal closure in K^c of K. If the group $G = Gal(F/K)$ and the group $G^{'} = Gal(F/L)$, then the map $\mu : G \to Perm(X)$ is an embedding. $G^{'}$acts as a stabilizer for the inclusion $L \hookrightarrow F$, and the embeddings of K linear of L into K^c or F is acted transitively by G.

Consider the cocommutative K Hopf algebra H. Let $\epsilon : H \to K$ be the counit map for K and $\nu : H \to H \otimes_K H$ be the comultiplication map for $\nu(\xi) = \sum_{(\xi)} \xi_{(1)} \otimes \xi_{(2)}$. If we have $\xi(ab) = \sum_{(\xi)} \xi_{(1)}(a)\xi_{(2)}(b)$ for $\xi \in H$ and $a, b \in L$, and $\xi(k) = \epsilon(\xi)k$ for all $\xi \in H$ and $k \in K$, L is said to have H module algebra.

In addition, if $\phi : L \otimes_K H \to End_K(L)$ described as the K module homomorphism by $\phi(a \otimes \xi)(b) = a\xi(b)$ is an isomorphism, we say that L/K is a H Galois extension or that H yields an HGS on L/K.

The PG G is necessary to obtain the HGSs on L/K. Greither and Pareigis' discovery is that the left translation group $\mu(G)$ normalizes the regular subgroups N of $Perm(X)$ that are isomorphic to the HGSs on L/K.

The Hopf algebra of K for each such group N, $H = F[N]^G$ acts on L via Galois descent, where $F[G]$ is acted by G as an automorphisms field of F and conjugates on N via μ. The HGS type is also known as the N isomorphism type. If $N = C$, where C is the normal complement of the subgroup $G^{'}$ of G, we get an HGS. The classical HGSs on L/K is then admitted by $F[N]^G$.

If the isomorphism $\phi : G \to Gal(F/K)$ with $\phi(G^{'}) = Gal(F/L)$ exists, we have G is realised by an SFE L/K. The point stabilizer is $G^{'}$, and the PG is G. In addition, we state that type N of an HGS realises G if it is admitted by L/K.

We count the type N of the number of HGSs as a group of order n such that G is realised by a SFE L/K, which corresponds to the number of regular subgroups N^{*} normalised by $\mu(G)$ and isomorphic to N of $Perm(X)$, according to Greither and Pareigis conclusions. Let $Hol(N) = N \rtimes Aut(N)$ be the holomorph of N. As a result, the number of HGSs on L/K can be determined using Byott's result in [3] and the formula

$$f(G, N) = \frac{|Aut(G, G^{'})|}{|Aut(N)|} f^{'}(G, N), \tag{1}$$

we denote $f^{'}(G, N)$ as the number of regular subgroups B with transitive on N of $Hol(N)$ and $B \cong G$ by an isomorphism with the stabilizer $B^{'}$of 1_N in B to $G^{'}$. If HGS of type N realises G, then $G \cong B$ of $Hol(N)$.

Because the preceding approach deals with $Hol(N)$ instead of the $Perm(X)$ group, counting HGSs is made easy.

We write the elements of $Hol(N)$ by $[x, \alpha]$ where $x \in N$ and $\alpha \in Aut(N)$. Thus $Hol(N)$ acts on N as permutations by $[x, \alpha].y = x\alpha(y)$. Then, in $Hol(N)$ the normal subgroup N specifies $\mu(N)$ of the left translations, and the stabilizer of 1_N creates the subgroup $Aut(N)$. In $Hol(N)$, the multiplication is defined as follows

$$[x, \alpha][y, \beta] = [x\alpha(y), \alpha\beta].$$

We commonly refer to x and α instead of $[x, i_N]$ and $[1_N, \alpha]$ the elements of $Hol(N)$, respectively. For example, we have the identification $\alpha x = \alpha(x)\alpha$.

Now we have some general results from [6] about holomorphs, the group N and $Aut(N)$.

Proposition 1. *Assume that N and $Aut(N)$ are abelian group and abelian automorphism respectively. Suppose the two subgroups of $Hol(N)$, $B = N \rtimes A$ and $B^{'} = N \rtimes A^{'}$ such that $A, A^{'}$ two subgroups of $Aut(N)$. Let $\psi : B \to B^{'}$ be an isomorphism such that $\psi(N) = N$, therefore $B = B^{'}$.*

Proposition 2. *Suppose that N is a group such that A is a subgroup of $Aut(N)$. Suppose that the subgroup of $Hol(N)$ is $B = N \rtimes A$ such that N is characteristic in B. Then the normalizer of A in automorphism of N is isomorphic to the group $Aut(B, A) := \{\phi \in Aut(B) : \phi(A) = A\}$. In particular, the group $Aut(B, A)$ is isomorphic to $Aut(N)$ if the group $Aut(N)$ is abelian.*

The next result from [15] shows the total number of PGs which admit HGS of cyclic case of degree pqw.

Theorem 3. *The total number of PGs G which admits HGS of cyclic case is $12(r + i + 1)[\sigma_0(s) + \sigma_1(j) + \sigma_0(s)\sigma_1(j)]$ of isomorphism types of degree pqw. The regular group is the cyclic of order pqw. Any field extension L/K admits a unique HGS of cyclic case G and is essentially classically Galois and for all groups G.*

More general, since any Sylow subgroup is cyclic. Hence, the square free groups of order n can exist and classify.

3. Results

We will concentrate the rest of the work on HGSs on SFEs of degree pqw with $p = 2w+1, q$ and w are odd primes of square free. As a result, w and p are Sophie Germain prime and safe prime, respectively. We have $w-1 = 2^i j, q-1 = 2^r s$ with $i, r \geq 1$ and s, j are odd numbers. We write $gcd(j, 2pw) = 1$ and $gcd(s, 2pq) = 1$. However, we have no more presumptions regarding the prime factors of j and s. There are six groups N of order pqw up to isomorphism, but in this work we deal with the CG C_{pqw} and precisely the nonabelian case. As a result, the transitive subgroups of $Hol(C_{pqw})$ must be determined. Assume that N is a CG of order pqw with the following form

$$N = \langle \sigma, \tau : \sigma^e = \tau^w = 1, \tau\sigma = \sigma\tau \rangle, \text{ where } e = pq.$$

We write $Aut(N) \cong Aut(\langle\sigma\rangle) \times Aut(\langle\tau\rangle)$, since we have the two characteristic subgroups $\langle\sigma\rangle$ and $\langle\tau\rangle$ in N, where $Aut(\langle\sigma\rangle)$ of order $(p-1)(q-1) = 2w2^r s$ and $Aut(\langle\tau\rangle)$ of order $w-1 = 2^i j$ are cyclic. Suppose that $\alpha, \beta, \gamma, \delta$ are automorphisms of the group N of order $w, 2, 2^r, s$ respectively that make τ fix, and assume that η, θ are automorphisms of the group N of order $2^i, j$ respectively that fix σ. The direct product $\langle\alpha\rangle\langle\beta,\gamma,\eta\rangle\langle\delta,\theta\rangle$ is decomposed by $Aut(N)$, where the factors have coprime orders $w, 2^{(r+i+1)}$ and sj respectively. A subgroup of $Aut(N)$ decomposes into one subgroup from each of these factors as a direct product. The number of divisors in s is $\sigma_0(s), \sigma_1(j)$ in j and $\sigma_0(s)\sigma_1(j)$ in sj.

Proposition 3. *Let $J_l = \langle\sigma, [\tau, \alpha^l]\rangle$ with $1 \leq l \leq w-1$. Then J_l is a nonabelian regular subgroup of $Hol(N)$. In addition, Table 1 shows the transitive subgroups G for the nonabelian group J_l of $Hol(N)$.*

Proof. It is clear that J_l is nonabelian and regular of order pqw on N, since we discover that $[\tau, \alpha^l]$ has order w (since α fixes τ) and $[\tau, \alpha^l]\sigma = \alpha^l(\sigma)[\tau, \alpha^l]$ in J_l. Given that holomorph of the N has a subgroup $Z = \langle\sigma, \tau, \alpha\rangle$ of order pqw^2 with index $2^i j$ relatively prime to pqw uniquely. So pqw divides the order of any transitive subgroup B, hence $B \cap Z$ must be transitive on N. As a result, either $B \cap Z$ is regular on N or $B \subset Z$. Now, there is one nonregular subgroup $\langle\sigma, \alpha\rangle$ in Z and the other subgroups of order pqw are N and J_l. For some l, we have $B \cap Z = Z$ or N or J_l. That means each individual transitive subgroup B has either N or some J_l. So any subgroup of $Aut(N)$ can be used to create a transitive subgroup B, since in $Hol(N)$ the group N is normal . If $\psi \in Aut(N)$ and $\psi(\tau) \neq \tau$, we obtain $\psi[\tau, \alpha^l]\psi^{-1} = [\psi(\tau), \alpha^l] \notin J_l$, so in $Aut(N)$ the group $\langle\alpha, \beta, \gamma^{2^{r-c_1}}, \delta^{s/d}\rangle$ is the normalizer of J_l. As a result, any transitive subgroup B contains J_l, has the forms $J_l, J_l \rtimes \langle\beta\rangle, J_l \rtimes \langle\gamma^{2^{r-c_1}}\rangle$ or $J_l \rtimes \langle\delta^{s/d}\rangle$. Therefore, the Table 1 of transitive subgroups is derived from $Aut(N)$ subgroups.

Lemma 1. *Table 1 shows that there are $w-1$ groups in case 1 and 2 which are PGs and isomorphic. But, in cases 3 and 4 there are $(w-1)(r+1)$ and $(w-1)\sigma_0(s)$ groups respectively which are isomorphic as PGs.*

Proof. Let $1 \leq l \leq w-1$ and $\psi \in Aut(N)$ with $\psi(\tau) = \tau^l$ be the two variables. Consequently, $\psi[\tau, \alpha^l]\psi^{-1} = [\psi(\tau), \alpha^l] = [\tau, \alpha]^l$. In addition, $\psi\beta\psi^{-1} = \beta$. As a result, conjugating by ψ yields the isomorphism $J_l \rtimes \langle\beta\rangle \to J_1 \rtimes \beta$, which is a PG isomorphism because the stabilizer $\langle\beta\rangle$ of 1_N is fixed. It also determines to $J_l \to J_1$ isomorphism. As a result, in case (2) all the groups are PGs as are isomorphic, and simultaneously (1). Let $1 \leq l \leq w-1, 0 \leq c_1 \leq r$ so $\psi\gamma^{2^{r-c_1}}\psi^{-1} = \gamma^{2^{r-c_1}}$ and then conjugating by ψ obtains the isomorphism $J_l \rtimes \langle\gamma^{2^{r-c_1}}\rangle \to J_1 \rtimes \gamma^{2^{r-c_1}}$. Finally, let $1 \leq l \leq w-1, d \mid s$ so $\psi\delta^{s/d}\psi^{-1} = \delta^{s/d}$ and then conjugating by ψ obtains the isomorphism $J_l \rtimes \langle\delta^{s/d}\rangle \to J_1 \rtimes \delta^{s/d}$. As a result, all the groups in case (3) and case (4) are PGs as are isomorphic.

Table 1: The transitive subgroups for the nonabelian group J_l.

Key	Order	Parameters	# Groups	Groups
1	pqw	$1 \leq l \leq w-1$	$w-1$	J_l
2	$2pqw$	$1 \leq l \leq w-1$	$w-1$	$J_l \rtimes \langle\beta\rangle$
3	$2^{c_1}pqw$	$1 \leq l \leq w-1, 0 \leq c_1 \leq r$	$(w-1)(r+1)$	$J_l \rtimes \langle\gamma^{2^{r-c_1}}\rangle$
4	$pqwd$	$1 \leq l \leq w-1, d \mid s$	$(w-1)\sigma_0(s)$	$J_l \rtimes \langle\delta^{s/d}\rangle$

Lemma 2. *The number of HGSs for the nonabelian group J_l is as in Table 3.*

Proof. In the cases are shown in Table 3, the stabilizer of 1_N in B is $B' = B \cap Aut(N)$. We start with case 1, a single isomorphism class is formed by the $w-1$ regular groups J_l. The automorphism ψ of J_l must induce an automorphism of the characteristic subgroup C_{pqw} in order for ψ to be compatible with the relation $\tau\sigma = \sigma^u\tau, u > 1$, therefore $\psi(\sigma) = \sigma^a$ and $\psi(\tau) = \sigma^b\tau^c$ with $1 \leq a \leq (q-1)(p-1), 0 \leq b \leq (q-1)(p-1)$, and $1 \leq c \leq (w-1)$ are required $c = 1$.
Thus $\mid Aut(J_l) \mid= (p-1)(q-1)[(p-1)(q-1)+1]$, and we have $\mid B' \mid= 1$ for $B = J_l$. As a result, the number of cyclic HGSs on a J_l-extension by using Byott's formula (1) is $(p-1)(q-1)+1$.
We assume in case 2 that $B = J_l \rtimes \langle\beta\rangle$ with J_l instead of N and $B' = \langle\beta\rangle$, we use Proposition 2. Conjugating by β fixes the generator $F = [\tau, \alpha^l]$ of order w by inverting σ. If $\psi \in Aut(J_l)$, we have $\psi(\sigma) = \sigma^a$ and $\psi(F) = \sigma^b F$ for $1 \leq a \leq (q-1)(p-1) \leq$ and $0 \leq b \leq (q-1)(p-1)$ respectively.
Then, $b = 0$ if and only if ψ normalizes B' in $Aut(J_l)$. As a result, $\mid Aut(B, B') \mid= (p-1)(q-1)$, and the $w-1$ conjugate subgroups yield that the number of HGS is 1.

Table 2: The structures of transitive subgroups for J_l.

Key	Restrictions	Order	Structure
1		pqw	$C_{pq} \rtimes C_w$
2		$2pqw$	$C_{pq} \rtimes C_{2w}$
3	$c_1 \neq (0,1), c_1 = 0, c_1 = 1$	$2^{c_1}pqw, pqw, 2pqw$	$C_{pq} \rtimes C_{2^{c_1}w}, C_{pq} \rtimes C_w, C_{pq} \rtimes C_{2w}$
4	$d \neq 1, d = 1$	$pqwd, pqw$	$C_{pq} \rtimes C_{wd}, C_{pq} \rtimes C_w$

Table 3: The number of HGSs for the nonabelian group J_l.

Key	Order	$\mid Aut(B, B') \mid$	# iso. class	# HGS per iso. class
1	pqw	$(p-1)(q-1)[(p-1)(q-1)+1]$	1	$(p-1)(q-1)+1$
2	$2pqw$	$(p-1)(q-1)$	1	1
3	$2^{c_1}pqw$	$(p-1)(q-1)$	$r+1$	$r+1$
4	$pqwd$	$(p-1)(q-1)$	$\sigma_0(s)$	$\sigma_0(s)$

We assume in cases 3 and 4 that $B = J_l \rtimes \langle \gamma^{2^{r-c_1}} \rangle$ and $B = J_l \rtimes \langle \delta^{s/d} \rangle$ respectively with J_l instead of $N, B' = \langle \gamma^{2^{r-c_1}} \rangle$ and $B' = \langle \delta^{s/d} \rangle$ respectively, we also use Proposition 2. Conjugating by $\langle \gamma^{2^{r-c_1}} \rangle$ in case 3 and $\langle \delta^{s/d} \rangle$ in case 4 fixes the generator $F = [\tau, \alpha^l]$ of order w by reversing σ. If $\psi \in Aut(J_l)$, we have $\psi(\sigma) = \sigma^a$ and $\psi(F) = \sigma^b F$ for $1 \leq a \leq (q-1)(p-1)$ and $0 \leq b \leq (q-1)(p-1)$ respectively. Then, $b = 0$ if and only if ψ normalize B' in $Aut(J_l)$. As a result, in case 3 and case 4 $\mid Aut(B, B') \mid = (p-1)(q-1)$. $(w-1)(r+1)$ in case 3 and $(w-1)\sigma_0(s)$ in case 4 conjugate subgroups yield that the number of HGS in case 3 is $(r+1)$ and in case 4 is $\sigma_0(s)$.

The conclusions of the nonabelian group are summed up in the theorem below.

Theorem 4. *The total number of isomorphism types admits nonabelian HGSs is* $2 + (r + 1) + \sigma_0(s)$ *of PGs of degree* pqw*. The nonabelain of order* pqw *is the regular group.*

Proof. It is clear from summing the numbers of permutation groups G of degree pqw of isomorphism types in column four from Table 3 that the total number is $2+(r+1)+\sigma_0(s)$ which admits HGS of nonabelain case.

The following theorem summarizes the results of the cyclic case in Theorem 3 and the nonabelain case in Theorem 4.

Theorem 5. *A HGS of cyclic type can realise isomorphism types in total* $12(r + i + 1)[\sigma_0(s) + \sigma_1(j) + \sigma_0(s)\sigma_1(j)] + 2 + (r + 1) + \sigma_0(s)$ *of PGs* G *of degree* pqw *of both cases regular groups (where the Galois extensions have 1 HGS for the cyclic group and* $(p - 1)(q - 1) + 1, 1, r + 1,$ *and* $\sigma_0(s)$ *HGSs for the nonabelian group of the cyclic type).*

Proof. It is clear from summing the numbers of PGs G of degree pqw of isomorphism types in column four from Table 3 and Theorem 3 that the total number is $12(r + i + 1)[\sigma_0(s)+\sigma_1(j)+\sigma_0(s)\sigma_1(j)]+2+(r+1)+\sigma_0(s)$ which admits HGS of cyclic and nonabelain type G.

Example 1. *Assume that we have* $q = 5, w = 3, p = 2w + 1 = 7$ *three squarefree prime numbers. So, we have the conditions and notations of the group* N *as follow according to the primes above.*
$q - 1 = 2^r.s, r \geq 1, s$ *odd* $\Longrightarrow q - 1 = 5 - 1 = 4 = 2^2.1 \Longrightarrow r = 2, s = 1$. *Then* $d \mid s$ *has* $d = 1 \Longrightarrow \sigma_0(s) = 1$.
$w - 1 = 2^i.j, i \geq 1, j$ *odd* $\Longrightarrow w - 1 = 3 - 1 = 2 = 2^1.1 \Longrightarrow i = 1, j = 1$.
$0 \leq c_1 \leq r \Longrightarrow 0 \leq c_1 \leq 2$ *that means* $c_1 = 0, 1, 2$.

$1 \leq l \leq w-1 \Longrightarrow 1 \leq l \leq 2$ *that means* $l = 1, 2$.
As a result of these conditions, Table 1 and Table 3 have the following shape.

Table 4: The transitive subgroups for the nonabelian group J_l when $p = 7, q = 5, w = 3$.

Key	Order	Parameters	# Groups	Groups
1	105	$l = 1, 2$	2	J_1, J_2
2	210	$l = 1, 2$	2	$J_1 \rtimes \langle\beta\rangle$, $J_2 \rtimes \langle\beta\rangle$
3	105	$l = 1, 2, c_1 = 0, 1, 2$	6	$J_1 \rtimes \langle\gamma^4\rangle$, $J_2 \rtimes \langle\gamma^4\rangle$,
	210			$J_1 \rtimes \langle\gamma^2\rangle$, $J_2 \rtimes \langle\gamma^2\rangle$,
	420			$J_1 \rtimes \langle\gamma\rangle$, $J_2 \rtimes \langle\gamma\rangle$
4	105	$l = 1, 2, d \mid s = 1 \mid 1$	2	$J_1 \rtimes \langle\delta\rangle$, $J_2 \rtimes \langle\delta\rangle$

Table 5: The number of HGSs for the nonabelian group J_l when $p = 7, q = 5, w = 3$.

Key	Order	$\mid Aut(B, B') \mid$	# iso. class	# HGS per iso. class
1	105	600	1	25
2	210	24	1	1
3	105, 210, 420	24	3	3
4	105	24	1	1

We can see from Theorem 4 that there is 6 isomorphism types admits nonabelian HGSs of PGs of degree 105. According to the results obtained in Theorem 5, A HGS of cyclic type can realise 150 isomorphism types of PGs G of degree 105 of both cases regular groups (cyclic and nonabelian), where the extension has 1 HGS for the cyclic groups and 25, 1, 3, 1 HGSs for the nonabelian groups of the cyclic type.

4. Discussion

Comparing the work to the results of other references, we can see from the tables and results that similar behaviour exists for square free degree $n = pqw$ as the field extension of degree $n = pq$ in [6]. It is clear through Tables 1, 2 and 3 that no similar abstract group can be found for any two distinct PGs admitted HGSs. Thus we partially answer the question in [6] related to the behaviour of square free degree in general.

5. Conclusion

We investigate the group permutations G for the nonabelian case of degree pqw where $q, w \geq 3$ and $p = 2w + 1$ are all square free primes then for each G we enumerate the HGSs. There exists four G such that the field extensions L/K admit the HGSs in this case. Furthermore, we have obtained the total number of HGSs of nonabelian case as $2 + (r + 1) + \sigma_0(s)$ of PGs G of types of isomorphism of degree pqw and we have found the number of both cases (nonabelian and cyclic) in total as $12(r+i+1)[\sigma_0(s)+\sigma_1(j)+\sigma_0(s)\sigma_1(j)]+2+(r+1)+\sigma_0(s)$ PGs of isomorphism types which admit HGSs. Finally, we have found that any two distinct PGs admitted HGSs can not have the same abstract group.

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