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Inverse Boundary Value Problem Solution for Deflected Beams Joined Together by Elastic Medium

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Abstract. In this paper, we extend the Euler-Bernoulli beam theory for bending boundary value problem into mechanically coupled system. We follow the inverse approach to find the exerted force on two beams separated by elastic material. The theory was utilized in two ways: in the first approach, we calculate the force exerted on the beams using known values for the stiffness constant and measured values for the beam deflections. In the second method, we calculate the stiffness constant using a single known force and measured deflections. These problems are typically ill-posed problems whose solution does not depend continuously on the boundary data. To minimize the variational functional, we develop an iterative algorithm based on the system of three equations: the direct, adjoint, and control equations. Then, we present numerical examples to obtain the solutions.

2020 Mathematics Subject Classifications: 65L09, 31A25, 34A55, 35R30, 65M32

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1. Introduction

Devices with touchscreens capability such as smartphones, tablets, televisions, and medical equipment have become a necessity nowadays [6]. Their popular usage and daily need have made it crucial for manufacturers of such devices to understand how their glass screens react to stresses of various kinds, especially for hand-held devices because of their extensive use [1, 4, 6, 14, 17]. To minimize the lengthy and costly lab testing experiments of evaluating the sensitivity of these devices to accidents such as falling on varies surfaces and manually recording the data on the responses and stresses generated, bending models can be developed to predict the response and stress generated using numerical computations [7, 18]. The model can deliver scientific expectations and results in a more efficient and

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cost-effective design for new gadgets. A general model to describe touchscreens is presented in Figure 1. A framework wherein two beams are joined by persistent elastic layer of stiffness k. The two beams are basically bolstered at the edges. The top beam is loaded by a point force f.

In this paper, we develop and implement a numerical scheme to represent the model in Figure 1 in non-denominational approach using the one-dimensional Euler-Bernoulli beam theory for bending boundary value problem [5, 8, 11, 12]. We utilize this scheme in two ways: First we consider the inverse problem: Given the spring stiffness k and known finite measurements of the resulting deflections v_1 and v_2 for the upper and lower beams respectively, then we find the imposed force f. Secondly, we use given forces $f(x)$ and finite measurements of the resulting deflections $v_1(x)$ and $v_2(x)$ to find the spring stiffness k.

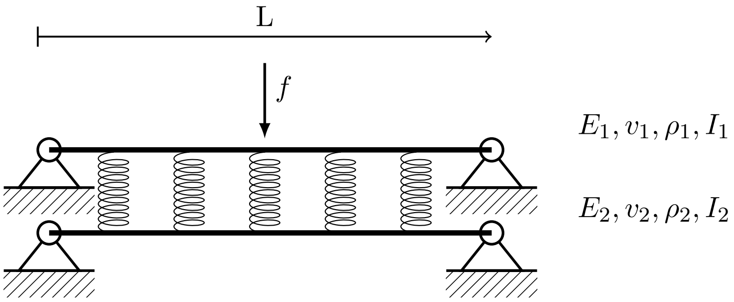

Figure 1: Representation of the physical problem

Following reference [10] and to avoid the complication of the shear stress imposed by the two dimensional nature of the plates, we consider the one-dimensional cross-section of the full problem, where we assume that the deflection of the glass plates can be described by beams. Therefore, we modeled the edge of the plate with a one-dimensional fixable beam, assuming the same bending from all parts of the plate without any shear stress.

We use Euler-Bernoulli beam theory for the transverse displacements $v_1(x,t)$ and $v_2(x,t)$ (vertical displacements) of two simply supported one-dimensional beams of length L, thickness h, density ρ, flexural rigidity, I the moment of inertia, and E is Young's modulus, for the top beam and for the lower beam (as shown in Figure 1),

Considering the two elastic beams joined by continuous elastic layer of stiffness k. Both beams are pinned to a rigid frame along the boundary. The top beam is loaded by a point force f. The assumption of point load is based on the fact the affected area by the force is much smaller than the whole area of the plates. The governing set of equations for such system in this case is given as:

$$\rho_1 h_1 v_{1tt} = -(E_1 I_1 v_{1xx})_{xx} + k(v_2 - v_1) - f(x,t) \tag{1}$$

$$\rho_2 h_2 v_{2tt} = -(E_2 I_2 v_{2xx})_{xx} - k(v_2 - v_1), \tag{2}$$

Our interest will be in the response to external load $f(x,t)$ on the upper beam. The boundary conditions are based on the assumption that the plates are pinned to a rigid frame along the boundary which maintains a fixed separation between them:

$$v_1(0,t) = v_1(L,t) = v_2(0,t) = v_2(L,t) = 0, \tag{3}$$
$$v_{1xx}(0,t) = v_{1xx}(L,t) = v_{2xx}(0,t) = v_{2xx}(L,t) = 0, \tag{4}$$

The problem of steady state solution of the one-dimensional, two beam system was discussed an analyzed by Adriazola et al. [10], they solved the forward problem using analysis of Green's functions for system response to point loads, they also solved the inverse problem of recovering the spring stiffness using Fourier decomposition.

In this paper we consider the same time-independent (stationary) problem, therefore the system above, (1),(2), (3), and (4) becomes:

$$-E_1 I_1 v_1^{(4)} + k(v_2 - v_1) = f(x), \tag{5}$$
$$-E_2 I_2 v_2^{(4)} - k(v_2 - v_1) = 0. \tag{6}$$

Subtracting (6) from (5) and letting $u = v_1 - v_2$ and $L = 1$ reduces the system to

$$u^{(4)} - Ku = g(x) \tag{7}$$

$$u(0,t) = u(1) = 0 = u_{xx}(0) = u_{xx}(1) = 0 \tag{8}$$

where:

$$K = \frac{k}{E_1 I_1} + \frac{k}{E_2 I_2}, \text{ and}$$
$$g = \frac{-f}{E_1 I_1}$$

Obtaining the solution $u(x)$ for this system (7) and (8), we then substitute the relation $v_1 = u + v_2$ into (6) to get a modified equation for v_2,

$$-E_2 I_2 v_2^{(4)} - ku = 0.$$

or

$$v_2^{(4)} = \tilde{g}.$$

where $\tilde{g} = \frac{-u}{E_2 I_2}$ is the forcing on the lower beam due to the top beam,
After solving for v_2, we can recover v_1 from the relation $v_1 = u + v_2$.

First we consider the inverse problem: Given the spring stiffness k and measurements of the resulting deflections $v_1(x)$ and $v_2(x)$, find the imposed force $f(x)$, this is the same as finding the function $g \in L^2[0,1]$ in (7),(8) if K is known and a finite observation is given about the function u.

Second we consider the inverse problem: Given a single imposed force $f(x)$ and measurements of the resulting deflections $v_1(x)$ and $v_2(x)$, find the spring stiffness k, this is the same as recovering the coefficient K, in (7),(8), if a continuous function $g(x)$ is given.

2. Formulation of the two inverse problems

The problem of recovering a source function g or a coefficient K in (7)(8) are typical ill-posed problems whose solution does not depend continuously on the boundary data. That is, a small error in the specified data may result in an enormous error in the numerical solution [13]. We employ Tikhonov regularization technique to restore the stability of the numerical solution [3, 15]. Stable and efficient numerical methods are of high importance. We assume that the only available information is finite observations of the solution u_{obs} at $x_i, i = 1, 2, 3, \dots n$. Henceforth, u_{obs} is the interpolating cubic spline of the finite observations of the function u.

2.1. Inverse problem 1

Suppose that the functions $g = g_0$ and K are known in (7)(8), we solve the direct problem and obtain the exact solution $u^{(0)} \neq u_{obs}$. Then we seek a solution u_1 in the neighborhood of $u^{(0)}$ and $g_1 = g_0 + v$ in neighborhood of g_0, where, u_1 meets the boundary conditions (8).

We define the operator A as follows:

$$\begin{aligned} Au &= u^{(4)} - Ku, \\ D(A) &= \{u \in W_4^2[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}, \\ Au &= g, \quad g \in L^2[0,1], \end{aligned}$$

where

$$W_k^p(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall \alpha, |\alpha| \leq k\},$$

and define the inner product

$$(u, v) = \int_0^1 uv\ dx$$

The inverse problem is to find $u_1 \in W_4^2[0,1]$ and $v \in L^2$ such that:

$$Au_1 = g_1 = g_0 + v.$$

Along the above, we consider the variational problem:

$$\inf_{v\in L^2}\left\{\alpha\|v\|^2_{L^2}+\|u_1-u_{obs}\|^2_{W_4^2}\right\}.$$

Now we can reformulate the problem above as the following inverse problem of the function $u=u_1-u^{(0)}$. For given u_{obs} and $u^{(0)}$, find $u\in W_4^2[0,1]$ and $v\in L^2$ such that:

$$\begin{aligned}Au&=v,\\ u(0)=u(1)=u''(0)=u''(1)&=0,\\ \inf_{v\in L^2}J(u,v),\end{aligned}$$

where,

$$J=\inf_{v\in L^2}\left\{\alpha\|v\|^2_{L^2}+\|u-(u_{obs}-u^{(0)})\|^2_{W_4^2}\right\}.$$

In the last expression we seek u and v such that the minimum is attained. Here α is a constant. To minimize the functional J, we consider its variational δJ and equals it to zero, we obtain:

$$\delta J=2\alpha(v,\delta v)+2(u-(u_{obs}-u^{(0)}),\delta u)=0,$$

here, δu satisfies $Au=v$, we get:

$$A\delta u=\delta v,$$

now we introduce the adjoint operator, A^* and obtain:

$$(A^*q,\delta u)=(q,A\delta u)=(q,\delta v).$$

Clearly, $A^*=A$, if we set

$$A^*q=u-(u_{obs}-u^{(0)}),$$

we obtain the control equation:

$$\alpha v+q=0.$$

So, the algorithm can be written as follows:

$$Au_n=v_n,\ u_n(0)=u_n(1)=u_n''(0)=u_n''(1)=0, \tag{9}$$

$$A^*q_n=u_n-(u_{obs}-u^{(0)}),\ q_n(0)=q_n(1)=q_n''(0)=q_n''(1)=0, \tag{10}$$

$$v_{n+1} = v_n - \tau(\alpha v_n + q_n). \tag{11}$$

Where arbitrary initial value $v_0 \in L^2$ and τ is a constant (stabilizer).

For fixed n iteration steps, the computer program of the algorithm (9)-(11) is approximated by a finite difference scheme justified in [2, 9, 16], that can be written in the following form:

Set $\Delta x = \frac{1}{N}$, where, N is the number of grid points. The fourth derivative is approximated as:

$$D^{(4)}(z(x_i)) \approx \frac{z(x_{i+2}) - 4z(x_{i+1}) + 6z(x_i) - 4z(x_{i-1}) + z(x_{i-2})}{(\Delta x)^4},$$

and the computer program is approximated as follows:

$$D^{(4)}(u(x_i)) - Ku(x_i) = v(x_i),\ i = 1, 2, \ldots, N,$$

$$D^{(4)}(q(x_i)) - Kq(x_i) = u(x_i) - (u_{obs}(x_i) - u^{(0)}(x_i)),\ i = 1, 2, \ldots, N,$$

We find u and q from above, after that we solve the control equation (11), which is approximated as follows:

$$v_{n+1}(i) = v_n(i) - \alpha(\tau v_n(i) - q(x_i)),\ i = 1, 2, \ldots, N.$$

Then, we are ready to pass to the next, $n + 1$ iteration step.

2.1.1. Numerical solution for Inverse Problem 1

The aim of Inverse Problem 1 is to recover the right hand side function of (7) in the neighborhood of known $g_0(x)$ over $[0, 1]$. Given the known functions, $K = 1$, and $g_0(x) = \frac{1}{2}\sin(2\pi x)$, by solving the forward problem (7) we obtain the initial solution $u^{(0)}$. If we set $v(x) = \frac{1}{4}\sin(\pi x)$ and calculate the exact solution u_e for $K = 1$ and $g_e = g_0(x) + v(x)$, then we can compare this solution with the calculated solution using our algorithm. The exact solution u_e and the initial starting solution $u^{(0)}$ are shown in Figure 7, we set u_{obs} to be the interpolating cubic spline of n finite values of u_e after adding noise. Figure 2 shows a side by side comparison of the exact solution u_e and the noisy measurements (u_{obs}). We then pretend that we know neither the value of $v(x)$ nor $u_e(x)$, we run the algorithm to recover $v(x)$ based on an initial guess of $v_0 = v_{guess} = 0$ shown in Figure 5. Figure 6 shows a side by side comparison of the functions v_e (expected value of v), and v_c, the calculated value (recovered) of v. Figure 8 shows a side by side comparison of the functions u_e (expected value of u), and v_c, the calculated value of u. As shown in the figure the graphs of both recovered functions, u and v, are almost identical to the expected values.

It is worth mentioning that Figure 3 shows the expected value of g_e based on the exact solution u_e is as expected ($Au_e = g_e$), however, when we plug the noisy solution u_{obs} in

Au_{obs}, we get instability as shown in Figure 4. Our method still works well with such noisy measurements. This is clear in Figure 9 that shows side by side comparison with Au_c and Au_e.

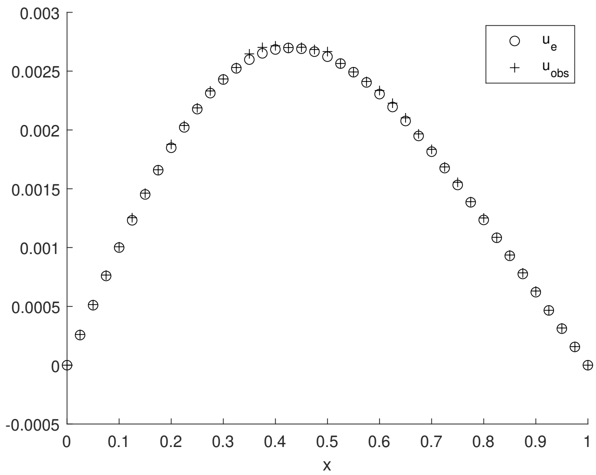

Figure 2: Inverse problem 1, the graph shows noisy u_{obs} compare to exact u_e

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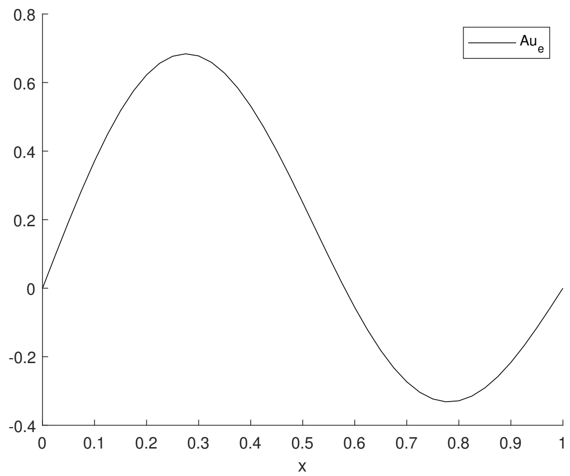

Figure 3: Inverse problem 1, the graph shows Au_e

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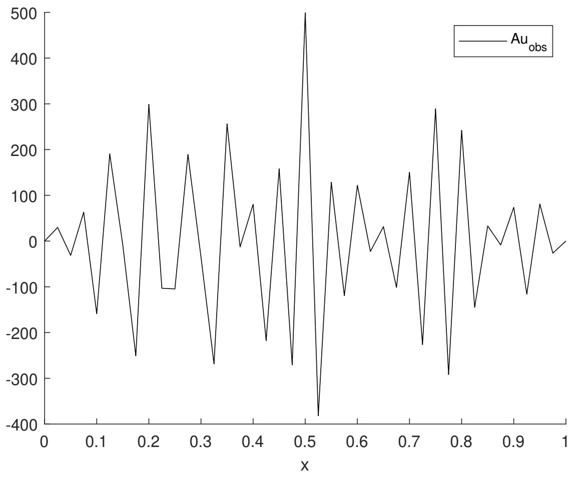

Figure 4: Inverse problem 1, the graph shows Au_{obs}

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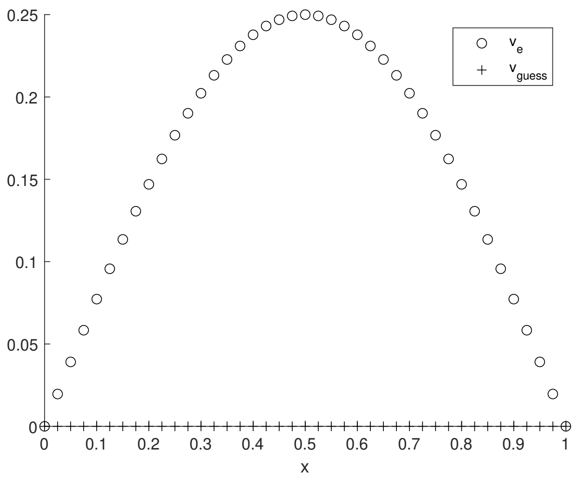

Figure 5: Inverse problem 1, the graph shows the initial process of expected value of v,(v_e) and the starting value of v, ($v_0 = v_{guess}$).

2.2. Inverse problem 2

Suppose that the functions g in (7) is known, we use series expansion method of K and the subsequent transformation of this problem to the first problem (recovery of the right-hand side).

$$\text{let } u(x) = r(x) + \beta s(x) \text{ , and}$$
$$K = k_0 + \beta k_1(x).$$

From (7),

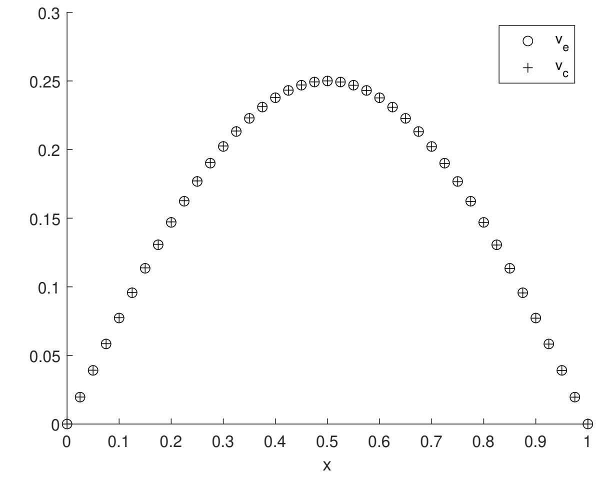

Figure 6: Inverse problem 1, the graph shows a side by side comparison of the functions v_e (expected value of v), and the calculated value of of v, (v_c)

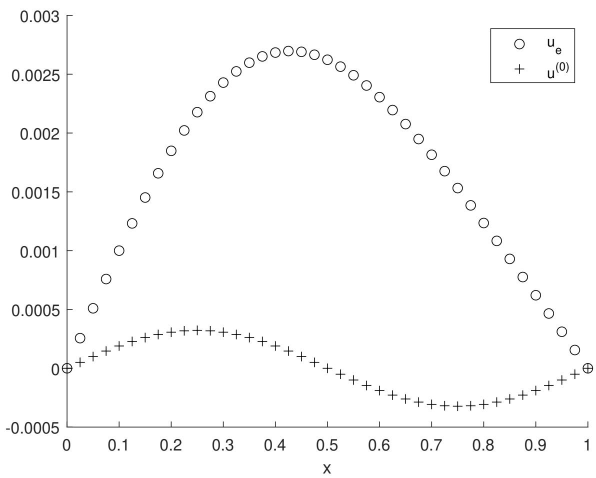

Figure 7: Inverse problem 1, the graph shows a side by side comparison of the functions u_e (expected value of u), and the initial value of $u, (u^{(0)})$ before we start the calculation.

$$r^{(4)} + \beta s^{(4)} - (k_0 + \beta k_1)(r + \beta s) = g$$

combining the terms:

$$\begin{aligned} r^{(4)} - k_0 r &= g \\ r(0) = r(1) = r''(0) &= r''(1) = 0. \end{aligned} \tag{12}$$

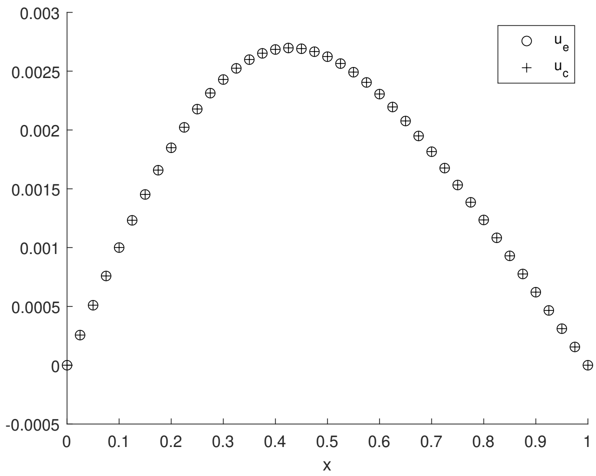

Figure 8: Inverse problem 1,the graph shows a side by side comparison of the functions u_e (expected value of u), and the calculated value of of u, (u_c)

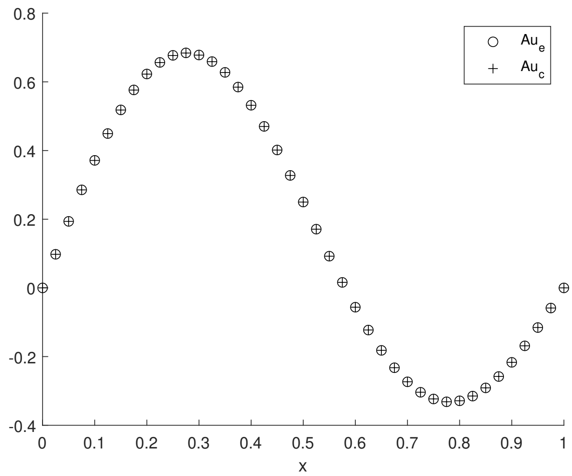

Figure 9: Inverse problem 1, the graph shows a side by side comparison of the functions Au_e (expected value of $g(x)$), and the calculated value of of Au_c

$$
\begin{gathered}
s^{(4)} - k_0 s = k_1 r \\
s(0) = s(1) = s''(0) = s''(1) = 0. \\
s_{obs} = \frac{u_{obs} - r}{\beta}
\end{gathered} \tag{13}
$$

Note that Equation (12) is a linear forward problem since $g(x)$ and k_0 are known, therefore it can be solved directly to find the solution $r(x)$. Also, s_{obs} in (13) is calculated based on the known finite observation u_{obs}

We utilize the algorithm developed above to solve the inverse problem (13) to recover the right-hand side $v = k_1 r$. Note that the only unknown in this right-hand-side is k_1, therefore, k_1 will be derived from the recovered right-hand-side in every iteration of the algorithm developed and justified in the previous sections.

2.2.1. Numerical solution for Inverse Problem 2

Our goal is to recover the coefficient function $K = 1$ of (7) over $[0, 1]$ for the given function $g(x) = \sin(\pi x)$. These values were used in the calculations: $\beta = 0.05$, $k_0 = 0.5$, initial guess for v is $v_{guess} = 0$. As before, we pretend that we don't know the function K, we run the algorithm to recover $v(x) = k_1 r$ based on the mentioned initial guess. Figure 10 shows side by side comparison of exact value of $u = r + \beta s$ (u_e) and calculated value of u (u_c) . Figure 11 shows a side by side comparison of the exact function $K = k_0 + \beta k_1$ and the calculated value of K, (K_c). As we can see the graphs of both functions are almost identical as expected.

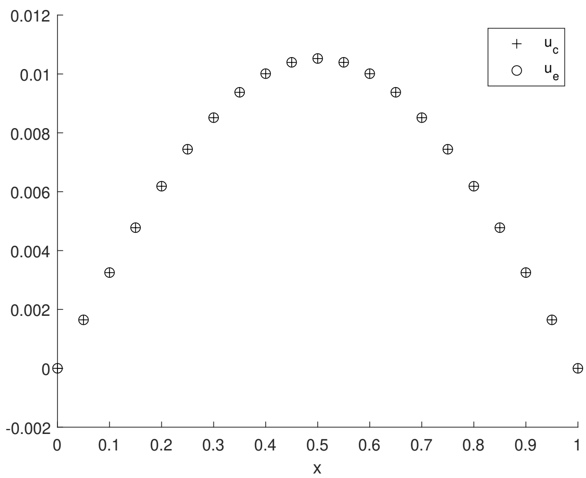

Figure 10: Inverse problem 2 with input parameters: $g(x) = \sin(\pi x)$, $\beta = 0.05$, $k_0 = 0.5$, initial guess for v is $v_{guess} = 0$, the graph shows the expected value of u, (u_e) and the calculated value of u, (u_c)

3. Conclusion

We developed and implemented a numerical scheme to represent coupled mechanical system. We considered a one-dimensional model for two beams joined together by elastic layer. First we considered the inverse problem: Given the spring stiffness k and finite measurements of the resulting deflections $v_1(x)$ and $v_2(x)$, find the imposed force $f(x)$, second we considered the inverse problem: Given a single imposed force $f(x)$ and finite measurements of the resulting deflections $v_1(x)$ and $v_2(x)$, find the spring stiffness k. This development includes a numerical algorithm, that takes as input parameters the physical

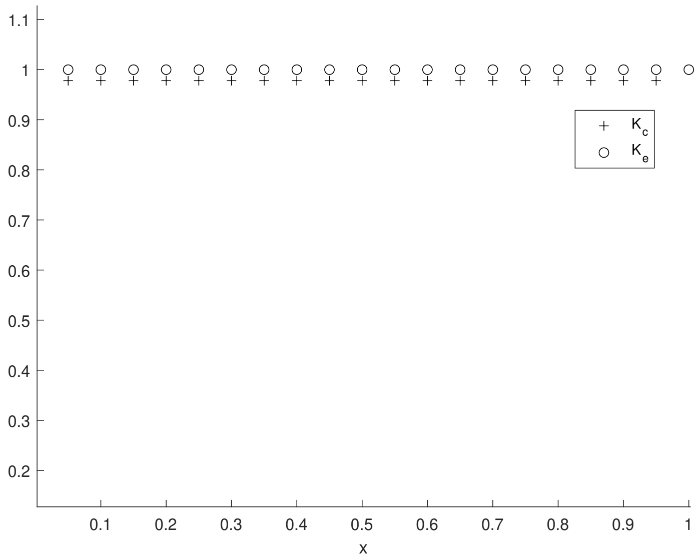

Figure 11: Inverse problem 2 with input parameters: $g(x) = \sin(\pi x)$, $\beta = 0.05$, $k_0 = 0.5$, initial guess for v is $v_{guess} = 0$, the graph shows the expected value of K, (K_e) and the calculated value of K, (K_c)

measurements of the resulting deflections of the beams and outputs the external forces or spring stiffness. This computational framework can serve as a preliminary tool in the product development process, allowing scientists and engineers to simulate desired physical situations without costly testing of prototypes. The algorithm recovered the right-hand side $f(x)$ of Equations (7), (8), this method is based on minimizing the defect in the functional between the calculated data and the measured data. We then used this development to also recover the coefficient k by reformulating the problem of recovering the coefficient to the problem of recovering the right-hand-side. The numerical experiments demonstrated that the proposed algorithm was able to recover the unknowns very closely to the exact solution.
In the algorithm, two additional parameters are used: a parameter α that regularizes the problem, and parameter τ that stabilizes the numerical algorithm. In the algorithm, the relationship between parameters α and τ was analyzed and the values of α and τ were optimized on the base of the computational tests of different values of α, and τ.

References

[1] Alaa A Abdelrahman, Mohamed S Abdelwahed, Hani M Ahmed, Amin Hamdi, and Mohamed A Eltaher. Investigation of size-dependent vibration behavior of piezoelectric composite nanobeams embedded in an elastic foundation considering flexoelectricity effects. *Mathematics*, 11(5):1180, 2023.

[2] VI Agoshkov. Control theory approaches in data assimilation processes, inverse prob-

lems and hydrodynamics. *Computer Mathematics and its Applications, HMS/CMA*, 1:21–39, 1994.

[3] VI Agoshkov and PB Dubovski. Solution of the reconstruction problem of a source function in the coagulation-fragmentation equation. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 17(4):319–330, 2002.

[4] Leon Gray. *How Does a Touch Screen Work?* The Rosen Publishing Group, 2013.

[5] Leopold Herrmann. Vibration of the euler-bernoulli beam with allowance for dampings. *nature*, 5:433, 2008.

[6] Satwant Kaur. The revolution of tablet computers and apps: A look at emerging trends. *IEEE Consumer Electronics Magazine*, 2(1):36–41, 2013.

[7] Dieter Krause and Horst Loch. *Mathematical simulation in glass technology.* Springer Science & Business Media, 2002.

[8] YX Li and LZ Sun. Transverse vibration of an undamped elastically connected double-beam system with arbitrary boundary conditions. *Journal of Engineering Mechanics*, 142(2):04015070, 2016.

[9] GI Marchuk and VI Lebedev. *Numerical methods in the theory of neutron transport.* 1986.

[10] Andre Gianesini Odu, Emmanuel Maitre, Matthew McCurdy, Arthur Newell, Don Schwendeman, and Thomas Witelski. Deflection and stresses in beams joined together by a continuous spring layer.

[11] Z Oniszczuk. Forced transverse vibrations of an elastically connected complex simply supported double-beam system. *Journal of Sound and Vibration*, 264(2):273–286, 2003.

[12] Z Oniszczuk. Forced transverse vibrations of an elastically connected complex rectangular simply supported double-plate system. *Journal of sound and vibration*, 270(4-5):997–1011, 2004.

[13] V Shutyaev. Control operators and iterative algorithms in variational data assimilation problems. *Journal of Inverse and Ill-posed Problems*, 9(2):177–188, 2001.

[14] Guillermo Suarez-Tangil, Juan E Tapiador, Pedro Peris-Lopez, and Arturo Ribagorda. Evolution, detection and analysis of malware for smart devices. *IEEE Communications Surveys & Tutorials*, 16(2):961–987, 2013.

[15] AN Tikhonov. *Ill-Posed Problems in Natural Sciences: Proceedings of the International Conference Held in Moscow, August 19–25, 1991.* Walter de Gruyter GmbH & Co KG, 2020.

[16] GM Vainikko and A Yu Veretennikov. Iteration procedures in ill-posed problems, 1986.

[17] Geoff Walker. Fundamentals of touch technologies. *Sunday Short Course (S-4), SID Display Week*, 2013.

[18] Zhen Zhang, Wuyi Ming, Yi Zhang, Ling Yin, Tao Xue, Haishen Yu, Zhijun Chen, Dunming Liao, and Guojun Zhang. Analyzing sustainable performance on high-precision molding process of 3d ultra-thin glass for smart phone. *Journal of Cleaner Production*, 255:120196, 2020.