EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 3, 2023, 1817-1829
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Connected Outer-Hop Independent Dominating Sets in Graphs Under Some Binary Operations

Jahiri U. Manditong[1], Javier A. Hassan[1,*], Ladznar S. Laja[1], Amy A. Laja[1], Nurijam Hanna M. Mohammad[1], Sisteta U. Kamdon[1]

[1] *Mathematics and Sciences Department, College of Arts and Sciences, MSU Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*

Abstract. Let G be a connected graph. A set $D \subseteq V(G)$ is called a connected outer-hop independent dominating if D is a connected dominating set and $V(G) \setminus D$ is a hop independent set in G. The minimum cardinality among all connected outer-hop independent dominating sets in G, denoted by $\gamma_c^{ohi}(G)$, is called the connected outer-hop independent domination number of G. In this paper, we initiate the study and investigation of connected outer-hop independent domination in some families of graphs and graphs under some binary operations. We construct properties and determine its connections with other known concepts and parameters in graph theory. Moreover, we characterize this type of sets in the join and corona of two graphs, and we use these results to determine the exact values or bounds of the parameters of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Outer-hop independent, connected outer-hop independent dominating set, connected outer-hop independent domination number

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1. Introduction

The concept of domination in a graph has been one of the interesting topics of research in graph theory. Let G be a graph. A subset D of $V(G)$ is called a *dominating* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, a set D is called a dominating set of G if $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in G. Researchers have been studied this concept and introduced new variants by imposing additional conditions to the usual concept of domination. Some studies on domination and its variants can be found in these references [1–6, 8–14].

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i3.4766

Email addresses: jahirimanditong@msutawi-tawi.edu.ph (J. Manditong), javierhassan@msutawi-tawi.edu.ph (J. Hassan), ladznarlaja@msutawi-tawi.edu.ph (L. Laja), amylaja@msutawi-tawi.edu.ph (A. Laja), hannamohammad@msutawi-tawi.edu.ph (N.H. Mohammad) sistetakamdon@msutawi-tawi.edu.ph (S. Kamdon)

Recently, Hassan et al. [7] introduced the concept of hop independent sets in a graph. Let G be a graph. A subset S of $V(G)$ is called a *hop independent* if for every pair of distinct vertices $v, w \in S$, $d_G(v, w) \neq 2$. The maximum cardinality of a hop independent set in G, denoted by $\alpha_h(G)$, is called the hop independence number of G. They have shown that the maximum hop independent set in a graph is a hop dominating set, that is, the hop independence number is at least equal to the hop domination number. Moreover, they have found that that hop independence number is incomparable to the independence number of a graph. In fact, they have shown that the absolute difference between the independence number and hop independence number of a graph can be made arbitrarily large.

In this study, the concept of connected outer-hop independent domination in a graph will be introduced and investigated. This will be investigated for some special graphs including those graphs obtained from some binary operations. Moreover, exact values or bounds for the parameter will be given for some families of graphs and graphs under some binary operations.

2. Terminology and Notation

Let G be a simple graph. Two vertices u, v of a graph G are *adjacent*, or *neighbors*, if uv is an edge of G. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A subset D of $V(G)$ is called a *dominating* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in G. Any dominating set D with cardinality equal to $\gamma(G)$ is called a γ-set of G.

A *dominating set* D of G is called a connected dominating set if the induced subgraph $\langle D \rangle$ of D is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G. Any connected dominating set D with cardinality equal to $\gamma_c(G)$ is called a γ_c-set of G.

A subset B of $V(G)$ is an *independent* if for every pair of distinct vertices $v, w \in B$, $d_G(v, w) \neq 1$. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the independence number of G. Any independent set B with cardinality equal to $\alpha(G)$ is called an α-set of G.

Let G be a connected graph. Then $D \subseteq V(G)$ is called a connected outer-independent dominating set if D is connected dominating set and $V(G) \setminus D$ is an independent set in G. The minimum cardinality of a connected outer-independent dominating set in G, denoted by, $\gamma_c^{oi}(G)$ is called the connected outer-independent domination number of G. Any connected outer-independent dominating set with cardinality equal to $\gamma_c^{oi}(G)$ is called a γ_c^{oi}-set of G.

A subset S of $V(G)$ is called a *hop independent* if for every pair of distinct vertices

$v, w \in S$, $d_G(v, w) \neq 2$. The maximum cardinality of a hop independent set in G, denoted by $\alpha_h(G)$, is called the hop independence number of G. Any hop independent set S with cardinality equal to $\alpha_h(G)$ is called a α_h-set of G.

Let G and H be two graphs. The *join* of G and H, denoted by $G + H$, is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} + H^v \rangle$.

3. Results

We begin this section by introducing the concept of connected outer-hop independent domination in a graph.

Definition 1. Let G be a connected graph. Then $D \subseteq V(G)$ is called a connected outer-hop independent dominating set if D is connected dominating set and $V(G) \setminus D$ is a hop independent set in G. The minimum cardinality of a connected outer-hop independent dominating set in G, denoted by, $\gamma_c^{ohi}(G)$ is called the connected outer-hop independent domination number of G. Any connected outer-hop independent dominating set D with cardinality equal to $\gamma_c^{ohi}(G)$ is called a γ_c^{ohi}-set of G.

It is worth mentioning that every connected graph G admits a connected outer-hop independent domination. The following first remark is the result concerning the relationship between connected domination number and connected outer-hop independent domination number of a graph G.

Remark 1. *Let G be a connected graph. Then $\gamma_c(G) \leq \gamma_c^{ohi}(G)$.*

It is clear since every connected outer-hop independent dominating set is connected dominating.

Remark 2. *The bound given in Remark 1 is tight. Moreover, strict inequality can be attained.*

For the equality, consider the graph H given in Figure 1. Let $D = \{d, g, h, k\}$. Then D is both γ_c-set and γ_c^{ohi}-set of H. Thus, $\gamma_c(H) = 4 = \gamma_c^{ohi}(H)$.

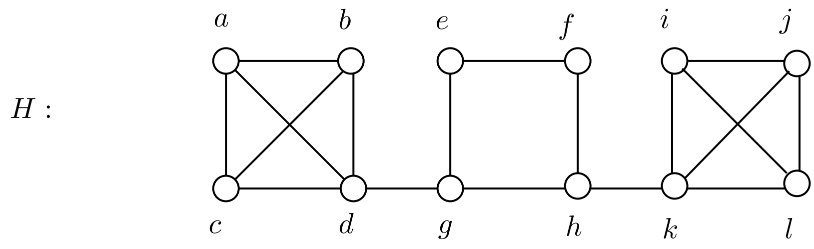

Figure 1: A graph G with $\gamma_c(H) = \gamma_c^{ohi}(H)$

For strict inequality, consider the graph G given in Figure 2. Let $C = \{b, c, f\}$ and $C' = \{b, c, f, g, h\}$. Then C and C' are γ_c-set and γ_c^{ohi}-set of G, respectively. Hence, $\gamma_c(G) = 3 < 5 = \gamma_c^{ohi}(G)$.

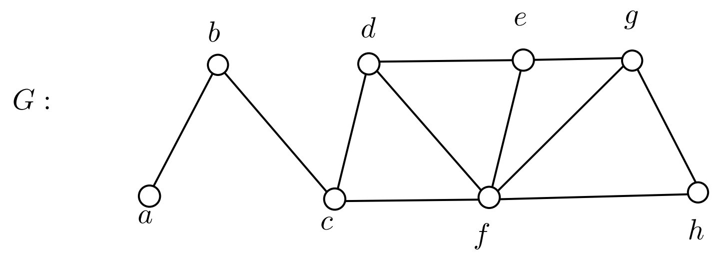

Figure 2: A graph G with $\gamma_c(G) < \gamma_c^{ohi}(G)$

Theorem 1. *Let G be a connected graph on n vertices. Then $1 \leq \gamma_c^{ohi}(G) \leq n - 1$. Moreover, each of the following statements holds.*

(i) $\gamma_c^{ohi}(G) = 1$ *if and only if G is complete.*

(ii) $\gamma_c^{ohi}(G) = 2$ *if and only if for each pair of adjacent vertices $a, b \in V(G)$ such that $N_G[a] \neq N_G[b]$, $D = \{x, y\}$ is a dominating set of G and $V(G) \setminus D$ is hop independent set in G.*

Proof. Let G be any connected graph. Since $\varnothing$ is not a connected outer-hop independent dominating set in G, it follows that $\gamma_c^{ohi}(G) \geq 1$. Let a be a non-cutting vertex of G. Then $V(G) \setminus \{a\}$ is a connected outer-hop independent dominating set in G. Thus, $\gamma_c^{oho}(G) \leq n - 1$. Consequently, $1 \leq \gamma_c^{ohi}(G) \leq n - 1$.

(i) Assume that $\gamma_c^{ohi}(G) = 1$. Suppose G is not a complete graph. Then there exists $a, b \in V(G)$ such that $d_G(a, b) = 2$. Let $c \in N_G(a) \cap N_G(b)$. Clearly, $\gamma_c^{ohi}(G) \geq 2$, a contradiction. Hence, G is complete.

Conversely, suppose G is complete. Then every $v \in V(G)$ is a connected outer-hop independent dominating vertex of G. Thus, $\gamma_c^{ohi}(G) \leq 1$. Consequently, $\gamma_c^{ohi}(G) = 1$.

(ii) Assume that $\gamma_c^{ohi}(G) = 2$. Let a and b be two distinct adjacent vertices of G such that $N_G[a] \neq N_G[b]$. Suppose there exists $x \in V(G) \setminus (N_G[a] \cup N_G[b])$. Since a and b are arbitrary, it follows that $\gamma_c^{ohi}(G) \geq 3$, a contradiction. Therefore, $\{a, b\}$ is a dominating set of G. By letting $D = \{a, b\}$ to be the γ_c^{ohi}-set of G, it would imply that $V(G) \setminus D$ is a hop independent set in G.

Conversely, suppose that for each pair of distinct adjacent vertices $a, b \in V(G)$ such that $N_G[a] \neq N_G[b]$, $\{a, b\}$ is a dominating set of G and $D = \{a, b\}$ is a hop independent set in G. Then G is non-complete and D is a connected outer-hop independent dominating set of G. Hence, by (i), $\gamma_c^{ohi}(G) = 2$. □

The next result follows from Theorem 1.

Corollary 1. *Let G be a non-trivial connected graph on n vertices such that $\overline{G}$ is connected. Then each of the following statements holds.*

(i) *$\gamma_c^{ohi}(G) \geq 2$ if and only if G is non-complete.*

(ii) *If G is non-complete, then*

  (a) $4 \leq \gamma_c^{ohi}(G) + \gamma_c^{ohi}(\overline{G}) \leq 2n - 2$, *and*

  (b) $4 \leq \gamma_c^{ohi}(G) \cdot \gamma_c^{ohi}(\overline{G}) \leq n^2 - 2n + 1$.

Proposition 1. *For any positive integer $n \geq 1$,*

$$\gamma_c^{ohi}(P_n) = \begin{cases} 1 \text{ if } n = 1, 2 \\ 2 \text{ if } n = 3 \\ n - 2 \text{ if } n \geq 4 \end{cases}$$

Proof. Clearly, $\gamma_c^{ohi}(P_n) = 1$ for $n = 1, 2$ and $\gamma_c^{ohi}(P_3) = 2$. Suppose $n \geq 4$. Let $P_n = [v_1, v_2, \ldots, v_n]$ and let $D = \{v_2, \cdots, v_{n-1}\}$. Clearly, D is a connected dominating set of P_n. Since $n \geq 4$, it follows that $d_{P_n}(v_1, v_n) \geq 3$. Thus, $V(P_n) \setminus D = \{v_1, v_n\}$ is a hop independent set of P_n. Thus, D is a connected outer-hop independent dominating set in P_n, and so $\gamma_c^{ohi}(P_n) \leq n - 2$. Observe that every connected dominating set of P_n contains D. Therefore, $\gamma_c^{ohi}(P_n) = n - 2$ by Remark 1. □

Proposition 2. *For any positive integer $n \geq 3$,*

$$\gamma_c^{ohi}(C_n) = \begin{cases} 1 \text{ if } n = 3 \\ n - 2 \text{ if } n \geq 4 \end{cases}$$

Proof. Clearly, $\gamma_c^{ohi}(C_3) = 1$. Suppose that $n \geq 4$. Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ and consider $D' = \{v_1, v_2, \cdots, v_{n-2}\}$. Then D' is a connected dominating set of C_n. Since $d_{C_n}(v_{n-1}, v_n) = 1$, it follows that $V(C_n) \setminus D' = \{v_{n-1}, v_n\}$ is a hop independent set in C_n. Thus, D' is a connected outer-hop independent dominating set in C_n, and so

$\gamma_c^{ohi}(C_n) \leq n-2$. Since $\gamma_c(C_n) = n-2$ for all $n \geq 4$, it follows that $\gamma_c^{ohi}(C_n) = n-2$ for all $n \geq 4$ by Remark 1. □

The next theorem is a realization result involving connected domination number and connected outer-hop independent domination number of a graph.

Theorem 2. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_c(G) = a$ and $\gamma_c^{ohi}(G) = b$. In other words, $\gamma_c^{ohi}(G) - \gamma_c(G)$ can be made arbitrarily large.*

Proof. For $a = b$, consider a path graph P_{a+2}. Then $\gamma_c(P_{a+2}) = a = \gamma_c^{ohi}(P_{a+2})$ by Proposition 1.

Suppose $a < b$. Consider the following two cases:

Case 1: a is odd.

Let $m = b - a$ and consider the graph G_1 given in Figure 3. Let $D_1 = \{d_1, d_2, \ldots, d_a\}$ and $D_2 = \{d_1, d_2, \ldots, d_a, v_1, v_2, \ldots, v_m\}$. Then D_1 and D_2 are γ_c-set and γ_c^{ohi}-set of G_1, respectively. Hence, $\gamma_c(G_1) = a$ and $\gamma_c^{ohi}(G_1) = m + a = b$.

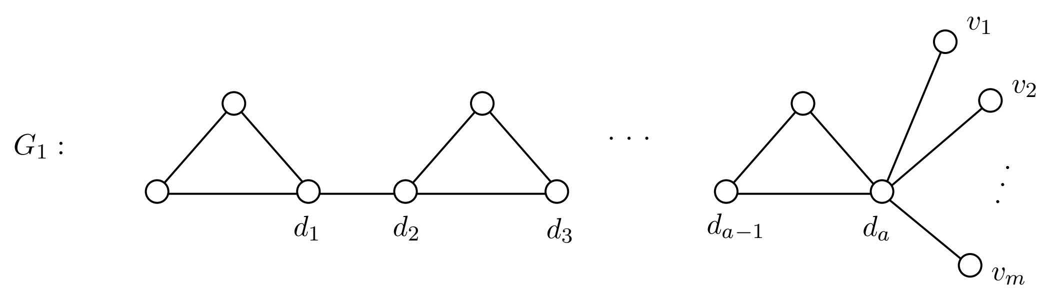

Figure 3: A graph G_1 with $\gamma_c(G_1) < \gamma_c^{ohi}(G_1)$

Case 2: a is even.

Let $m = b - a$ and consider the graph G_2 given in Figure 4. Let $D = \{u_1, u_2, \ldots, u_a\}$ and $D^* = \{u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_m\}$. Then D and D^* are γ_c-set and γ_c^{ohi}-set of G_2, respectively. Therefore, $\gamma_c(G_2) = a$ and $\gamma_c^{ohi}(G_2) = m + a = b$.

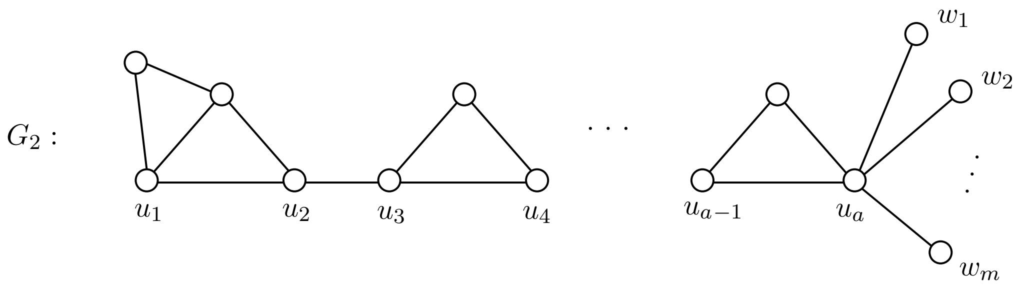

Figure 4: A graph G_2 with $\gamma_c(G_2) < \gamma_c^{ohi}(G_2)$

□

The next theorem is a realization result involving connected outer-independent domination number and connected outer-hop independent domination number of a graph.

Theorem 3. *Let a and b be positive integers such that $2 \leq a \leq b$. Then*

(*i*) *there exists a connected graph G such that $\gamma_c^{ohi}(G) = a$ and $\gamma_c^{oi}(G) = b$.*

(*ii*) *there exists a connected graph G such that $\gamma_c^{oi}(G) = a$ and $\gamma_c^{ohi}(G) = b$.*

In other words, $|\gamma_c^{oi}(G) - \gamma_c^{ohi}(G)|$ can be made arbitrarily large.

Proof. (*i*) Suppose $a < b$. Let $m = b - a$ and consider the graph G in Figure 5. Let $D_1 = \{x_1, x_2, \ldots, x_a\}$ and $D_2 = \{x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_m\}$. Then D_1 and D_2 are γ_c^{ohi}-set and γ_c^{oi}-set of G, respectively. Hence, $\gamma_c^{ohi}(G) = a$ and $\gamma_c^{oi}(G) = m + a = b$.

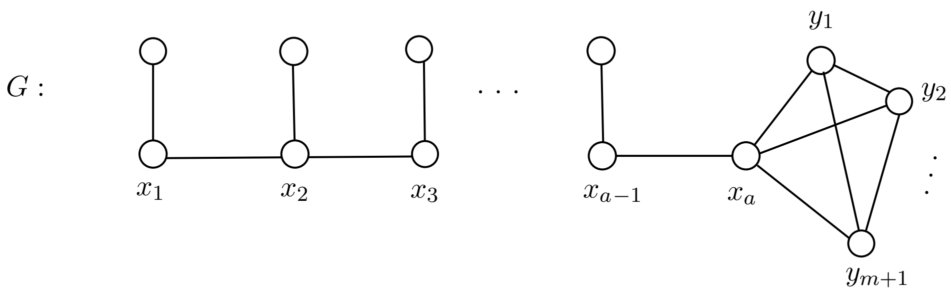

Figure 5: A graph G with $\gamma_c^{ohi}(G) < \gamma_c^{oi}(G)$

(*ii*) Suppose $a < b$. Let $m = b - a$ and consider the graph G^* in Figure 6. Let $D' = \{x_1, x_2, \ldots, x_a\}$ and $D'' = \{x_1, x_2, \ldots, x_a, z_1, z_2, \ldots, z_m\}$. Then D' and D'' are γ_c^{oi}-set and γ_c^{ohi}-set of G^*, respectively. Therefore, $\gamma_c^{oi}(G^*) = a$ and $\gamma_c^{ohi}(G^*) = m + a = b$.

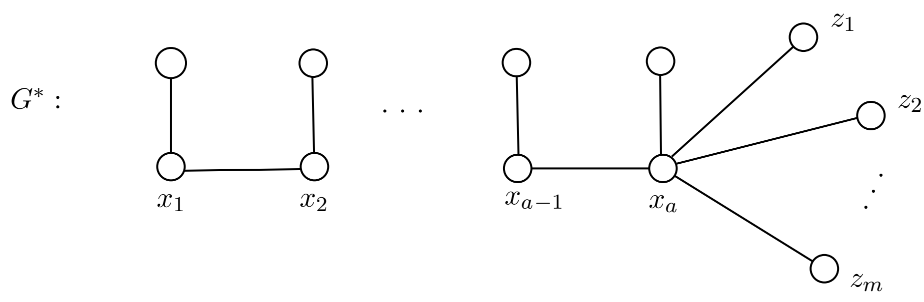

Figure 6: A graph G^* with $\gamma_c^{oi}(G^*) < \gamma_c^{ohi}(G^*)$

□

The following concept will be used in characterizing the connected outer-hop independent dominating sets in the join and corona of two graphs.

Definition 2. Let G be a non-complete graph. A non-empty subset $O \subseteq V(G)$ is called an outer-clique set if $V(G) \setminus O$ is clique in G. The smallest cardinality of an outer-clique set of G, denoted by $\tilde{\omega}(G)$, is called the *outer-clique number* of G. Any outer-clique set O of G with cardinality equal to $\tilde{\omega}(G)$, is called an $\tilde{\omega}$-set of G.

Remark 3. *Let $n \geq 2$ be any positive integer. Then each of the following holds.*

(i) $\tilde{\omega}(G) = n - 1$ *if* $G = \overline{K}_n$

(ii) $\tilde{\omega}(P_n) = \begin{cases} 1 & if\ n = 3 \\ n-2 & if\ n \geq 4; \end{cases}$ *and*

(iii) $\tilde{\omega}(C_n) = n - 2$ *for all* $n \geq 4$.

Theorem 4. *Let G and H be two non-complete graphs. Then $D \subseteq V(G + H)$ is a connected outer-hop independent dominating in $G+H$ if and only if $D = D_G \cup D_H$, where D_G and D_H are outer-clique sets in G and H, respectively.*

Proof. Suppose $D \subseteq V(G + H)$ is a connected outer-hop independent dominating set in $G + H$. Let $D_G = V(G) \cap D$ and $D_H = V(H) \cap D$. Since G and H are non-complete, it follows that $D_G \neq \varnothing$ and $D_H \neq \varnothing$. Suppose $V(G) \setminus D_G$ is not a clique in G. Then there exist $a, b \in V(G) \setminus D_G$ such that $d_G(a, b) = 2 = d_{G+H}(a, b)$. Since $V(G) \setminus D_G \subseteq V(G+H) \setminus D$, it follows that $V(G+H) \setminus D$ is not a hop independent set, a contradiction to the fact that D is a connected outer-hop independent dominating set in $G + H$. Therefore, $V(G) \setminus D_G$ is clique in G. Similarly, $V(H) \setminus D_H$ is clique in H.

Conversely, suppose $D = D_G \cup D_H$, where D_G and D_H are outer-cliques in G and H, respectively. Clearly, D is a connected dominating set of $G+H$. Suppose that $V(G+H)\setminus D$

is not a hop independent set in $G+H$. Then there exist $x, y \in V(G+H) \setminus D$ such that $d_{G+H}(x,y) = 2$. This means that either $x, y \in V(G) \setminus D_G$ or $x, y \in V(H) \setminus D_H$, and this is a contradiction to our assumption that D_G and D_H are outer-cliques in G and H, respectively. Therefore, $V(G+H) \setminus D$ is a hop independent set in $G+H$. Consequently, D is a connected outer-hop independent dominating in $G+H$. □

The next result follows from Theorem 4

Corollary 2. *Let G and H be two non-complete graphs. Then*

$$\gamma_c^{ohi}(G+H) = \tilde{\omega}(G) + \tilde{\omega}(H).$$

In particular, we have

(i) $\gamma_c^{ohi}(P_n + P_m) = n + m - 4$ *for all* $n, m \geq 3$;

(ii) $\gamma_c^{ohi}(C_n + C_m) = n + m - 4$ *for all* $n, m \geq 4$; *and*

(iii) $\gamma_c^{ohi}(P_n + C_m) = n + m - 4$ *for all* $n, m \geq 4$.

The following concept will be used in characterizing connected outer-hop independent dominating sets in the join of complete and non-complete graphs.

Definition 3. Let G be a connected graph. A connected dominating set $C \subseteq V(G)$ is called a *connected outer-clique dominating* if $V(G) \setminus C$ is a clique set in G. The *connected outer-clique domination number* of G, denoted by $\gamma_c^{oc}(G)$, is the minimum cardinality of a connected outer-clique dominating set of G. Any connected outer-clique dominating set C with cardinality equal to $\gamma_c^{oc}(G)$, is called a γ_c^{oc}*-set* of G.

Theorem 5. *Let G be a complete graph and H be any non-complete connected graph. Then $D \subseteq V(G+H)$ is a connected outer-hop independent dominating set in $G+H$ if and only if $D = D_G \cup D_H$ and satisfies one of the following conditions:*

(i) If $D_G = \varnothing$, then D_H is a connected outer-clique dominating set in H.

(ii) If $D_G \neq \varnothing$, then D_H is an outer-clique set in H.

Proof. Suppose $S \subseteq V(G+H)$ is a connected outer-hop independent dominating in $G+H$. Then $V(G+H) \setminus S$ is a hop independent set in $G+H$. Let $D_G = \varnothing$. Suppose on the contrary that D_H is not a connected outer-clique dominating set in H. Then D_H is either not a connected, not a dominating or $V(H) \setminus D_H$ not a clique sets in H, respectively. Assume first that D_H is not a dominating set in H. Then there exists $a \in V(H) \setminus D_H$ such that $a \notin N_H[D_H]$. Since $D_G = \varnothing$, it follows that $a \notin N_{G+H}[D]$, a contradiction. Therefore, D_H is a dominating set in H. Similarly, a contradiction follows if D_H is not a connected or $V(H) \setminus D_H$ is not a clique in H. Hence, (i) holds. Next, suppose that $D_G \neq \varnothing$ and suppose that D_H is not an outer-clique set in H. Then there exist

$x, y \in V(H) \setminus D_H \subseteq V(G+H) \setminus D$ such that $d_H(x,y) = d_{G+H}(x,y) = 2$, a contradiction. Hence, D_H must be an outer-clique set in H showing that (ii) holds.

For the converse, suppose (i) holds. Since G is complete, it follows that D is an outer-hop independent set in $G+H$. Clearly, D is a connected dominating set in $G+H$. Hence, D is a connected outer-hop independent dominating set in $G+H$. Similarly, if (ii) holds, then D is a connected outer-hop independent dominating set in $G+H$. □

The next result follows from Theorem 5.

Corollary 3. *Let G be a complete graph and H be any non-complete connected graph. Then*

$$\gamma_c^{ohi}(G+H) = \gamma_c^{oc}(H).$$

In particular, we have

(i) $\gamma_c^{ohi}(W_n) = \gamma_c^{ohi}(K_1 + C_n) = n-2$ *for all* $n \geq 4$;

(ii) $\gamma_c^{ohi}(F_n) = \gamma_c^{ohi}(K_1 + P_n) = n-1$ *for all* $n \geq 3$;

(iii) $\gamma_c^{ohi}(K_n + C_m) = m-2$ *for all* $n \geq 2, m \geq 4$; *and*

(iv) $\gamma_c^{ohi}(K_n + P_m) = m-1$ *for all* $n \geq 2, m \geq 3$.

Theorem 6. *Let G be a non-trivial connected graph and H be any non-complete graph. A set $D \subseteq V(G \circ H)$ is a connected outer-hop independent dominating set in $G \circ H$ if and only if $D = V(G) \cup (\bigcup_{v \in V(G)} D_v)$, where $D_v \subseteq V(H^v)$ and $V(H^v) \setminus D_v$ is clique in H^v for each $v \in V(G)$.*

Proof. Assume that D is a connected outer-hop independent dominating set in $G \circ H$ and let $D_v = V(H^v) \cap D$ for each $v \in V(G)$. Since $\langle D \rangle$ is connected and H is non-complete, it follows that $D = V(G) \cup (\bigcup_{v \in V(G)} D_v)$. Suppose $V(H^v) \setminus D_v$ is not a clique in H^v for some $v \in V(G)$. Then there exists $u, w \in V(H^v) \setminus D_v \subseteq V(G \circ H) \setminus D$ such that $d_{H^v}(u,w) = d_{G \circ H}(u,w) = 2$ for some $v \in V(G)$, a contradiction to the fact that D is an outer-hop independent set in $G \circ H$. Therefore, $V(H^v) \setminus D_v$ is clique in H^v for every $v \in V(G)$.

Conversely, suppose $D = V(G) \cup (\bigcup_{v \in V(G)} D_v)$, where $D_v \subseteq V(H^v)$ and $V(H^v) \setminus D_v$ is clique of H^v for each $v \in V(G)$. Clearly, D is a connected dominating set of $G \circ H$. Since $V(H^v) \setminus D_v$ is clique of H^v for each $v \in V(G)$, it follows that $V(G \circ H) \setminus D = \bigcup_{v \in V(G)} (V(H^v) \setminus D_v)$ is a hop independent set of $G \circ H$. Therefore, D is a connected outer-hop independent dominating set in $G \circ H$. □

The next result follows from Theorem 6.

Corollary 4. *Let G be a non-trivial connected graph with $|V(G)| = s$ and H be any non-complete graph with $|V(H)| = t$. Then*

$$\gamma_c^{ohi}(G \circ H) = s + s(\tilde{\omega}(H)).$$

In particular, we have

(i) $\gamma_c^{ohi}(P_s \circ P_t) = \gamma_c^{ohi}(C_s \circ C_t) = \gamma_c^{ohi}(P_s \circ C_t) = s + s(t-2)$ *for all* $s \geq 2$ *and* $t \geq 4$;

(ii) $\gamma_c^{ohi}(K_s \circ P_t) = \gamma_c^{ohi}(K_s \circ C_t) = s + s(t-2)$ *for all* $s \geq 2$ *and* $t \geq 4$;

(iii) $\gamma_c^{ohi}(G \circ W_t) = |V(G)| + |V(G)|(t-2)$ *for all* $t \geq 4$; *and*

(iv) $\gamma_c^{ohi}(G \circ F_t) = |V(G)| + |V(G)|(t-2)$ *for all* $t \geq 3$.

Theorem 7. *Let* G *be a non-trivial connected graph and* H *be any complete graph. A set* $D \subseteq V(G \circ H)$ *is a connected outer-hop independent dominating set in* $G \circ H$ *if and only if* $D = V(G) \cup (\bigcup_{v \in V(G)} D_v)$, *where* $D_v \subseteq V(H^v)$ *such that* $D_v = \varnothing$ *or* $D_v \neq \varnothing$ *for each* $v \in V(G)$.

Proof. Assume that D is a connected outer-hop independent dominating set in $G \circ H$ and let $D_v = V(H^v) \cap D$ for each $v \in V(G)$. Since $\langle D \rangle$ is connected, it follows that $D = V(G) \cup (\bigcup_{v \in V(G)} D_v)$. Since H is complete, either $D_v = \varnothing$ or $D_v \neq \varnothing$ holds for each $v \in V(G)$.

Conversely, suppose that $D = V(G) \cup (\bigcup_{v \in V(G)} D_v)$, where $D_v \subseteq V(H^v)$. If $D_v = \varnothing$ for each $v \in V(G)$, then $D = V(G)$. Since H is complete, it follows that $D = V(G)$ is connected outer-hop independent dominating set in $G \circ H$. Similarly, if $D_v \neq \varnothing$ for each $v \in V(G)$, then D connected outer-hop independent dominating set in $G \circ H$. □

The next result follows from Theorem 7.

Corollary 5. *Let* G *be a non-trivial connected graph with* $|V(G)| = s$ *and* H *be any complete graph. Then*

$$\gamma_c^{ohi}(G \circ H) = s.$$

In particular, we have

(i) $\gamma_c^{ohi}(P_n \circ K_m) = n = \gamma_c^{ohi}(C_n \circ K_m)$ *for all* $n \geq 3, m \geq 1$; *and*

(ii) $\gamma_c^{ohi}(F_n \circ K_m) = n + 1 = \gamma_c^{ohi}(W_n \circ K_m)$ *for all* $n \geq 3, m \geq 1$.

4. Conclusion

The concept of connected outer-hop independent domination in a graph has been introduced and investigated in this study. It was shown that the connected outer-hop independent domination number is at least equal to the connected domination number of a graph. Connected outer-hop independent dominating sets in some special graphs, join and corona of two graphs have been characterized. This results have been used in determining the exact values or bounds of the parameter of each of these graphs. Moreover, realization results involving connected outer-hop independent domination have been presented and its relationships with other known parameters have been determined. Interested researchers may study this concept in some product of graphs which were not considered in this paper. Furthermore, they may consider and study its bounds with respect to the other known parameters in graph theory.

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions that led to the improvement of the paper. Also, the authors would like to thank Mindanao State University - Tawi-Tawi College of Technology and Oceanography for funding this research.

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