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Weakly Convex Hop Dominating Sets in Graphs

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Abstract. Let G be an undirected connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A set $C \subseteq V(G)$ is called weakly convex hop dominating if for every two vertices $x, y \in C$, there exists an x-y geodesic $P(x, y)$ such that $V(P(x, y)) \subseteq C$ and for every $v \in V(G) \setminus C$, there exists $w \in C$ such that $d_G(v, w) = 2$. The minimum cardinality of a weakly convex hop dominating set of G, denoted by $\gamma_{wconh}(G)$, is called the weakly convex hop domination number of G. In this paper, we introduce and initially investigate the concept of weakly convex hop domination. We show that every two positive integers a and b with $3 \leq a \leq b$ are realizable as the weakly convex hop domination number and convex hop domination number of some connected graph. Furthermore, we characterize the weakly convex hop dominating sets in some graphs under some binary operations.

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Key Words and Phrases: Weakly convex set, weakly convex hop dominating set, weakly convex hop domination number

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1. Introduction

Domination has been one of the interesting and widely studied topics in graph theory. A number of concepts had already been used to introduce variations of the standard concept of domination. In particular, concepts such as convex and weakly convex had been considered to define convex domination and weakly convex domination, respectively (see [4], [12], [14], [15], [16], [17], [22], [23]). Some variations of domination can be found in [5], [6], [10], and [20].

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Recently, Natarajan et al. in [19] introduced and studied the concept of hop domination. Follow-up studies are done in [2], [3], [11], and [13]. Since its introduction, a number of variations of the concept have already been defined and studied (see [7], [8], [18], [21], and [24]). Previously, the authors (see [9]) introduced and made an initial investigation of convex hop domination. Motivated by the studies on weakly convex domination and hop domination, we introduce and study a new variant of hop domination that incorporates the concept of weakly convex. This new parameter lies between the connected hop domination number and convex hop domination number of a graph.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple undirected graph and let u and v be vertices of G. The distance $d_G(u, v)$ of u and v is the length of a shortest path joining them. Any u-v path of length $d_G(u, v)$ is called a *u-v geodesic*. The interval $I_G[u, v]$ consists of u, v, and all vertices lying on a u-v geodesic. The interval $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$. Vertices u and v are adjacent (or neighbors) if $uv \in E(G)$. The set of neighbors of vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. The *open neighborhood* of $X \subseteq V(G)$ is the set $N_G(X) = \bigcup_{w\in X} N_G(w)$. The *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V(G)$ is *dominating* (*total dominating*) in G if for every $v \in V(G) \setminus D$ (resp. $v \in V(G)$), there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$ (resp. $N_G(D) = V(G)$).

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N^2_G(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u is given by $N^2_G[u] = N^2_G(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N^2_G(X) = \bigcup_{u\in X} N^2_G(u)$. The *closed hop neighborhood* of X is the set $N^2_G[X] = N^2_G(X) \cup X$.

A set $S \subseteq V(G)$ is *hop dominating* (*total hop dominating*) in G if $N^2_G[S] = V(G)$ (resp. $N^2_G(S) = V(G)$), that is, for every $v \in V(G)\setminus S$ (resp. $v \in V(G)$), there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating (resp. total hop dominating) sets in G, denoted by $\gamma_h(G)$ (resp. $\gamma_{th}(G)$), is called the *hop domination number* (resp. *total hop domination number*) of G. Any hop dominating (resp. total hop dominating) set with cardinality equal to $\gamma_h(G)$ (resp. $\gamma_{th}(G)$) is called a γ_h-set (resp. γ_{th}-set). A hop dominating set S is *connected hop dominating* if $\langle S\rangle$ is connected. The minimum cardinality among all connected hop dominating sets in G, denoted by $\gamma_{ch}(G)$, is called the *connected hop domination number* of G. Any connected hop dominating set with cardinality equal to $\gamma_{ch}(G)$ is called a γ_{ch}-set.

A set $C \subseteq V(G)$ is *convex* if for every two vertices $x, y \in C$, $I_G[x, y] \subseteq C$. The largest cardinality of a proper convex set in G, denoted by $con(G)$, is called the *convexity number* of G. A set $C \subseteq V(G)$ is *convex dominating* (resp. *convex hop dominating*) if C is both convex and dominating (resp. convex and hop dominating). The minimum cardinality among all convex dominating (resp. convex hop dominating) sets in G, denoted by $\gamma_{con}(G)$ (resp. $\gamma_{conh}(G)$), is called the *convex domination number* (resp. *convex hop domination number*) of G. Any convex dominating (resp. convex hop dominating set) with cardinality equal to $\gamma_{con}(G)$ (resp. $\gamma_{conh}(G)$) is called a γ_{con}-set (resp. γ_{conh}-set).

A set $W \subseteq V(G)$ is *weakly convex* if for every two vertices $x, y \in W$, there exists an x-y geodesic $P(x, y)$ such that $V(P(x, y)) \subseteq W$. The largest cardinality of a proper weakly convex set in G, denoted by $wcon(G)$, is called the *weakly convexity number* of G. A set $W \subseteq V(G)$ is *weakly convex dominating* (resp. *weakly convex hop dominating*, *weakly convex total hop dominating*) if C is both weakly convex and dominating (resp. weakly convex and hop dominating, weakly convex and total hop dominating). The minimum cardinality among all weakly convex dominating (resp. weakly convex hop dominating, weakly convex total hop dominating) sets in G, denoted by $\gamma_{wcon}(G)$ (resp. $\gamma_{wconh}(G)$, $\gamma_{wconth}(G)$), is called the *weakly convex domination number* (resp. *weakly convex hop domination number*, *weakly convex total hop domination number*) of G. Any weakly convex dominating (resp. weakly convex hop dominating, weakly convex total hop dominating) set with cardinality equal to $\gamma_{wcon}(G)$ (resp. $\gamma_{wconh}(G)$, $\gamma_{wconth}(G)$) is called a γ_{wcon}-set (resp. γ_{wconh}-set, γ_{wconth}-set).

A set $C \subseteq V(G)$ is *pointwise non-dominating* if for every $v \in V(G) \setminus C$, there exists $u \in C$ such that $v \notin N_G(u)$. The minimum cardinality of a pointwise non-dominating set in G, denoted by $pnd(G)$, is called the *pointwise non-domination number* of G.

The *shadow graph* $S(G)$ of graph G is constructed by taking two copies of G, say G_1 and G_2, and then joining each vertex $u \in V(G_1)$ to the neighbors of its corresponding vertex $u' \in V(G_2)$.

For a graph G, the *complementary prism* $G\overline{G}$ is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex in $\overline{G}$ corresponding to v. In simple terms, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every vertex $v \in V(G)$.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} \rangle + H^v$. The *lexicographic product* $G[H]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(v, a)(u, b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. Any non-empty set $C \subseteq V(G) \times V(H)$ can be expressed as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Specifically, $T_x = \{a \in V(H) : (x, a) \in C\}$ for each $x \in S$.

3. Results

Proposition 1. *Let G be any connected graph G on $n \geq 2$ vertices. Then the following hold:*

(*i*) *If S is a weakly convex hop dominating set in G, then the induced graph $\langle S \rangle$ is connected.*

(*ii*) *$\gamma_{ch}(G) \leq \gamma_{wconh}(G) \leq \gamma_{conh}(G)$ and every equality and strict inequality can be attained.*

Proof. (*i*) Let S be a weakly convex hop dominating set in G and let $x, y \in S$ with $x \neq y$. Since S weakly convex, there exists an x-y geodesic $P(x, y)$ such that $V(P(x, y) \subseteq S$. This implies that the induced graph $\langle S \rangle$ is connected.

(*ii*) Since every weakly convex hop dominating set in G is connected hop dominating, $\gamma_{ch}(G) \leq \gamma_{wconh}(G)$. Also, since every convex hop dominating set in G is weakly convex hop dominating, $\gamma_{wconh}(G) \leq \gamma_{conh}(G)$.

For equality, consider the graph G in Figure 1. Let $W = \{v_1, v_2, v_3, v_4\}$. Then W is both a γ_{ch}-set, a γ_{wconh}-set, and a γ_{conh}-set of G. Thus, $\gamma_{ch}(G) = \gamma_{wconh}(G) = \gamma_{conh}(G) = 4$.

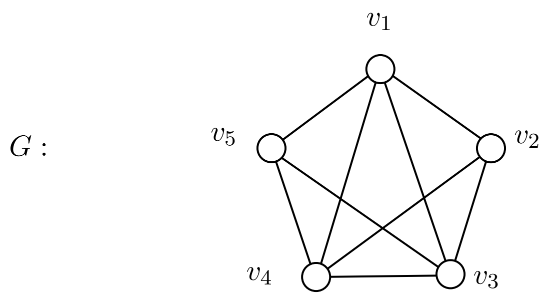

Figure 1: A graph G with $\gamma_{ch}(G) = \gamma_{wconh}(G) = \gamma_{conh}(G) = 4$

For strict inequalities, consider first C_{14}. Then it can be verified that $\gamma_{ch}(C_{14}) = 10$ and $\gamma_{wconh}(C_{14}) = 14$. Thus, $\gamma_{ch}(C_{14}) < \gamma_{wconh}(C_{14})$. Next, consider the graph G' in Figure 2. Let $W' = \{u_1, u_2, u_3\}$ and $W'' = \{u_1, u_2, u_3, u_4, u_5\}$. Then W' and W'' are γ_{wconh}-set and γ_{conh}-set of G', respectively. Accordingly, $\gamma_{wconh}(G') = 3 < 5 = \gamma_{conh}(G')$.

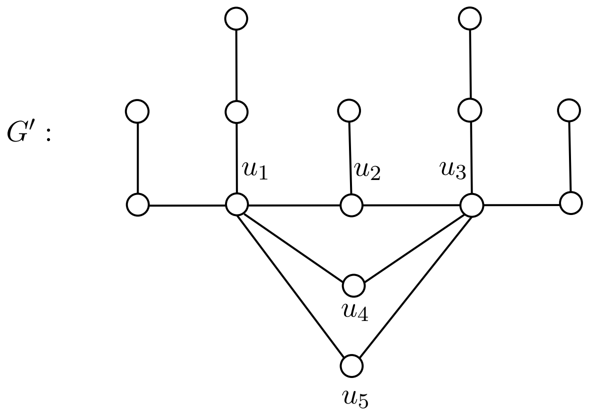

Figure 2: A graph G' with $\gamma_{wconh}(G') = 3 < 5 = \gamma_{conh}(G')$

This completes the proof of the assertion. □

Theorem 1. *Let G be any connected graph on $n \geq 2$ vertices. Then $2 \leq \gamma_{wconh}(G) \leq n$. Moreover, each of the following holds:*

(*i*) *$\gamma_{wconh}(G) = 2$ if and only if there exist two adjacent vertices x and y of G such that $N_G(x) \cap N_G(y) = \varnothing$ and for each $z \in V(G) \setminus N_G(\{x, y\})$, there exists $w \in [N_G(\{x, y\}) \setminus \{x, y\}] \cap N_G(z)$.*

(*ii*) *$\gamma_{conh}(G) = n$ if and only if for every connected hop dominating set $S \neq V(G)$, there exists a pair of distinct vertices x and y in S such that $d_G(x, y) < d_{\langle S\rangle}(x, y)$.*

Proof. Since G is a non-trivial graph, every hop dominating has at least two elements. Hence, $2 \leq \gamma_{wconh}(G) \leq n$.
(*i*) Suppose that $\gamma_{wconh}(G) = 2$, say $S = \{x, y\}$ is a γ_{wconh}-set in G. By Proposition 1(*i*), $xy \in E(G)$. Since S is a hop dominating set in G, $N_G(x) \cap N_G(y) = \varnothing$. Let $z \in V(G) \setminus N_G(\{x, y\})$. Since S is a hop dominating set, it follows that $z \in N^2_G(\{x, y\})$. We may assume that $z \in N^2_G(x)$. Let $[z, w, x]$ be a z-x geodesic. Since $z \notin N_G(\{x, y\})$, $w \neq y$. Thus, $w \in [N_G(\{x, y\}) \setminus \{x, y\}] \cap N_G(z)$.

For the converse, suppose there exist adjacent vertices x and y of G such that $N_G(x) \cap N_G(y) = \varnothing$ and for each $z \in V(G) \setminus N_G(\{x, y\})$, there exists $w \in [N_G(\{x, y\}) \setminus \{x, y\}] \cap N_G(z)$. Let $S = \{x, y\}$ and let $v \in V(G) \setminus S$. If $v \in N_G(\{x, y\})$, then by assumption $d_G(x, v) = 2$ or $d_G(y, v) = 2$. If $v \in V(G) \setminus N_G(\{x, y\})$, then there exists $w \in [N_G(\{x, y\}) \setminus \{x, y\}] \cap N_G(v)$ by assumption. With the assumption that $N_G(x) \cap N_G(y) = \varnothing$, $v \in N^2_G(\{x, y\})$. Therefore, S is a weakly convex hop dominating set in G and $\gamma_{wconh}(G) = |S| = 2$.

(ii) Suppose that $\gamma_{wconh}(G) = n$. Suppose further that there exists a connected hop dominating set $S \neq V(G)$ such that $d_G(a,b) = d_{\langle S\rangle}(a,b)$ for each pair of distinct vertices a and b in S. Since $\langle S\rangle$ is connected, there exists an a-b geodesic in $\langle S\rangle$ for each pair of distinct vertices a and b in S. Hence, S is a weakly convex hop dominating set in G, contrary to the assumption that $\gamma_{wconh}(G) = n$. Therefore, for each connected hop dominating set S in G, there exist distinct vertices x and y in S such that $d_G(x,y) < d_{\langle S\rangle}(x,y)$.

Conversely, suppose that for each connected hop dominating set S in G, there exist distinct vertices x and y in S such that $d_G(x,y) < d_{\langle S\rangle}(x,y)$. Let S_0 be a γ_{wconh}-set in G. Suppose $S_0 \neq V(G)$. Then, by Proposition 1(i), S_0 is a connected hop dominating set. Hence, by assumption, there exists a pair of distinct vertices $p, q \in S_0$ such that $d_G(p,q) < d_{\langle S_0\rangle}(p,q)$. This implies that every p-q path in $\langle S_0\rangle$ is not a geodesic in G. Hence, S_0 is not a weakly convex set, a contradiction. Therefore, $S_0 = V(G)$, showing that $\gamma_{wconh}(G) = n$. □

Since $\gamma_h(K_n) = n$, it follows that $\gamma_{wconh}(G) = n$. The same conclusion can be deduced from Theorem 1 because $V(K_n)$ is the only (connected) hop dominating set of K_n.

Corollary 1. *Let n be a positive integer. Then $\gamma_{wconh}(K_n) = n$.*

Observe that for a non-trivial connected graph G, $\gamma_{ch}(G) = 2$ (or $\gamma_{conh}(G) = 2$) is equivalent to the condition given in Theorem 1(i). The next result states this formally.

Corollary 2. *Let G be any connected graph on $n \geq 2$ vertices. Then the following statements are equivalent:*

(i) $\gamma_{ch}(G) = 2$.

(ii) $\gamma_{wconh}(G) = 2$.

(iii) $\gamma_{conh}(G) = 2$.

Theorem 2. *Let a and b be positive integers such that $3 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_{wconh}(G) = a$ and $\gamma_{conh}(G) = b$.*

Proof. For $a = b$, consider the complete graph $K_a = G$. Then

$$\gamma_{wconh}(G) = a = \gamma_{conh}(G).$$

Suppose $a < b$. Consider the following two cases:
Case 1: $a = 3$.
Let $m = b - a$ and consider the graph G in Figure 3. Let $W = \{w_1, w_2, w_3\}$ and $W' = \{w_1, w_2, w_3, v_1, v_2, \ldots, v_m\}$. Then W and W' are γ_{wconh}-set and γ_{conh}-set in G, respectively. Therefore, $\gamma_{wconh}(G) = a$ and $\gamma_{conh}(G) = a + m = b$.
Case 2: $a \geq 4$.
Let $m = b - a$ and consider the graph G' in Figure 4. Let $W_1 = \{x_1, x_2, \ldots, x_a\}$ and $W_2 = \{x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_m\}$. Then W_1 and W_2 are γ_{wconh}-set and γ_{conh}-set in G', respectively. Hence, $\gamma_{wconh}(G') = a$ and $\gamma_{conh}(G') = a + m = b$.

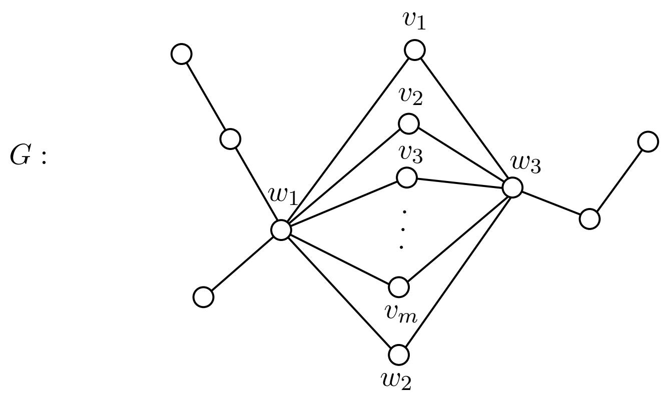

Figure 3: A graph G with $\gamma_{wconh}(G) < \gamma_{conh}(G)$.

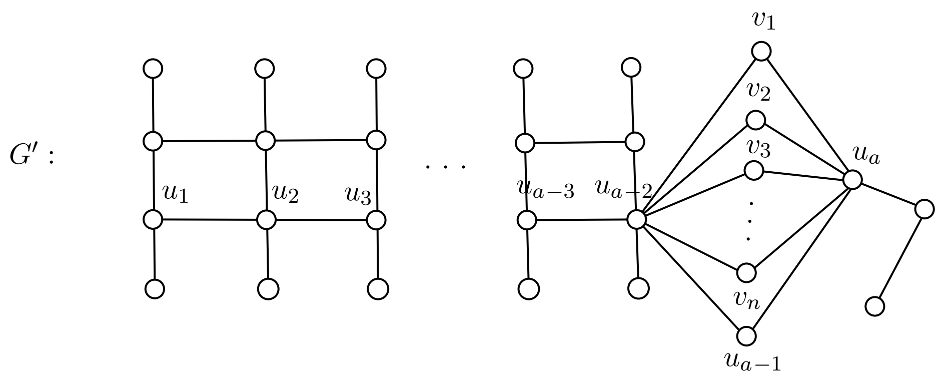

Figure 4: A graph G' with $\gamma_{wconh}(G') < \gamma_{conh}(G')$.

Therefore, the assertion holds. □

Corollary 3. *Let n be a positive integer. Then there exists a connected graph G such that $\gamma_{conh}(G) - \gamma_{wconh}(G) = n$. In other words, $\gamma_{conh} - \gamma_{wconh}$ can be made arbitrarily large.*

Proposition 2. *Let n be any positive integer. Then each of the following holds.*

$$(i)\ \ \gamma_{wconh}(P_n) = \begin{cases} 2 & if\ n = 2,3,4,5,6 \\ n-4 & if\ n \geq 7. \end{cases}$$

$$(ii)\ \ \gamma_{wconh}(C_n) = \begin{cases} 2 & if\ n = 4,5 \\ 3 & if\ n = 3 \\ n-4 & if\ 6 \leq n \leq 10 \\ n & if\ n \geq 11. \end{cases}$$

Proof. (i) Clearly, $\gamma_{wconh}(P_n) = 2$ for $n \in \{2,3,4,5,6\}$. Suppose $n \geq 7$. Let $P_n = [v_1, v_2, \dots, v_n]$ and consider $W = \{v_3, v_4, \dots, v_{n-3}, v_{n-2}\}$. Clearly, W is weakly convex set in P_n. Now, observe that $N_G^2[W] = V(P_n)$ and so W is a hop dominating set in P_n. Therefore, W is a weakly convex hop dominating set in P_n. Notice that every weakly convex hop dominating set in P_n contains W. It follows that W is a γ_{wconh}-set of P_n. Hence, $\gamma_{wconh}(P_n) = n - 4$ for all $n \geq 7$.

(ii) Clearly, $\gamma_{wconh}(C_n) = 2$ for $n \in \{4,5\}$ and $\gamma_{wconh}(C_3) = 3$. Suppose $6 \leq n \leq 10$. Let $C_n = [v_1, v_2, \dots, v_n, v_1]$ and let $W' = \{v_1, v_2, \dots, v_{n-4}\}$. Then W' is a minimum weakly convex hop dominating set in C_n. Thus, $\gamma_{wconh}(C_n) = n - 4$ for all n, where $6 \leq n \leq 10$. Next, suppose that $n \geq 11$. Clearly, $\gamma_{ch}(C_n) = n - 4$ and if S is a connected hop dominating set in C_n with $S \neq V(C_n)$, then S satisfies the property given in Theorem 1(ii). Hence, $\gamma_{wconh}(C_n) = n$. □

Theorem 3. *[9] Let G be a connected graph of order n. Then $\gamma_{conh}(G\overline{G}) = 2$. In particular, $\{u, \overline{u}\}$ is a γ_{conh}-set of $G\overline{G}$ for each $u \in V(G)$.*

The next result is immediate from Corollary 2 and Theorem 3.

Theorem 4. *Let G be any connected graph of order n. Then $\gamma_{wconh}(G\overline{G}) = 2$. In particular, $W = \{a, \overline{a}\}$ is a γ_{wconh}-set of $G\overline{G}$ for any $a \in V(G)$.*

Theorem 5. *[9] Let G be a non-trivial connected graph. Then S is a hop dominating set in $S(G)$ if and only if one of the following conditions holds:*

(i) *S is a hop dominating set in G_1.*

(ii) *S is a hop dominating set in G_2.*

(iii) *$S = S_{G_1} \cup S_{G_2}$ such that $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are hop dominating sets in G_1 and G_2, respectively.*

Theorem 6. *Let G be a non-trivial connected graph. Then W is a weakly convex hop dominating set in $S(G)$ if and only if one of the following conditions holds:*

(i) *W is weakly convex hop dominating set in G_1.*

(ii) *W is weakly convex hop dominating set in G_2.*

(iii) *$W = W_{G_1} \cup W_{G_2}$ such that $W_{G_1} \cup W'_{G_2}$ and $W'_{G_1} \cup W_{G_2}$ are weakly convex hop dominating sets in G_1 and G_2, respectively, where $W'_{G_2} = \{a \in V(G_1) : a' \in W_{G_2}\}$ and $W'_{G_1} = \{a \in V(G_2) : a' \in W_{G_1}\}$*

Proof. Let W be a weakly convex hop dominating set in $S(G)$. Let $W_{G_1} = W \cap V(G_1)$ and $W_{G_2} = W \cap V(G_2)$. If $W_{G_2} = \varnothing$, then $W = W_{G_1}$ is a weakly convex hop dominating set in G_1, showing that (i) holds. Similarly, if $W_{G_1} = \varnothing$, then $W = W_{G_2}$ is a weakly convex hop dominating set in G_2. Hence, (ii) holds. Next, suppose $W_{G_1} \neq \varnothing$ and $W_{G_2} \neq \varnothing$. Then $W_{G_1} \cup W'_{G_2}$ is a hop dominating set in G_1 by Theorem 5. Let $x, y \in W_{G_1} \cup W'_{G_2}$ where $x \neq y$. Suppose $x, y \in W_{G_1}$. Since W is weakly convex, there exists an x-y geodesic $P(x, y)$ in $S(G)$ such that $V(P(x, y)) \subseteq W$. Let $P(x, y) = [x_1, x_2, \cdots, x_k]$, where $x_1 = x$ and $x_k = y$. Let $j \in \{2, \cdots, k-1\}$ such that $x_j \in W_{G_2}$, say $x_j = y'_j$. Then $y_j \in W'_{G_2}$. Replacing each $x_j \in W_{G_2}$ by y_j, we obtain an x-y geodesic $P'(x, y)$ in G_1 such that $V(P'(x, y)) \subseteq W_{G_1} \cup W'_{G_2}$. Suppose now that $x, y \in W'_{G_2}$. Then $x', y' \in W_{G_2} \subset W$. By assumption, we may let $P(x', y') = [v_1, v_2, \cdots, v_t]$, where $x' = v_1$ and $y' = v_t$, be an x'-y' geodesic in $S(G)$ such that $V(P(x', y')) \subseteq W$. Let $r \in \{1, 2, \cdots, t\}$ such that $v_r \in W_{G_2}$, say $v_r = z'_r$. Then $z_r \in W'_{G_2}$. Note that $z_1 = x$ and $z_t = y$. Replacing each $v_r \in W_{G_2}$ by z_j, we obtain an x-y geodesic $P^*(x, y)$ in G_1 such that $V(P^*(x, y)) \subseteq W_{G_1} \cup W'_{G_2}$. Finally, suppose that $x \in W_{G_1}$ and $y \in W'_{G_2}$. Then $y' \in W_{G_2}$. Let $P(x, y') = [q_1, q_2, \cdots, q_m]$, where $x = q_1$ and $y' = q_m$, be an x-y' geodesic in $S(G)$ such that $V(P(x, y')) \subseteq W$. Replacing each $q_i = p'_i \in W_{G_2}$ by $p_i \in W'_{G_2}$ $(p_m = y)$, we obtain an x-y geodesic $P^{**}(x, y)$ such that $V(P^{**}(x, y)) \subseteq W_{G_1} \cup W'_{G_2}$. Therefore, $W_{G_1} \cup W'_{G_2}$ is a weakly convex set in G_1. Similarly, $W'_{G_1} \cup W_{G_2}$ is weakly convex in G_2. This shows that (iii) holds.

For the converse, suppose first that (i) or (ii) holds. Then W is hop dominating by Theorem 5. Since weakly convex sets in G_1 and G_2 are weakly convex sets in $S(G)$, W is weakly convex. Suppose (iii) holds. Again, by Theorem 5, W is a hop dominating set in $S(G)$. By the additional assumption, W is weakly convex in $S(G)$. Therefore, W is a weakly convex hop dominating set in $S(G)$. □

The next result follows from Theorem 6.

Corollary 4. *Let G be a non-trivial connected graph. Then $\gamma_{wconh}(S(G)) = \gamma_{wconh}(G)$.*

Theorem 7. *[1] Let G be a graph of order n. Then $1 \leq pnd(G) \leq n$. Moreover,*

(i) *$pnd(G) = 1$ if and only if G has an isolated vertex.*

(ii) *$pnd(G) = n$ if and only if $G = K_n$.*

Corollary 5. *[1] Let n be any positive integer. Then*

(i) *$pnd(P_n) = 2$ for any $n \geq 2$.*

(ii) *$pnd(C_n) = 2$ for any $n \geq 4$.*

Theorem 8. *[13] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is hop dominating set in $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets in G and H, respectively.*

Corollary 6. *[13] Let G and H be any two graphs. Then*

$$\gamma_h(G+H) = pnd(G) + pnd(H).$$

Theorem 9. *Let G and H be any two graphs. Then $W \subseteq V(G+H)$ is weakly convex hop dominating in $G+H$ if and only it is hop dominating in $G+H$.*

Proof. Suppose W is a weakly convex hop dominating set in $G+H$. Then, by definition, W is hop dominating in $G+H$.

For the converse, suppose that W is a hop dominating set in $G+H$. By Theorem 8, $W = W_G + W_H$ where W_G and W_H are pointwise non-dominating sets in G and H, respectively. Let $v, w \in W$ and $v \neq w$. If $d_{G+H}(v,w) = 1$, then $I_{G+H}[v,w] = \{v,w\} \subseteq W$. Suppose that $d_{G+H}(v,w) = 2$. Then $v, w \in W_G$ or $v, w \in W_H$. We may assume that $v, w \in W_G$. Pick any $z \in W_H$. Then $P(v,w) = [v,z,w]$ is a v-w geodesic in $G+H$ and $V(P(v,w)) = \{v,z,w\} \subseteq W$. Thus, W is weakly convex hop dominating in $G+H$. □

The next result follows from Theorem 9 and Corollary 6.

Corollary 7. *Let G and H be two graphs. Then*

$$\gamma_{wconh}(G+H) = pnd(G) + pnd(H).$$

In particular, we have

(i) $\gamma_{wconh}(P_n + P_m) = 4$ *for all* $n, m \geq 2$;

(ii) $\gamma_{wconh}(C_n + C_m) = 4$ *for all* $n, m \geq 4$;

(iii) $\gamma_{wconh}(F_n) = 3$ *for all* $n \geq 2$;

(iv) $\gamma_{wconh}(W_n) = 3$ *for all* $n \geq 4$; *and*

(v) $\gamma_{wconh}(K_{1,n}) = 2$ *for all* $n \geq 1$.

The result that follows is a restatement of a result in [13].

Theorem 10. *Let G and H be any two graphs. A set $C \subseteq V(G)$ is a hop dominating set in $G \circ H$ if and only if $C = A \cup (\cup_{v \in V(G)} C_v)$, where $A \subseteq V(G)$ and $C_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:*

(i) *For each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w,x) = 2$ or there exists $y \in N_G(w)$ with $C_y \neq \varnothing$.*

(ii) *C_w is a pointwise non-dominating set in H^w for each $w \in V(G) \setminus N_G(A)$.*

Theorem 11. *Let G be a non-trivial connected graph and let H be any graph. Then W is a weakly convex hop dominating set in $G \circ H$ if and only if $W = B \cup (\cup_{v \in V(G)} W_v)$, where $B \subseteq V(G)$, $W_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:*

(i) *B is a weakly convex dominating set in G.*

(ii) *$W_v = \varnothing$ for each $v \in V(G) \setminus B$.*

(iii) *For each $a \in V(G) \setminus B$, there exists $b \in B$ with $d_G(a,b) = 2$ or there exists $y \in B \cap N_G(a)$ such that $W_y \neq \varnothing$.*

(iv) W_x *is a pointwise non-dominating set in* H^x *for each* $x \in B \setminus N_G(B)$.

Proof. Suppose W is a weakly convex hop dominating set in $G \circ H$. Then W is hop dominating and $W = B \cup (\cup_{v \in V(G)} W_v)$ where the sets B and $W_v s$ satisfy (i) and (ii) of Theorem 10. Suppose $B = \varnothing$. Then W_v is pointwise non-dominating in H^v for each $v \in V(G)$ by Theorem 10(ii). Let $a, b \in V(G)$ $(a \neq b)$ and choose any $p \in W_a$ and $q \in W_b$. Since every p-q geodesic in $G \circ H$ contains a and b, W is not weakly convex in $G \circ H$, a contradiction. Hence, $B \neq \varnothing$. Let $w \in V(G) \setminus B$ and suppose that $w \notin N_G(B)$. By Theorem 10(ii), W_w is a pointwise non-dominating set in H^w. Pick any $p \in W_w$ and $z \in B$ (z exists because $B \neq \varnothing$). Then there exists no p-z geodesic $P(p, z)$ in $G \circ H$ with $V(P(p, z)) \subseteq W$, contrary to the assumption that W is weakly convex. Thus, B is a dominating set in G. Let x and y be distinct vertices in B. Then $x, y \in W$. Since W is weakly convex in $G \circ H$, there exists an x-y geodesic $P(x, y)$ in $G \circ H$ such that $V(P(x, y) \subseteq W$. Clearly, $P(x, y)$ is an x-y geodesic in G. Hence, $V(P(x, y)) \subseteq B$. Therefore, B is weakly convex in G, showing that (i) holds. Next, let $v \in V(G) \setminus B$. Suppose $W_v \neq \varnothing$, say $p \in W_v$. Choose any $z \in B$. Since every p-z geodesic in $G \circ H$ contains v as a vertex, it follows that W is not weakly convex in $G \circ H$, a contradiction. Therefore, $W_v = \varnothing$, showing that (ii) holds. This and (i) in Theorem 10 imply that (iii) holds. Moreover, because B is a dominating set, and (ii) in Theorem 10 holds, condition (iv) also holds.

For the converse, suppose that $W = B \cup (\cup_{v \in V(G)} W_v)$ and satisfies (i), (ii), (iii), and (iv). Then (i) and (ii) in Theorem 10 hold, that is, W is a hop dominating set in $G \circ H$. Next, let $x, y \in W$ and let $v, w \in V(G)$ such that $x \in V(v + H^v)$ and $y \in V(w + H^w)$. Consider the following cases:

Case 1. $x = v$ and $y = w$.

Then $x, y \in B$. Since B is weakly convex in G, there exists an x-y geodesic $P(x, y)$ in G (also an x-y geodesic in $G \circ H$) such that $V(P(x, y)) \subseteq B \subseteq W$.

Case 2. $x = v$ and $y \in W_w$ (or $x \in W_v$ and $y = w$).

Then $w \in B$ by (iv). By (ii), we may let $P'(x, w)$ be an x-w geodesic in G such that $V(P(x, w)) \subseteq B \subseteq W$. Let $P'(x, w) = [x_1, x_2, \cdots, x_k]$, where $x = x_1$ and $w = x_k$. Then $P^*(x, y) = [x_1, x_2, \cdots, x_k, y]$ is an x-y geodesic in $G \circ H$ and $V(P^*(x, y)) \subseteq W$.

Case 3. $x \in V(H^v)$ and $y \in V(H^w)$.

Then $v, w \in B$ by (iv). Property (ii) will imply that there exists a v-w geodesic $P(v, w)$ in $G \circ H$ such that $V(P(v, w)) \subseteq W$. Let $P(v, w) = [v_1, v_2, \cdots, v_k]$, where $v = v_1$ and $w = v_k$. Then $P(x, y) = [x, x_1, x_2, \cdots, x_k, y]$ is an x-y geodesic in $G \circ H$ and $V(P(x, y)) \subseteq W$.

Therefore, W is weakly convex in $G \circ H$.

Accordingly, W is weakly convex hop dominating in $G \circ H$. □

Corollary 8. *Let G be a non-trivial connected graph and let H be any graph. Then*

$$\gamma_{wcon}(G) \leq \gamma_{wconh}(G \circ H) \leq \gamma^h_{twcon}(G),$$

where

$$\gamma^h_{twcon}(G) = \min\{|S| : S \text{ is hop dominating and weakly convex total dominating in } G\}.$$

Note that the bounds in Corollary 8 are tight. In fact, $\gamma_{wconh}(C_4 \circ H) = 2 = \gamma_{wcon}(C_4)$ for any graph H. Consider the graph G in Figure 5 and the sets $W_1 = \{a, b, c\}$ and $W_2 = W_1 \cup \{v\}$. It can be verified that W_1 is weakly convex dominating, W_2 is weakly convex total dominating and hop dominating, and $\gamma_{wcon}(G) = |W_1|$ and $\gamma^h_{twcon}(G) = |W_2|$. For any graph H, we find that $\gamma_{wcon}(G) < \gamma^h_{twcon}(G) = \gamma_{wconh}(G \circ H) = 4$.

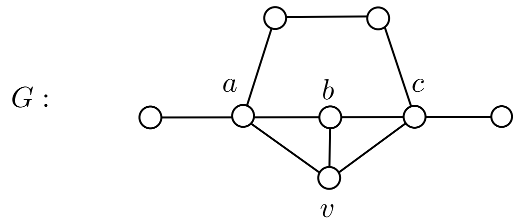

Figure 5: A graph G with $\gamma_{wcon}(G) = 3 < 4 = \gamma^h_{twcon}(G) = \gamma_{wconh}(G \circ H)$

The next result is found in [13].

Theorem 12. *Let G and H be connected non-trivial graphs. Then $C = \bigcup_{x \in S}[\{x\} \times T_x]$ is a hop dominating set in $G[H]$ if and only if the following conditions hold.*

(i) *S is a hop dominating set in G.*

(ii) *T_x is a pointwise non-dominating set in H for each $x \in S \setminus N^2_G(S)$.*

Theorem 13. *Let G and H be connected non-trivial graphs. Then $C = \bigcup_{x \in S}[\{x\} \times T_x]$ is a weakly convex hop dominating set in $G[H]$ if and only if the following conditions hold.*

(i) *S is a weakly convex hop dominating set in G.*

(ii) *T_x is a pointwise non-dominating set in H for each $x \in S \setminus N^2_G(S)$.*

Proof. Suppose C is weakly convex hop dominating in $G[H]$. Then S is a hop dominating set in G and property (ii) holds by Theorem 12(ii). Let $v, w \in S$ with $v \neq w$. Choose any $p \in T_v$ and $q \in T_w$. By assumption, there exists a (v, p)-(w, q) geodesic $P((v, p), (w, q)) = [(v_1, p_1), (v_2, p_2), \cdots, (v_k, p_k)]$ in $G[H]$, where $(v, p) = (v_1, p_1)$ and $(w, q) = (v_k, p_k)$, such that $V(P((v, p), (w, q))) \subseteq C$. This implies that $P(v, w) = [v_1, v_2, \cdots, v_k]$ is a v-w geodesic in G and $V(P(v, w)) \subseteq S$. This shows that S is weakly convex, showing that (i) holds.

For the converse, suppose that C satisfies conditions (i) and (ii). Then C is a hop dominating set $G[H]$ by Theorem 12. Let $(x,a),(y,b) \in C$ with $(x,a) \neq (y,b)$. Consider the following cases:

Case 1. $x \neq y$.
Since S is weakly convex in G, there exists an x-y geodesic $P(x,y) = [x_1, x_2, \cdots, x_t]$, where $x = x_1$ and $y = x_t$, such that $V(P(x,y)) \subseteq S$. Let $a_1 = a$ and $a_t = b$. For each $i \in \{2, \ldots, t-1\}$, choose any $a_i \in T_{x_i}$. Then $P((x,a),(y,b)) = [(x_1,a_1),(x_2,a_2),\cdots,(x_t,a_t)]$ is an (x,a)-(y,b) geodesic in $G[H]$ and $V(P((x,a),(y,b))) \subseteq C$.

Case 2. $x = y$.
Then $a \neq b$ and $d_{G[H]}((x,a),(y,b)) = 2$. Since $\langle S \rangle$ is non-trivial and connected, choose any $z \in S \cap N_G(x)$ and let $c \in T_z$. Then $P((x,a),(y,b)) = [(x,a),(z,c),(y,b)]$ is an (x,a)-(y,b) geodesic in $G[H]$ and $V(P((x,a),(y,b))) \subseteq C$.

Therefore, C is weakly convex in $G[H]$. □

The next result follows from Theorem 13.

Corollary 9. *Let G and H be non-trivial connected graphs. Then* $\gamma_{wconh}(G[H]) = \min\{|S \cap N_G^2(S)| + pnd(H)|S \setminus N_G^2(S)| : S$ *weakly convex hop dominating in* $G\}$. *Moreover, each of the following holds:*

(i) *If* $\gamma(G) \neq 1$, *then* $\gamma_{wconh}(G[H]) \leq \gamma_{wconth}(G)$.

(ii) *If* $G = K_n$, *then* $\gamma_{wconh}(G[H]) = n \cdot pnd(H)$.

Proof. Let $\alpha = \min\{|S \cap N_G^2(S)| + pnd(H)|S \setminus N_G^2(S)| : S$ is a weakly convex hop dominating set in $G\}$. Let S_0 be a weakly convex hop dominating set in G such that $\alpha = |S \cap N_G^2(S)| + pnd(H)|S \setminus N_G^2(S)|$. Choose any $a \in V(H)$ and let D be a *pnd*-set in H. Set $T_x = \{a\}$ if $x \in S_0 \cap N_G^2(S_0)$ and $T_x = D$ if $x \in S_0 \setminus N_G^2(S_0)$. Then $C = \bigcup_{x \in S_0}[\{x\} \times T_x]$ is a weakly convex hop dominating set in $G[H]$ by Theorem 13. Hence, $\gamma_{wconh}(G[H]) \leq |C| = |S \cap N_G(S)| + pnd(H)|S \setminus N_G^2(S)| = \alpha$. Next, suppose that $C_0 = \bigcup_{x \in A}[\{x\} \times R_x]$ is a γ_{wconh}-set in $G[H]$. By Theoren 13, A is weakly convex hop dominating in G and R_x is pointwise non-dominating in H for each $x \in A \setminus N_G^2(A)$. It follows that $\gamma_{wconh}(G[H]) = |C_0| \geq |A \cap N_G^2(A)| + pnd(H)|A \setminus N_G^2(A)| \geq \alpha$, showing the desired equality.

For (i), let S be a weakly convex total hop dominating set in G with $|S| = \gamma_{wconth}(G)$ (S exists because $\gamma(G) \neq 1$). Then $S \cap N_G^2(S) = S$ and $S \setminus N_G^2(S) = \varnothing$. Set $Q_x = \{p\}$, where $p \in V(H)$, for each $x \in S$. Then $C = \bigcup_{x \in S}[\{x\} \times Q_x]$ is a weakly convex hop dominating set in $G[H]$ by Theorem 13. It follows that $\gamma_{wconh}(G[H]) \leq |C| = \gamma_{wconth}(G)$.

For (ii), suppose that $G = K_n$. Let $C = \bigcup_{x \in S}[\{x\} \times T_x]$ be a γ_{wconh}-set in $G[H]$. By Theorem 13, S is a weakly convex hop dominating set in G and T_x is a pointwise non-dominating set in H for each $x \in S \setminus N_G^2(S)$. Since $V(G)$ is the only hop dominating set in G, it follows that $S = V(G)$. Also, since C is a γ_{wconh}-set, each T_x is a *pnd*-set in H. Hence, $\gamma_{wconh}(G[H]) = |C| = n \cdot pnd(H)$. □

Note that the bound in Corollary 9(i) is tight. To see this, consider $G = P_7$. For any connected graph H, $\gamma_{wconh}(P_7[H]) = 4 = \gamma_{wconth}(P_7)$.

4. Conclusion

Weakly convex hop domination was introduced and initially investigated in this study. Characterizations of weakly convex hop dominating sets in the shadow graph, and in the join, corona, and lexicographic product of two graphs were formulated. These characterizations were used to determine the weakly convex hop domination number of each of these graphs. Moreover, it was shown that any two positive integers a and b with $3 \leq a \leq b$ are realizable as weakly connected hop domination number and convex hop domination number, respectively, of some connected graph. The parameter may be studied further for trees and other graphs. Also, bounds involving other known parameters may be obtained. Moreover, interested readers may find the complexity aspect of the weakly convex hop dominating set problem worth considering.

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