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Perfect Isolate Domination in Graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple connected graph. A set $S \subseteq V(G)$ is said to be a perfect isolate dominating set of G if S is a perfect dominating set and an isolate dominating set of G. The minimum cardinality of a perfect isolate dominating set of G is called perfect isolate domination number, and is denoted by $\gamma_{p0}(G)$. A perfect isolate dominating set S with $|S| = \gamma_{p0}(G)$ is said to be γ_{p0}-set. In this paper, the author gives a characterization of perfect isolate dominating set of some graphs and graphs obtained from the join, corona and lexicographic product of two graphs. Moreover, the perfect isolate domination number of the forenamed graphs is determined and also, graphs having no perfect isolate dominating set are examined.

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Key Words and Phrases: Perfect domination, isolate domination, perfect isolate domination

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1. Introduction

In 1960, the study of domination in graphs began and it became the most interesting topic in graph theory because of its application in networking. In 1990, Livingston and Stout [10], introduced the concept of perfect dominating sets of G, denoted by $\gamma_p(G)$. They studied the existence and construction of perfect dominating sets in families of graphs arising from the interconnection networks of parallel computers. In 2014, Kwon and Lee [9] investigated some results related to perfect domination sets of Cayley graphs.

In 2013, Hamid and Balamurugan [4] studied the concept of isolate domination in graphs. In 2015, Ariola [2] looked at another aspect of the isolate dominating set and characterized the lower and upper bounds of the isolate domination number and those graphs resulting from some binary operations such as join and corona. In 2016, Hamid and Balamurugan [4] extended these parameters isolate domination number γ_0 and the upper isolate domination number Γ_0. In 2017, Rad [11] studied the complexity of the

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isolate domination in graphs, and obtain several bounds and characterizations on the isolate domination number. Furthermore, some variations and parameters of domination in graphs are studied in many classes (see [1], [8], [6] and [5]).

In this paper, we introduce the concept of perfect isolate domination number. We characterize the perfect isolate dominating set and determine the exact values of perfect isolate domination number of some special graphs and graphs under some binary operations such as join, corona and lexicographic product of two graphs. Not all graphs have perfect isolate dominating set and we called them non-γ_{p0}-graphs.

2. Terminology and Notation

This section contains definitions that are needed for the study.

Let $G = (V(G), E(G))$ be a simple *connected graph* where $V(G)$ is a *vertex-set* of G and $E(G)$ is an *edge-set* of G. The number of edges incident with v is called the *degree* of a vertex v and is denoted by $\deg(v)$. The *maximum* of $\{\deg(v) : v \in V(G)\}$ is denoted by $\Delta(G)$. The set of neighbors of a vertex u in G is called the *open neighborhood* of u in G and is denoted by $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$ and the *closed neighborhood* of a subset S of $V(G)$ is the set $N_G[S] = N[S] = \cup_{v\in S} N_G[v]$. The *subgraph induced* by a set S of vertices of a graph G is denoted by $\langle S\rangle$ with $V(\langle S\rangle) = S$ and $E(\langle S\rangle) = \{uv \in E(G) : u, v \in S\}$, Harary in [7].

A set $S \subseteq V(G)$ is said to be a *dominating set* if $N[S] = V(G)$. A dominating set S is a minimal dominating set if no proper subset $S' \subset S$ is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A dominating set S with $|S| = \gamma(G)$ is said to be a γ-set.

A set $S \subseteq V(G)$ is said to be a *perfect dominating set* if each vertex $v \in V(G) \setminus S$ is dominated by exactly one element in S. The minimum cardinality of a perfect dominating set of G is called *perfect domination number*, and is denoted by $\gamma_p(G)$. A perfect dominating set S with $|S| = \gamma_p(G)$ is said to be a γ_p-set.

A dominating set $S \subseteq G$ is said to be an *isolate dominating set* of G if $\langle S\rangle$ has at least one isolated vertex. An isolate dominating set S is said to be a minimal isolate dominating set if no proper subset of S is an isolate dominating set. The minimum cardinality of a minimal isolate dominating set of G is called the *isolate domination number* and is denoted by $\gamma_0(G)$. An isolate dominating set S with $|S| = \gamma_0(G)$ is said to be a γ_0-set.

A perfect dominating set $S \subseteq V(G)$ is said to be a *independent perfect dominating set* (ipds) if no two vertices in S are adjacent. The minimum cardinality of an independent perfect dominating set of G is called *independent perfect domination number* and is denoted by $\gamma_{ip}(G)$. A perfect dominating set S with $|S| = \gamma_{ip}(G)$ is said to be a γ_{ip}-set.

A set $S \subseteq V(G)$ is said to be a *perfect isolate dominating set* of G if S is a perfect dominating set and the induced subgraph $\langle S \rangle$ has at least one isolated vertex. The minimum cardinality of a perfect isolate dominating set of G is called *perfect isolate domination number* and is denoted by $\gamma_{p0}(G)$. A perfect isolate dominating set S with $|S| = \gamma_{p0}(G)$ is said to be γ_{p0}-set. If a graph G has no perfect isolate dominating set, then we say that the graph G is a non-γ_{p0}-graph.

Example 1. *Let G be the graph in Figure 1 and $S = \{v_1, v_2, v_7\}$. Then v_1 dominates v_3, v_2 dominates v_4, while v_7 dominates v_5 and v_6. This shows that for all vertices v_3, v_4, v_5, v_6 are elements in $V(G) \setminus S$ which are dominated by exactly one vertex in S. Thus, S is a perfect dominating set of G. Observe further that v_7 is an isolated vertex of the induced subgraph $\langle S \rangle$. Therefore, S is a perfect isolate dominating set of G and $\gamma_{p0}(G) = \gamma_p(G) = |S| = 3$.*

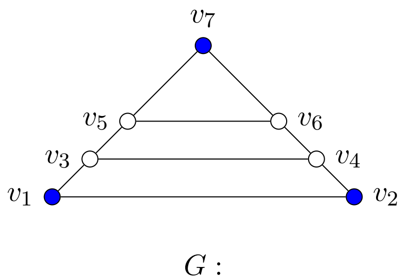

Figure 1: Graph G with $\gamma_{p0}(G) = \gamma_p(G) = 3$

3. Results

This section contains some known results involving the domination number, the perfect domination number and the isolate domination number. Also, it contains the the perfect isolate domination number of paths, cycles, complete graphs, fans, wheels, friendship graphs, windmill graphs and graphs resulting from some binary operations. Furthermore, some non-γ_{p0}-graphs are shown.

Proposition 1. [8] *For paths and cycles of order n, $\gamma_0(P_n) = \gamma_0(C_n) = \lceil \frac{n}{3} \rceil$.*

Proposition 2. [8] *For K_n, S_{n-1}, and W_{n-1} be complete, star, and wheel of $n \geq 2$ vertices, respectively. Then*

$$\gamma_0(K_n) = \gamma_0(S_{n-1}) = \gamma_0(W_{n-1}) = 1.$$

Theorem 1. [8] *For K_n, S_{n-1}, and W_{n-1} be complete, star, and wheel of $n \geq 2$ vertices, respectively. Then $\gamma_0(K_n) = \gamma_0(S_{n-1}) = \gamma_0(W_{n-1}) = 1$.*

Corollary 1. [3] *Let G and H be connected graphs. Then*

$$\gamma(G+H) = \begin{cases} 1, & \gamma(G) = 1 \text{ or } \gamma(H) = 1 \\ 2, & \gamma(G) \neq 1 \text{ and } \gamma(H) \neq 1. \end{cases}$$

The next four results follow directly from the definition of perfect isolate dominating set.

Proposition 3. *Let $S \subseteq V(G)$ be an isolate dominating set of G. Then S is a perfect isolate dominating set if and only if for every $v \in V(G) \setminus S$, $N_G(v) \cap S = \{u\}$ for some $u \in S$.*

Proposition 4. *If S is a dominating set or a perfect dominating set or an isolate dominating set of G with $|S| = 1$, then S is a perfect isolate dominating set of G. In particular, $\gamma(G) = \gamma_p(G) = \gamma_0(G) = 1$ if and only if $\gamma_{p0}(G) = 1$.*

Proposition 5. *If $\gamma_p(G) = c$ where $c \in \mathbb{Z}^+$ and if S is a γ_p-set of G such that $\langle S \rangle$ has an isolated vertex, then $\gamma_{p0}(G) = c$.*

Proposition 6. *If $\gamma_0(G) = c$ where $c \in \mathbb{Z}^+$ and if S is a γ_0-set of G such that every vertex $v \in V(G) \setminus S$ is dominated by exactly one vertex in S, then $\gamma_{p0}(G) = c$.*

Proposition 7. *Let G be any graph such that G has a perfect isolate dominating set. Then*

$$\gamma_0(G) \leq \gamma_{p0}(G).$$

Theorem 2. *For any positive integers a and b with $1 \leq a \leq b$, there exists a connected graph G such that $\gamma_0(G) = a$ and $\gamma_{p0}(G) = b$.*

Proof. Consider the following cases:
Case 1: $a = b$
Let G be the graph shown in Figure 2. Clearly, the set $S_1 = \{v_1, v_2, v_3, \ldots, v_a\}$ is both γ_0-set and γ_{p0}-set of G. Therefore, $\gamma_0(G) = \gamma_{p0}(G) = a = b$.

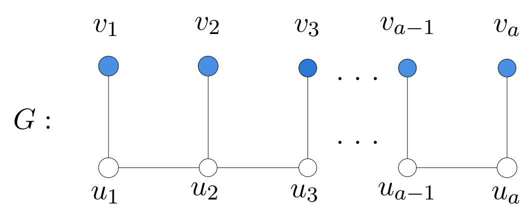

Figure 2: Graph G with $\gamma_0(G) = \gamma_{p0}(G) = a$

Case 2:$a < b$.

Let G be the graph shown in Figures 3 or Figure 4. Let $m = b - a + 4$ and $j, k, l \in \mathbb{Z}^+$. Observe that $S_1 = \{v_1, v_2, v_3 \ldots, v_{a-4}\} \cup \{z_1, z_2, z_3, z_m\}$ is a γ_0-set of G and $S_2 = \{v_1, v_2, v_3 \ldots, v_{a-4}\} \cup \{z_i : i = 1, 2, \ldots, m\}$ is a γ_{p0}-set of G. It follows that $\gamma_0(G) = |S_1| = a - 4 + 4 = a$ and $\gamma_{p0}(G) = |S_2| = a - 4 + m = a - 4 + (b - a + 4) = b$. Therefore, $\gamma_0(G) = a < b = \gamma_{p0}(G)$.

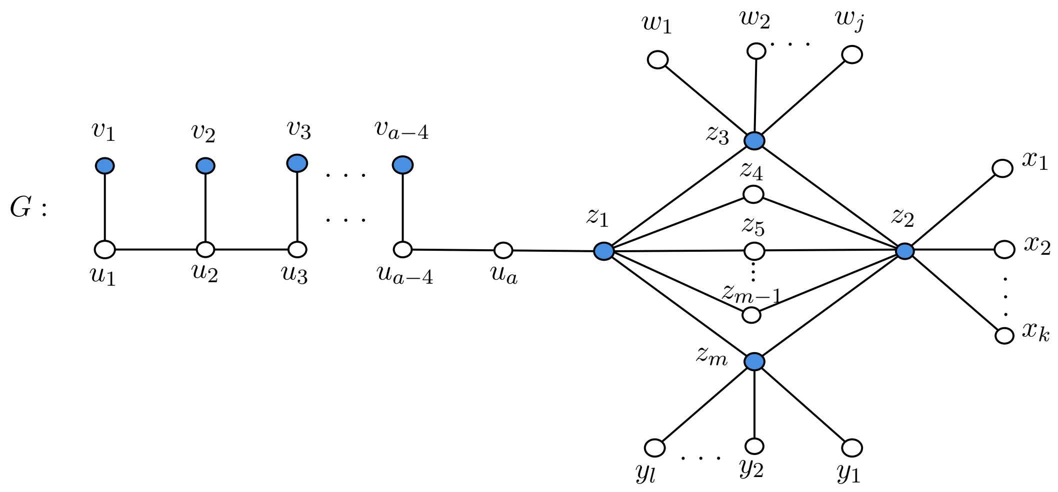

Figure 3: Graph G with $\gamma_0(G) = a$

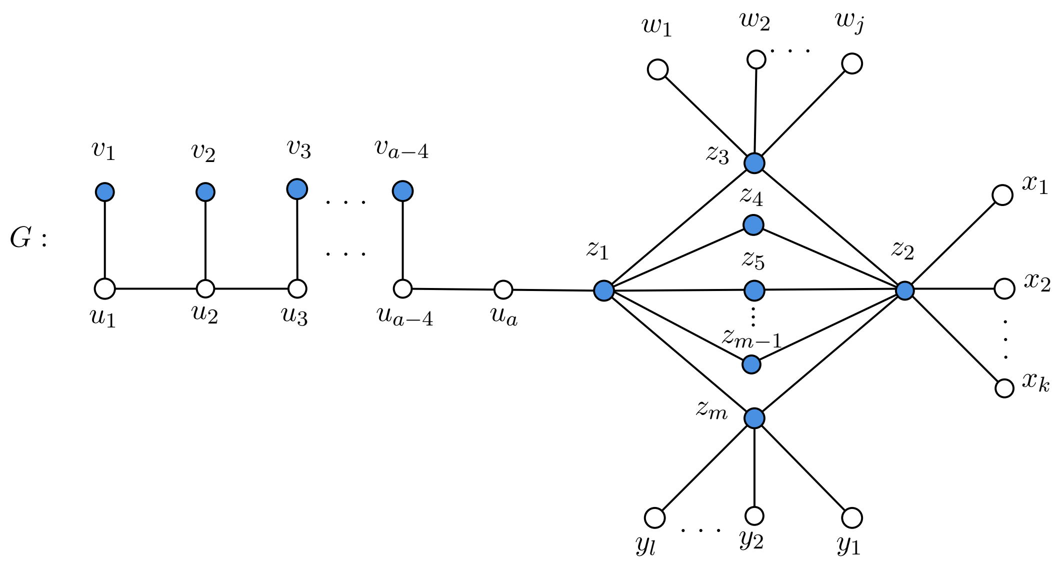

Figure 4: Graph G with $\gamma_{p0}(G) = b$

This proves the assertion. □

Corollary 2. *The difference γ_{p0}- γ_0 can be made arbitrarily large.*

The next result follows from Proposition 1 and Proposition 4.

Corollary 3. *Let G be a simple connected graph of order 3, that is, either $G = P_3$ or $G = C_3$. Then $\gamma_{p0}(G) = 1$.*

Theorem 3. *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{p0}(G) = 1$ if and only if $\Delta(G) = n - 1$.*

Proof. Suppose that $\gamma_{p0}(G) = 1$. Let $S = \{u\}$ be the perfect isolate dominating set of G. If G is trivial, then $\deg(u) = 0$. So, we are done. Assume that G is nontrivial. Then every vertex $v \in V(G) \setminus S$ is adjacent to $u \in S$. Hence, $deg(u) = n - 1$ since $|V(G)|= n$. Therefore, $\Delta(G) = n - 1$. For the converse, let $\Delta(G) = n - 1$. Then there exists a vertex $u \in V(G)$ such that $\deg(u) = n - 1$. Since every vertex $v \in V(G) \setminus \{u\}$ is dominated by exactly one vertex u, $S = \{u\}$ is a perfect dominating set of G. By Proposition 4, $\gamma_{p0}(G) = |S| = 1$. □

The next result follows from Theorem 3 since $\Delta(K_n) = n - 1$.

Corollary 4. *For the complete graph K_n, where $n \geq 1$, $\gamma_{p0}(K_n) = 1$.*

Theorem 4. *Let G be connected graph of order $n \geq 4$. Then $\gamma_{p0}(G) = 2$ if and only if there exists two vertices $x, y \in S \subseteq V(G)$ such that $N_G(x) \cap N_G(y) = \varnothing$ and $N_G[x] \cup N_G[y] = V(G)$.*

Proof. Suppose that $\gamma_{p0}(G) = 2$. Let $S \subseteq V(G)$ be a perfect isolate dominating set of G. Then $|S| = 2$ and so, there exist two vertices $x, y \in S$ which are isolated vertices in $\langle S \rangle$ such that each vertex $v \in V(G) \setminus S$ is dominated only by either x or y but not both. Thus, $N_G(x) \cap N_G(y) = \varnothing$ and $N_G[S] = N_G[x] \cup N_G[y] = V(G)$.

For the converse, suppose that $N_G(x) \cap N_G(y) = \varnothing$ and $N_G[x] \cup N_G[y] = V(G)$. It follows that the graph G cannot be dominated by 1 vertex only, that is, $\gamma_{p0}(G) > 1$ and each vertex $v \in V(G) \setminus S$ is dominated only by either x or y but not both and $\langle S \rangle$ has two isolated vertices. Clearly, $S = \{x, y\}$ is a perfect isolate dominating set of G. Therefore, $\gamma_{p0}(G) = |S| = 2$. □

Theorem 5. *For any path P_n of order $n \geq 1$,*

$$\gamma_{p0}(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Proof. Suppose that $V(P_n) = \{u_1, u_2, \ldots, u_{n-1}, u_n\}$ such that $\deg(u_1) = \deg(u_n) = 1$ and $\deg(u_i) = 2$ for all $i = 2, 3, \ldots, n - 1$. Note that by Proposition 1, $\gamma_0(P_n) = \lceil \frac{n}{3} \rceil$. Consider the following cases:

Case 1: Suppose that $n \equiv 0 \pmod 3$. If $n = 3$, by Corollary 3, $\gamma_{p0}(P_3) = 1 = \lceil \frac{3}{3} \rceil$. Suppose that $n > 3$. Let $r = \frac{n}{3}$ and $j = 1, 2, \ldots, r-1, r$. Group the vertices of P_n into r disjoint subsets S_j

$$\begin{aligned}
S_1 &= \{u_1, u_2, u_3\} \\
S_2 &= \{u_4, u_5, u_6\} \\
S_3 &= \{u_7, u_8, u_9\} \\
S_4 &= \{u_{10}, u_{11}, u_{12}\} \\
&\vdots \\
S_{r-1} &= \{u_{n-5}, u_{n-4}, u_{n-3}\} \\
S_r &= \{u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

Clearly, the set $S = \{u_2, u_5, u_8, u_{11}, \ldots, u_{n-4}, u_{n-1}\}$ is a γ_0-set of P_n since $N[S] = V(P_n)$, $\langle S \rangle$ has isolated vertices and $|S| = \lceil \frac{n}{3} \rceil$ by Proposition 1. Clearly, all other vertices $u_i \in V(P_n) \setminus S$ for all $i = 1, 3, 4, 6, \ldots, n-5, n-3, n-2, n$ are dominated by exactly one vertex in S. By Proposition 6, $\gamma_{p0}(P_n) = \lceil \frac{n}{3} \rceil$.

Case 2: Suppose that $n \equiv 1 \pmod 3$. If $n = 1$, then $\gamma_{p0}(P_1) = 1 = \lceil \frac{1}{3} \rceil$. Suppose that $n = 4$. Clearly, $S = \{u_1, u_4\}$ is a γ_0-set of P_4 since $|S| = \lceil \frac{4}{3} \rceil = 2$. Clearly, $N(u_1) \cap N(u_4) = \varnothing$ and $N[u_1] \cup N[u_4] = V(P_4)$. By Theorem 4, $\gamma_{p0}(P_4) = 2 = \lceil \frac{4}{3} \rceil$. Suppose that $n > 4$. Let $r = \frac{n+2}{3}$ and $j = 1, 2, \ldots, r-1, r$. Group the vertices of P_n into r disjoint subsets S_j

$$\begin{aligned}
S_1 &= \{u_1\} \\
S_2 &= \{u_2, u_3, u_4\} \\
S_3 &= \{u_5, u_6, u_7\} \\
S_4 &= \{u_8, u_9, u_{10}\} \\
&\vdots \\
S_{r-1} &= \{u_{n-5}, u_{n-4}, u_{n-3}\} \\
S_r &= \{u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

Clearly, the set $S = \{u_1, u_4, u_7, u_{10}, \ldots, u_{n-3}, u_n\}$ is a γ_0-set of P_n since $N[S] = V(P_n)$, $\langle S \rangle$ has isolated vertices and $|S| = \lceil \frac{n}{3} \rceil$ by Proposition 1. Clearly, all other vertices $u_i \in V(P_n) \setminus S$ for all $i = 2, 3, 5, 6, \ldots, n-5, n-4, n-2, n-1$ are dominated by exactly one vertex in S. By Proposition 6, $\gamma_{p0}(P_n) = \lceil \frac{n}{3} \rceil$.

Case 3: Suppose that $n \equiv 2 \pmod 3$. Suppose that $n = 2$. Clearly, $\Delta(P_2) = 1$. By Theorem 3, $\gamma_{p0}(P_2) = 1 = \lceil \frac{2}{3} \rceil$. Suppose that $n > 2$. Let $r = \frac{n+1}{3}$ and $j = 1, 2, \ldots, r-1, r$. Group the vertices of P_n into r disjoint subsets S_j

$$\begin{aligned}
S_1 &= \{u_1, u_2\} \\
S_2 &= \{u_3, u_4, u_5\} \\
S_3 &= \{u_6, u_7, u_8\} \\
S_4 &= \{u_9, u_{10}, u_{11}\} \\
&\vdots \\
S_{r-1} &= \{u_{n-5}, u_{n-4}, u_{n-3}\} \\
S_r &= \{u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

Clearly, the set $S = \{u_1, u_4, u_7, u_{10}, \ldots, u_{n-4}, u_{n-1}\}$ is a γ_0-set of P_n since $N[S] = V(P_n)$, $\langle S \rangle$ has isolated vertices and $|S| = \lceil \frac{n}{3} \rceil$ by Proposition 1. Clearly, all other vertices $u_i \in V(P_n) \setminus S$ for all $i = 2, 3, 5, 6, \ldots, n-5, n-3, n-2, n$ are dominated by exactly one vertex in S. By Proposition 6, $\gamma_{p0}(P_n) = \lceil \frac{n}{3} \rceil$. Therefore, in any case, $\gamma_{p0}(P_n) = \lceil \frac{n}{3} \rceil$. □

Theorem 6. *For a cycle C_n of order $n = 3$ or $n \geq 6$,*

$$\gamma_{p0}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2 \pmod 3 \\ \lceil \frac{n}{3} \rceil, & \text{if } n \not\equiv 2 \pmod 3 \end{cases}$$

Proof. Suppose that $V(C_n) = \{u_1, u_2, \ldots, u_{n-1}, u_n\}$ such that $\deg(u_i) = 2$ for all $i = 1, 2, 3, \ldots, n-1, n$. By Proposition 1, $\gamma_0(C_n) = \lceil \frac{n}{3} \rceil$. Consider the following cases:

Case 1: Suppose that $n \equiv 0 \pmod 3$. If $n = 3$, by Corollary 3, $\gamma_{p0}(C_3) = 1 = \lceil \frac{3}{3} \rceil$. Suppose that $n > 3$. Let $r = \frac{n}{3}$ and $j = 1, 2, \ldots, r-1, r$. Group the vertices of C_n into r disjoint subsets T_j

$$\begin{aligned}
T_1 &= \{u_1, u_2, u_3\} \\
T_2 &= \{u_4, u_5, u_6\} \\
T_3 &= \{u_7, u_8, u_9\} \\
T_4 &= \{u_{10}, u_{11}, u_{12}\} \\
&\vdots \\
T_{r-1} &= \{u_{n-5}, u_{n-4}, u_{n-3}\} \\
T_r &= \{u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

Clearly, the set $S = \{u_2, u_5, u_8, u_{11}, \ldots, u_{n-4}, u_{n-1}\}$ is a γ_0-set of C_n since $N[S] = V(C_n)$, $\langle S \rangle$ has isolated vertices and $|S| = \lceil \frac{n}{3} \rceil$ by Proposition 1. Clearly, all other vertices $u_i \in V(C_n) \setminus S$ for all $i = 1, 3, 4, 6, \ldots, n-5, n-3, n-2, n$ are dominated by exactly one vertex in S. By Proposition 6, $\gamma_{p0}(C_n) = \lceil \frac{n}{3} \rceil$.

Case 2: Suppose that $n \equiv 1 \pmod 3$. Suppose that $n = 7$. By Proposition 1, $\gamma_0(C_7) = 3$. Clearly, $S = \{u_1, u_2, u_5\}$ is a γ_0-set since $N[S] = V(C_7)$, $\langle S\rangle$ has an isolated vertex u_5 and $|S| = 3$. Also,$u_3, u_4, u_6, u_7 \in V(C_7) \setminus S$ are dominated by exactly one vertex in S. By Proposition 6, $\gamma_{p0}(C_7) = 3 = \lceil \frac{7}{3} \rceil$. Suppose that $n > 7$. Let $r = \frac{n+2}{3}$ and $j = 1, 2, \ldots, r-1, r$. Group the vertices of C_n into r disjoint subsets T_j

$$\begin{aligned}
T_1 &= \{u_1\} \\
T_2 &= \{u_2, u_3, u_4\} \\
T_3 &= \{u_5, u_6, u_7\} \\
T_4 &= \{u_8, u_9, u_{10}\} \\
&\vdots \\
T_{r-1} &= \{u_{n-5}, u_{n-4}, u_{n-3}\} \\
T_r &= \{u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

Clearly, the set $S = \{u_1, u_2, u_5, u_8, \ldots, u_{n-5}, u_{n-2}\}$ is a γ_0-set of C_n since $N[S] = V(C_n)$, $\langle S\rangle$ has isolated vertices and $|S| = \lceil \frac{n}{3} \rceil$ by Proposition 1. Clearly, all other vertices $u_i \in V(C_n) \setminus S$ for all $i = 3, 4, 6, 7, \ldots, n-4, n-3, n-1, n$ are dominated by exactly one vertex in S. By Proposition 6, $\gamma_{p0}(C_n) = \lceil \frac{n}{3} \rceil$.

Case 3: Suppose that $n \equiv 2 \pmod 3$. Suppose that $n = 8$. Clearly, $S = \{u_1, u_3, u_6\}$ is a γ_0-set of C_8 since $N[S] = V(C_8)$ and $|S| = \lceil \frac{8}{3} \rceil = 3$. Note that u_2 is dominated by both u_1 and u_3. Hence, S is not a perfect dominating set. Let $T = S \cup \{u_2\} = \{u_1, u_2, u_3, u_6\}$. Clearly, $|T| = 4 = \lceil \frac{8}{3} \rceil + 1$ and $u_4, u_5, u_7, u_8 \in V(C_8) \setminus T$ are dominated by exactly one vertex in T. Clearly, T is a γ_{p0}-set in C_8 and so, $\gamma_{p0}(C_8) = 4 = \lceil \frac{8}{3} \rceil + 1$. Suppose that $n > 8$. Let $r = \frac{n+1}{3}$ and $j = 1, 2, \ldots, r-1, r$. Group the vertices of C_n into r disjoint subsets T_j

$$\begin{aligned}
T_1 &= \{u_1, u_2\} \\
T_2 &= \{u_3, u_4, u_5\} \\
T_3 &= \{u_6, u_7, u_8\} \\
T_4 &= \{u_9, u_{10}, u_{11}\} \\
&\vdots \\
T_{r-1} &= \{u_{n-5}, u_{n-4}, u_{n-3}\} \\
T_r &= \{u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

Clearly, the set $S = \{u_1, u_3, u_6, u_9, u_{12}, \ldots, u_{n-5}, u_{n-2}\}$ is a γ_0-set of C_n since $N[S] = V(G)$, $\langle S\rangle$ has isolated vertices and $|S| = \lceil \frac{n}{3} \rceil$ by Proposition 1. Clearly, u_2 is dominated by both u_1 and u_3. Hence, S is not a perfect dominating set. Let $T = S \cup \{u_2\} = \{u_1, u_2, u_3, u_6, \ldots, u_{n-5}, u_{n-2}\}$. Clearly, $|T| = \lceil \frac{8}{3} \rceil + 1$ and all other vertices $u_i \in V(C_n) \setminus T$ for all $i = 4, 5, 7, 8, \ldots, n-4, n-3, n-1, n$ are dominated by exactly one vertex in T. Clearly, T is a γ_{p0}-set in C_n and so, $\gamma_{p0}(C_n) = \lceil \frac{n}{3} \rceil + 1$. $\square$

By Proposition 1 and Theorem 6, $\gamma_0(C_n) \leq \gamma_{p0}(C_n)$ for $n = 3$ or $n \geq 6$.

Theorem 7. *The cycle graphs C_4 and C_5 are non-γ_{p0}-graphs.*

Proof. Let $V(C_4) = \{u_1, u_2, u_3, u_4\}$ and $V(C_5) = \{u_1, u_2, u_3, u_4, u_5\}$.

For the graph C_4, clearly, the sets $S_1 = \{u_1, u_2\}$, $S_2 = \{u_2, u_3\}$, $S_3 = \{u_3, u_4\}$, and $S_4 = \{u_1, u_4\}$ are perfect dominating sets (γ_p-sets) of C_4 but none of them are isolate dominating sets since $\langle S_i \rangle$ has no isolated vertex for all $i = 1, 2, 3, 4$. Also, the sets $T_1 = \{u_1, u_3\}$ and $T_2 = \{u_2, u_4\}$ are isolate dominating sets (γ_0-sets) of C_4 by Proposition 1 but none of them are perfect dominating sets since all vertices in $V(C_4) \setminus T_j$ are dominated by two vertices in T_j for all $j = 1, 2$. Choosing 3 vertices or 4 vertices in C_4 for γ_{p0}-sets is not possible since their induced subgraphs has no isolated vertex. Thus, in any case, C_4 contains a perfect dominating set or an isolate dominating set but not both. Therefore, C_4 is a non-γ_{p0}-graph.

For the graph C_5, clearly, the sets $S_1 = \{u_1, u_2, u_3\}$, $S_2 = \{u_2, u_3, u_4\}$, $S_3 = \{u_3, u_4, u_5\}$, $S_4 = \{u_4, u_5, u_1\}$, and $S_5 = \{u_5, u_1, u_2\}$ are perfect dominating sets (γ_p-sets) of C_5 but none of them are isolate dominating sets since $\langle S_i \rangle$ has no isolated vertex for all $i = 1, 2, 3, 4, 5$. Also, the sets $T_1 = \{u_1, u_3\}$, $T_2 = \{u_1, u_4\}$, $T_3 = \{u_2, u_4\}$, $T_4 = \{u_2, u_5\}$ and $T_5 = \{u_3, u_5\}$ are isolate dominating sets (γ_0-sets) of C_5 by Proposition 1 but none of them are perfect dominating sets since some vertices in $V(C_5) \setminus T_j$ are dominated by two vertices in T_j for all $j = 1, 2, 3, 4, 5$. Note that the sets $R_1 = \{u_1, u_3, u_5\}$, $R_2 = \{u_1, u_3, u_4\}$, $R_3 = \{u_2, u_3, u_5\}$, $R_4 = \{u_2, u_4, u_5\}$, and $R_5 = \{u_1, u_2, u_4\}$ are isolate dominating sets but none of them are perfect dominating sets since all vertices in $V(C_5) \setminus R_k$ are dominated by two vertices in R_k for all $k = 1, 2, 3, 4, 5$. Choosing 4 or 5 vertices in C_5 for γ_{p0}-sets is not possible since their induced subgraphs has no isolated vertex. Thus, in any case, C_5 contains a perfect dominating set or an isolate dominating set but not both. Therefore, C_5 is a non-γ_{p0}-graph. □

4. The Perfect Isolate Dominating set in the Join of Graphs

This section contains results when the join $G + H$ has a γ_{p0}-set or has no γ_{p0}-set and its perfect isolate domination number.

The *join* of two graphs G and H, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 8. *Let G and H be any graphs with $\gamma(G) = 1$ or $\gamma(H) = 1$. Then*

$$\gamma_{p0}(G + H) = 1.$$

Proof. By Corollary 1, $\gamma(G + H) = 1$. By Proposition 4, $\gamma_{p0}(G + H) = 1$. □

The next result follows from Theorem 8 since $\gamma(K_1) = \gamma(P_2) = \gamma(P_3) = \gamma(C_3) = 1$.

Corollary 5. *The following are graphs having $\gamma_{p0}(G) = 1$.*

i. *Star graph $S_n = K_1 + \overline{K}_n$, $n \geq 2$*

ii. *Fan graph $F_n = K_1 + P_n$, $n \geq 2$*

iii. *Wheel graph $W_n = K_1 + C_n$, $n \geq 3$*

iv. *Friendship graph $F_n = K_1 + nP_2$, $n \geq 2$*

v. *Windmill graph $W_n^m = K_1 + mK_{n-1}$, $n \geq 3$ and $m \geq 2$*

vi. *Complete bipartite graph $K_{m,n} = \overline{K}_m + \overline{K}_n$, either $m = 1$ or $n = 1$.*

vii. *Generalized Fan Graph $F_{m,n} = \overline{K}_m + P_n$, $m \geq 1$ and $n = 2, 3$*

viii. *Generalized Wheel Graph $W_{m,3} = \overline{K}_m + C_3$, $m \geq 1$.*

Theorem 9. *Let G and H be any graphs with $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. Then the graph $G + H$ is a non-γ_{p0}-graph.*

Proof. By Corollary 1, $\gamma(G + H) = 2$. Let S be a γ_{p0}-set of $G + H$. By Proposition 4, $\gamma_{p0}(G + H) > 1$. Thus, $|S| > 1$. Suppose that $x, y \in S$. Consider the following cases:

Case 1: $x, y \in S \subseteq V(G)$.
Clearly, for all $z \in V(H)$, z is dominated in $G + H$ by the two vertices x and y in S . Thus, S is not a perfect dominating set of $G + H$. Thus, S is not a γ_{p0}-set of $G + H$. This is a contradiction. Similarly, if $x, y \in S \subseteq V(H)$, then S is not γ_{p0}-set of $G + H$. This is a contradiction. Therefore, having two vertices in $V(G)$ or in $V(H)$ is not possible for a set S to be a γ_{p0}-set of $G + H$.

Case 2: $x, y \in S$ such that $x \in V(G)$ and $y \in V(H)$.
Clearly, the induced subgraph $\langle S \rangle$ has no isolated vertex since $xy \in E(G + H)$. Hence, S is not an isolate dominating set of $G + H$. Thus, S is not a γ_{p0}-set of $G + H$, a contradiction. Furthermore, adding a vertex in S which is either from $V(G)$ or $V(H)$ is not possible by case 1.

Hence, in any case, S is not a γ_{p0}-set of $G + H$. Therefore, the graph $G + H$ is a non-γ_{p0}-graph if $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. □

The next result follows from Theorem 9.

Corollary 6. *The following graphs are non-γ_{p0}-graphs .*

i. *Complete bipartite graph $K_{m,n} = \overline{K}_m + \overline{K}_n$, $m \geq 2$ and $n \geq 2$.*

ii. *Generalized Fan Graph $F_{m,n} = \overline{K}_m + P_n$, $m \geq 2$ and $n \geq 4$*

iii. *Generalized Wheel Graph $W_{m,n} = \overline{K}_m + C_n$, $m \geq 2$ and $n \geq 4$*

5. The Perfect Isolate Dominating set in the Corona of Graphs

This section contains results when the corona $G + H$ has a γ_{p0}-set or has no γ_{p0}-set and its perfect isolate domination number.

The *corona* of graphs G and H, $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v$, $v \in V(G)$, Harary in [7].

Theorem 10. *Let G be a connected graph and H be any graph. Then a subset S of $V(G \circ H)$ is a perfect isolate dominating set of $G \circ H$ if and only if for every $v \in V(G)$, $S = \bigcup_{v \in V(G)} S_v$ where S_v is a minimal dominating set of H^v and $\gamma(H) = 1$.*

Proof. Let S be a perfect isolate dominating set of $G \circ H$. Suppose that $S = V(G)$. Clearly, S is a perfect dominating set since $N[S] = V(G \circ H)$ and every vertex in $V(H^v)$ is dominated by exactly one vertex $v \in V(G)$ but S is not an isolate dominating set since G is connected, a contradiction. Hence, $S \neq V(G)$. Thus, there exists $v \in V(G) \subseteq V(G \circ H)$ such that $v \notin S$, S contain a vertex or vertices that dominates H^v. Let S_v be a dominating set of H^v and $S_v \subseteq S$. Also, suppose that there exists $y \in V(G)$ such that $y \in S$ and $vy \in E(G) \subseteq E(G \circ H)$. Clearly, $v \in V(G \circ H) \setminus S$ is dominated by y and a vertex in S_v, a contradiction since S is a perfect dominating set. Since y is arbitrary, for all $y \in V(G)$, y must not be an element in S, that is, S must not contain a vertex in $V(G)$. Hence, $S = \cup_{v \in V(G)} S_v$ such that S_v is a dominating set in H^v. Suppose that S_v contains two or more vertices, say x and z, where $x, z \in V(H^v)$. Thus, $v \in V(G \circ H) \setminus S$ is dominated by x and z in $S_v \subseteq S$, a contradiction since S is a perfect dominating set. Thus, $|S_v| = 1$, and so, S_v is a minimal dominating set of H^v and $\gamma(H) = 1$.

Conversely, suppose that for every $v \in V(G)$, $S = \cup_{v \in V(G)} S_v$ where S_v is a minimal dominating set of H^v and $\gamma(H) = 1$. Clearly, $N[S] = V(G \circ H)$ and for every vertex $v \in V(G) = V(G \circ H) \setminus S$, v is dominated by exactly one vertex in $S_v \subseteq S$. Thus, S is a perfect dominating set. Also, since any two vertices in S is not adjacent in $G \circ H$, $\langle S \rangle$ has isolated vertices. Thus, S is also an isolate dominating set. Therefore, $S \subseteq V(G \circ H)$ is a perfect isolate dominating set of $G \circ H$. □

The next two results follow from Theorem 10.

Corollary 7. *Let G be a connected graph of order n and H be any graph where $\gamma(H) = 1$. Then*

$$\gamma_{p0}(G \circ H) = n.$$

Corollary 8. *Let G be a connected graph and H be any graph where $\gamma(H) \geq 2$. Then $G \circ H$ is a non-γ_{p0}-graph.*

The next two results follow from Corollary 7.

Corollary 9. *Let G be a connected graph and K_n be a complete graph. Then*

$$\gamma_{p0}(G \circ K_n) = |V(G)|.$$

Corollary 10. *Let G be a connected graph and H be any graph described in Corollary 5. Then*

$$\gamma_{p0}(G \circ H) = |V(G)|.$$

The next result follows from Corollary 8.

Corollary 11. *Let G be a connected graph and H be any graph described in Corollary 6. Then $G \circ H$ is non-γ_{p0}-graphs.*

The next result follows from Corollary 7 and Corollary 8.

Corollary 12. *Let G be a connected graph. Then for $n \geq 4$, $G \circ P_n$ and $G \circ C_n$ are non-γ_{p0}-graphs and for $n < 4$,*

$$\gamma_{p0}(G \circ P_n) = \gamma_{p0}(G \circ C_3) = |V(G)|.$$

6. The Perfect Isolate Dominating set in the Lexicographic Product of Graphs

This section contains results when the lexicographic product $G[H]$ has a γ_{p0}-set or has no γ_{p0}-set and its perfect isolate domination number.

The *lexicographic product* or *composition* of two graphs G and H is the graph $G[H]$ with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(x,u)(y,v) \in E(G[H])$ if and only if $xy \in E(G)$ or $x = y$ and $uv \in E(H)$. Any subset C of $V(G[H])$ can be expressed as $C = \bigcup_{x\in S} (\{x\} \times T_x)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. S is called as *G-projection* of C and $\cup_{x\in S}T_x$ is called as the *H-projection* of C.

Theorem 11. *Let G and H be connected nontrivial graphs. A subset $C = \bigcup_{x\in S} (\{x\} \times T_x)$ of $V(G[H])$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a perfect isolate dominating set of $G[H]$ if and only if S is an independent perfect dominating set (ipds) and T_x is a dominating set of H with $|T_x| = 1$ for all $x \in S$.*

Proof. Suppose that C is a perfect isolate dominating set of $G[H]$. Let $u \in V(G) \setminus S$. Pick any $v \in V(H)$. Since C is a perfect dominating set, there exists $(y,z) \in C$ such that $N_{G[H]}(u,v) \cap C = \{(y,z)\}$. This implies that $N_G(u) \cap S = \{y\}$. This implies that every $u \in V(G) \setminus S$ is dominated by exactly one vertex in S and so, S is a perfect dominating set. Suppose that $p, q \in S$ such that $pq \in E(G)$. Since H is connected, there exists vertices

$m, n \in V(H)$ such that $mn \in E(H)$, $(p,m)(q,m) \in E(G[H])$ and $(p,m),(q,m) \in C$, a contradiction since the vertices (p,n) and (q,n) are both dominated by the two vertices (p,m) and (q,m) in C. Hence, for any two vertices $p, q \in S$, $pq \notin E(G)$. Thus, S is an independent set. Let $x \in S$ and suppose that $|T_x| \geq 2$. Let $c, d \in T_x$ such that $c \neq d$. Thus, $(x,c),(x,d) \in C$. Let $e \in V(G)$ such that $xe \in E(G)$. Since S is independent, $e \notin S$. Clearly, $(e,c) \notin C$ and (e,c) is dominated by the two vertices (x,c) and (x,d) in C, a contradiction since C is a perfect dominating set. Hence, $|T_x| = 1$ for each $x \in S$. Furthermore, since S is an independent dominating set and C is a dominating set, T_x is a dominating set in H for each $x \in S$.

Conversely, suppose that S is an independent perfect dominating set (ipds) and T_x is a dominating set of H with $|T_x| = 1$ for all $x \in S$. Let $T_x = \{t\}$ such that t dominates all other vertices of H. Let $(p,q) \in V(G[H]) \setminus C$ and consider the following cases:

Case 1: $p \notin S$.
Since S is a perfect dominating set, there exists $x \in S$ such that $S \cap N_G(p) = \{x\}$. Then $\{(x,t)\} = N_{G[H]}((p,q)) \cap C$, that is, the point $(p,q) \in V(G[H]) \setminus C$ is dominated by exactly one vertex (x,t) in C. Hence, C is a perfect dominating set.

Case 2: $p \in S$.
Since $T_x = \{t\}$ and S is a perfect dominating set, $\{(p,t)\} = N_{G[H]}((p,q)) \cap C$, that is, the point $(p,q) \in V(G[H]) \setminus C$ is dominated by exactly one vertex (p,t) in C. Hence, C is a perfect dominating set.

Since S is independent set and T_x is a dominating set with $|T_x| = 1$, C is also independent set and so, $\langle C \rangle$ contains isolated vertices. Accordingly, C is a perfect isolate dominating set of $G[H]$. □

The next two results follow from Theorem 11.

Corollary 13. *Let G and H be connected graphs. Then*

$$\gamma_{p0}(G[H]) = \begin{cases} \gamma_{p0}(G), & \text{if } H = K_1 \\ \gamma_{p0}(H), & \text{if } G = K_1 \\ \gamma_{ip}(G), & \text{if } G \text{ has an ipds and } \gamma(H) = 1. \end{cases}$$

Corollary 14. *Let G and H be connected graphs. Then $G[H]$ is a non-γ_{p0}-graph if and only if one of the two cases is satisfied:*
(i) $\gamma(H) \geq 2$
(ii) $\gamma(H) = 1$ and G has no independent perfect dominating set.

The next two results follow from Corollary 13.

Corollary 15. *Let G be a connected graph such that G has an ipds and K_n be a complete graph. Then*

$$\gamma_{p0}(G[K_n]) = \begin{cases} \gamma_{p0}(G), & \text{if } n = 1 \\ \gamma_{ip}(G), & \text{if } n > 1. \end{cases}$$

Corollary 16. *Let G be a connected graph such that G has an ipds and H be any graph described in Corollary 5. Then*

$$\gamma_{p0}(G[H]) = \gamma_{ip}(G).$$

The next result follows from Corollary 14 (ii).

Corollary 17. *Let G be a connected graph such that G has no ipds and H be any graph. Then $G[H]$ is non-γ_{p0}-graphs.*

The next result follows from Corollary 14 (i).

Corollary 18. *Let G be a connected graph and H be any graph described in Corollary 6. Then $G[H]$ is non-γ_{p0}-graphs.*

The next result follows from Corollary 13 and Corollary 14.

Corollary 19. *Let G be a connected graph such that G has an ipds. Then for $n \geq 4$, $G[P_n]$ and $G[C_n]$ are non-γ_{p0}-graphs and for $n = 2, 3$,*

$$\gamma_{p0}(G[P_n]) = \gamma_{p0}(G[C_3]) = \gamma_{ip}(G).$$

7. Conclusion

The paper has introduced the concept of perfect isolate dominating sets of some graphs resulting from some binary operations such as join, corona, and lexicographic product of two graphs. The existence of the perfect isolate dominating sets of some graphs and some binary operations are examine because not all graphs have this set. For future investigation, the authors recommend to explore this parameter to determine the exact values of some graphs and some binary operations that have not discussed in the study. Moreover, look for the relationship with other parameters of domination which are related to this parameter.

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