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A New Scheme for Solving a Fractional Differential Equation and a Chaotic System

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Abstract. The subject of this study is the solution of a fractional Bernoulli equation and a chaotic system by using a novel scheme for the fractional derivative and comparison of approximate and exact solutions. It is found that the suggested method produces solutions that are identical to the exact solution. We can therefore generalize the strategy to different systems to get more accurate results. We think that the novel fractional derivative scheme that has been offered and the algorithm that has been suggested will be utilized in the future to construct and simulate a variety of fractional models that can be used to solve more difficult physics and engineering challenges.

2020 Mathematics Subject Classifications: 34C28, 34A08, 65P20

Key Words and Phrases: Numerical solutions, numerical scheme for ABC operator, analytical solutions, Laplace decomposition method, chaos

1. Introduction

Due to the modeling of diffusion, control, and viscoelasticity in fractional calculus, applied mathematics has grown in popularity over the past few decades. In physics and engineering research, fractional differential equations are used [29, 32]. There are several techniques for resolving fractional differential equations, see [11, 17]. The body of research on modeling chaotic and hyperchaotic systems has been exploded recently with several applications in disciplines as diverse as electrical circuits, biology, and physics [12, 18, 31, 34]. Electrical circuit modeling, which is described in multiple works, is one of the most well-known applications of chaos. It is justified to employ chaotic models given how difficult

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it is to predict many real-world situations. Asymptotic stability, which identifies the precise nature of the chaos, is only one of the many unique techniques for analyzing chaotic systems that have emerged in recent years. The mathematical and scientific domains of fractional calculus [2, 16, 25, 40] are extremely diverse: mathematics, biology, and other domains[19, 39, 44] are rapidly expanding cutting-edge applications in the area of fractional calculus [15, 21, 22, 42].

This research is significant since fractional operators have many different meanings. Derivatives with exponential and Mittag-Leffler kernels are examples of singular-free derivatives [20]. The fractional derivatives are helpful since they account for the influence of long-term memory [24]. Recent research has shown that there are several convincing grounds for using fractional derivatives in practical contexts [41]. Chaotic systems violently respond to both initial conditions and small changes in their parameters, as is well known [30].

The main goal of this research is to introduce a new approach for solving fractional differential equations [4, 5, 8, 23, 27, 38, 45]. Also, we study the Chaotic behavior of the studied problems.

Moreover, we sketch some figures to illustrate the efficiency of the proposed method.

This article is organized as follows, in the next section, we introduce the basic definitions and properties. In Section 3, we introduce the numerical scheme of the ABC operator. In Section 4, we introduce some applications and finally, we present the conclusion section.

2. Basic Principles

Definition 1. *The Riemann Liouville integral (RLI) order of* $0 < \alpha < 1$ *and* $v(\tau)$ *is provided by [33]:*

$$D^{\alpha}v\left(t\right) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \left(t-\tau\right)^{n-\alpha-1}v^n\left(\tau\right)d\tau \ = I^{n-\alpha}v^n\left(t\right), \quad t>0. \tag{1}$$

Definition 2. *The Riemann-Liouville fractional integral of order* $\alpha > 0$*, given by [1]:*

$$I^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \quad t>a. \tag{2}$$

Definition 3. *For a function* $y\left(\tau\right)$ *Caputo derivative of order* $0 < \alpha < 1$ *is given by [10]:*

$$I^{\alpha}y\left(t\right) \ = \frac{1}{\Gamma\ (\alpha)}\int_0^t \left(t-\tau\right)^{\alpha-1}y\left(\tau\right)d\tau\ , \quad t>0. \tag{3}$$

Definition 4. *The Mittag Leffler function can be expressed as follows [6]:*

$$E_{\alpha}\left(t\right) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}. \tag{4}$$

Definition 5. *(The Lagrange's polynomial interpolation)*
The Lagrange's polynomial interpolation which defined by [8]

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x),$$

where

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_i}{x_i - x_j}.$$

Definition 6. *The ABC operator,* $y(t)$ *in the RLI is given by [7]:*

$$ {}_{0}^{ABC}D_t^{\alpha} y(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t y(\tau) E_\alpha \left(\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right) d\tau, \quad 0 < \alpha < 1. \tag{5}$$

Where $B(\alpha)$ *satisfies the condition* $B(1) = B(0) = 1$.

3. Numerical scheme for ABC

The goal of this section is to investigate chaotic models in the sense of the ABC fractional derivative, of the form:

$$\begin{aligned} &{}_{0}^{ABC}D_0^{\alpha} v(t) = g(t, v(t)), \\ &v(0) = v_0. \end{aligned} \tag{6}$$

A fractional integral equation can be derived from the equation above

$$v(t) - v(0) = \frac{(1-\alpha)g(t, v(t))}{ABC(\alpha)} + \frac{\alpha}{\Gamma(\alpha+1) \times ABC(\alpha)} \int_0^t g(\tau, v(\tau))(t-\tau)^{\alpha-1} d\tau, \tag{7}$$

where $n = 0, 1, 2, 3 \ldots$, reformulated as

$$\begin{aligned} v(t_{n+1}) - v(0) &= \frac{(1-\alpha)g(t_n, v(t_n))}{ABC(\alpha)} \\ &\quad + \frac{\alpha}{ABC(\alpha) \times \Gamma(\alpha+1)} \int_0^{t_{n+1}} g(\tau, v(\tau)) (t_{n+1} - \tau)^{\alpha-1} d\tau \\ &= \frac{(1-\alpha)g(t_n, v(t_n))}{ABC(\alpha)} \\ &\quad + \frac{\alpha}{ABC(\alpha) \times \Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} g(\tau, v(\tau)) (t_{n+1} - \tau)^{\alpha-1} d\tau. \end{aligned} \tag{8}$$

The following can be approximated using two-step Lagrange polynomial interpolation:

$$\begin{aligned} P_k(\tau) &= \frac{(\tau - t_{k-1}) g(t_k, v(t_k))}{t_k - t_{k-1}} - \frac{(\tau - t_k) g(t_{k-1}, v(t_{k-1}))}{t_k - t_{k-1}} \\ &= \frac{g(t_k, v(t_k))(\tau - t_{k-1})}{h} - \frac{g(t_{k-1}, v(t_{k-1}))(\tau - t_k)}{h} \\ &\simeq \frac{g(t_k, v_k)(\tau - t_{k-1})}{h} - \frac{g(t_{k-1}, v_{k-1})(\tau - t_k)}{h}, \end{aligned} \tag{9}$$

$$
\begin{aligned}
v_{n+1} &= v_0 + \frac{(1-\alpha)}{ABC(\alpha)} g(t_n, v(t_n)) \\
&+ \frac{\alpha}{ABC(\alpha)\times\Gamma(\alpha)} \sum_{k=0}^{n} \left(\frac{g(t_k, v_k)}{h} \int_{t_k}^{t_{k+1}} (\tau +\ t_{k-1}t)(t_{n+1}-\tau)^{\alpha-1} d\tau \right. \\
&- \left. \frac{g(t_{k-1}, v_{k-1})}{h} \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1}-\tau)^{\alpha-1} d\tau \right).
\end{aligned} \tag{10}
$$

For simplicity

$$
A_{\alpha,k,1} = \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1}-\tau)^{\alpha-1} d\tau, \tag{11}
$$

$$
\begin{aligned}
A_{\alpha,k,2} &= \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+2}-\tau)^{\alpha-1} d\tau \\
A_{\alpha,k,1} &= h^{\alpha+1} \frac{(n+1-k)^{\alpha}(n-k+2+\alpha) - (n-k)^{\alpha}(n-k+2+2\alpha)}{\alpha(\alpha+1)} \\
A_{\alpha,k,2} &= (h^{\alpha+1}) \frac{(n+1-k)^{\alpha+1} - (n-k)^{\alpha}(n-k+1+\alpha)}{\alpha(\alpha+1)}.
\end{aligned} \tag{12}
$$

By combining equations (11) and (12) and substituting in (10),

$$
\begin{aligned}
v_{n+1} &= v(1) + \frac{(1-\alpha)}{ABC(\alpha)} g\left(t_n, v\left(t_n\right)\right) + \frac{\alpha}{ABC(\alpha)} \\
&\sum_{j=0}^{n} \left(\frac{h^{\alpha} g\left(t_k, v_k\right)}{\Gamma(\alpha+1)} \left((1+n-j)^{\alpha}\left(2+\alpha+n-k\right) + (j-n)^{\alpha}(2+n-k+2\alpha)\right) \right. \\
&- \left. \frac{h^{\alpha} g\left(t_{j-1}, v_{j-1}\right)}{\Gamma(1+\alpha)} \left((n-j+1)^{\alpha+1} + (j-n)^{\alpha}(n-j+1+\alpha)\right) \right).
\end{aligned} \tag{13}
$$

4. Applications

In this part, we explore the usefulness of the novel scheme for ABC fractional derivative for solving an initial value problem (IVP) numerically.

Problem 1. *We start with the Bernoulli equation [7]:*

$$
{}_0^{ABC}D_t^{\alpha} y(t) = 2y(t) - 4y^2(t), \tag{14}
$$

where $0 < \alpha \leq 1$ and $y(0) = 1$, ${}_0^{ABC}D_t^{\alpha}$ is ABC operator, given in Eq. (7). The exact solution of the Bernoulli equation is

$$
y(t) = \frac{-1}{e^{-2t} - 1}, \tag{15}
$$

under $y(0) = 1$, where ${}_0^{ABC}D_t^{\alpha}$ is defined by Eq. (5) with the parameter α, when $\alpha = 1$, the Bernoulli Equation (14) has an exact solution according to the proposed the numerical scheme for ABC fractional derivative, and we show this results in the Table 1 and Table 2.

Table 1: The numerical solutions of Eq. (14) when $\alpha = 1$.

h	$t = 10$	$t = 12$	$t = 14$
1/10	0.500000000938279	0.500000000018475	0.500000000000364
1/20	0.500000000627998	0.500000000011706	0.500000000000218
1/40	0.500000000554842	0.500000000010206	0.500000000000188
1/80	0.500000000531467	0.500000000009744	0.500000000000179
1/160	0.500000000522537	0.500000000009573	0.500000000000175
1/320	0.500000000518709	0.50000000000950	0.500000000000174
1/640	0.500000000516948	0.500000000009468	0.500000000000173
y_{Exact}	0.500000000515288	0.500000000009438	0.500000000000173

Table 2: The numerical solutions of Eq. (14) when $\alpha = 0.99$.

h	$t = 10$	$t = 12$	$t = 14$
1/10	0.500000003717573	0.500000002289498	0.500000001921669
1/20	0.500000003391435	0.500000002272609	0.500000001916067
1/40	0.500000003312222	0.500000002268483	0.500000001909773
1/80	0.500000003291813	0.500000002273635	0.500000001940562
1/160	0.500000003340172	0.500000002329872	0.500000001934836
1/320	0.500000003462054	0.500000002529303	0.500000002330563
1/640	0.500000004155025	0.500000003461087	0.500000004209577

In Table 1, we provide numerical results from our novel scheme for ABC fractional derivative to fractional Bernoulli equation Eq. (14) when $\alpha = 1$ at $t = 10$, $t = 12$ and $t = 14$, and when $\alpha = 0.99$ at $t = 10$, $t = 12$ and $t = 14$ in Table 2. The numerical answers we provided matched the exact solution perfectly, and the step size h is small enough.

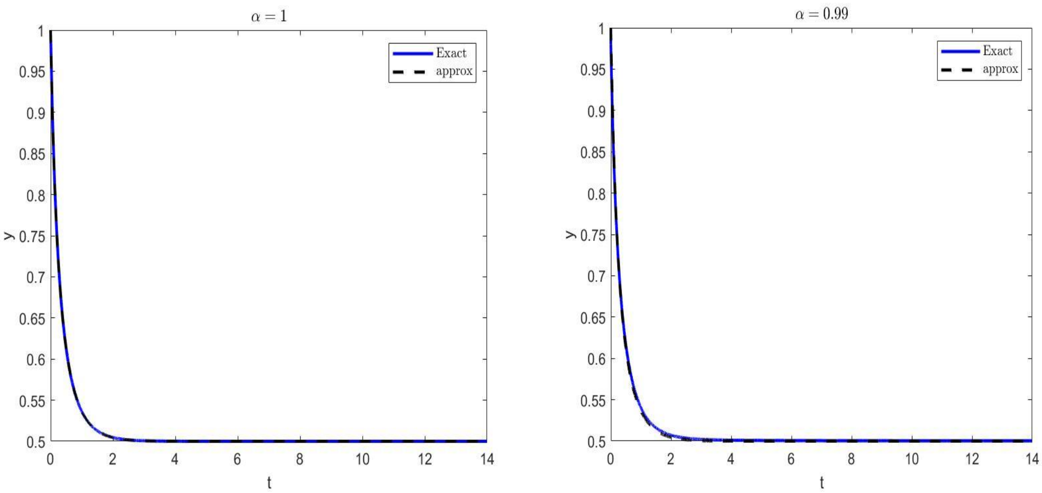

Figure 1: A comparison between the exact and approximate solutions of Eq. (14).

Problem 2. *We discuss the Chen system:*

$$\begin{cases} {}_0^{ABC}D_t^{\alpha} u\left(t\right) = a(v\left(t\right) - u(t)), \\ {}_0^{ABC}D_t^{\alpha} v\left(t\right) = \left(c-a\right)u\left(t\right) - u\left(t\right)w\left(t\right) + cv(t), \\ {}_0^{ABC}D_t^{\alpha} w\left(t\right) = u\left(t\right)v\left(t\right) - bw\left(t\right). \end{cases} \tag{16}$$

With $u\left(0\right) = -5$, $v\left(0\right) = -1$ and $w\left(0\right) = -1$, where $a, b, c \in \mathbb{R}$, $t > 0$, and ${}_0^{ABC}D_t^{\alpha}$ is the ABC operator, the parameters $a = 7.5$, $b = 1.0$ and $c = 5$. We show this results in the Table 3 and Table 4.

Table 3: The numerical solutions of Eq. (16) at $t = 14$ and $\alpha = 1$.

h	x	y	z
1/10	0.996232907806605	1.581099450806809	0.495241231468361
1/20	1.043578882387420	1.651420448120412	0.544251070796891
1/40	1.058429831301523	1.675181183975680	0.560064850780525
1/80	1.065523942097168	1.687112985888534	0.567652302039966
1/160	1.069190461371803	1.693385027057710	0.571579495559338
1/320	1.071078400638952	1.696632550666957	0.573603308346396
1/640	1.072039151580891	1.698288510008531	0.574633680132214

Table 4: The numerical solutions of Eq. (16) at $t = 14$ and $\alpha = 0.99$.

h	x	y	z
1/10	1.621482381001720	1.976822616207164	1.315702543995343
1/20	1.614809101418269	1.966952972142942	1.304706684275194
1/40	1.615906131167622	1.964093205028113	1.306415886448812
1/80	1.617681111000321	1.962647251183329	1.309263163719868
1/160	1.618884483012851	1.961844891929927	1.311201894320728
1/320	1.619565828368396	1.961413183136856	1.312301530738447
1/640	1.619926462286785	1.961188279022434	1.312884047918658

In Table 3 and Table 4 provide numerical results from the novel scheme for ABC fractional derivative to fractional Chen system Eq. (16), when $\alpha = 1$ and $t = 14$ in Table 3, and when $\alpha = 0.99$ and $t = 14$ in Table 4.

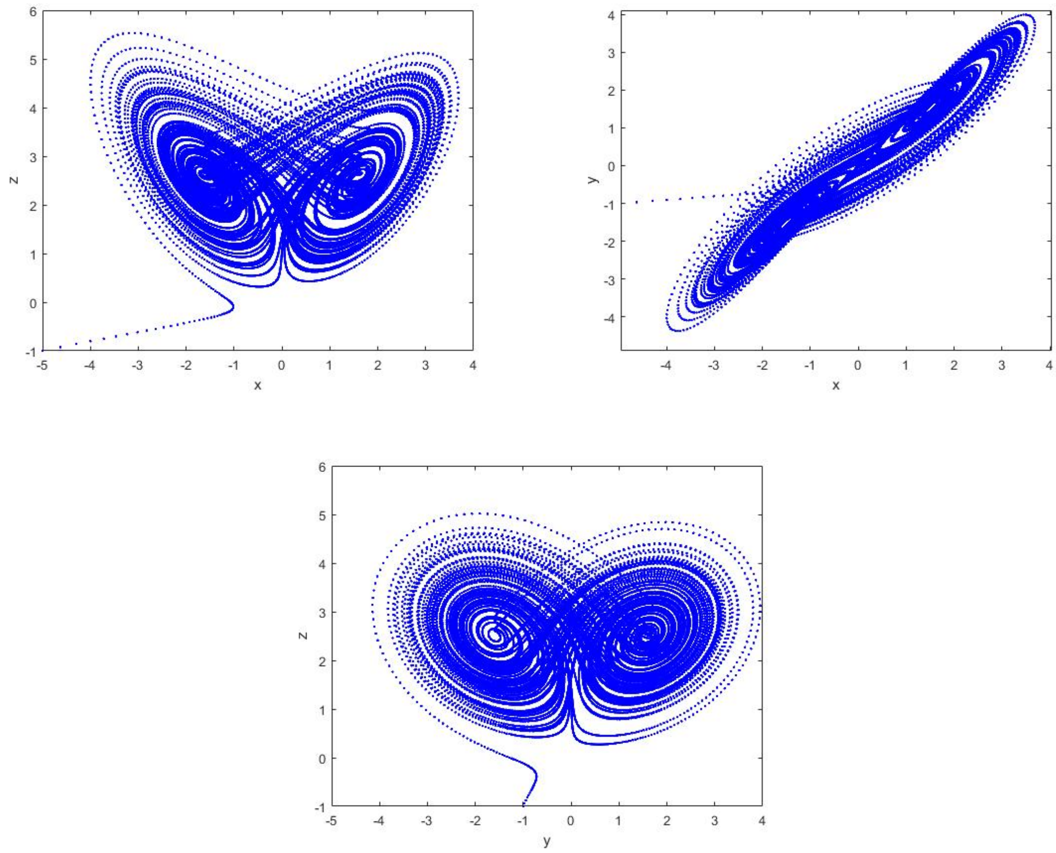

Figure 2: Chaotic attractor of Eq. (16), when $\alpha = 1$.

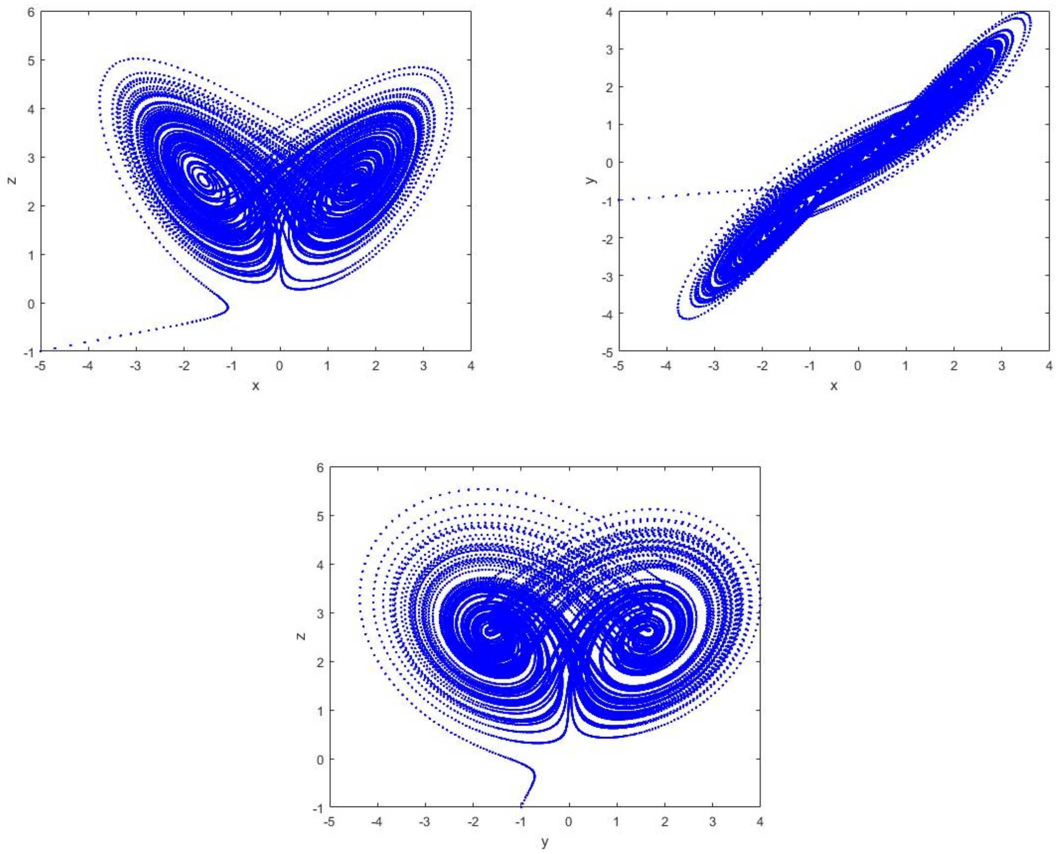

Figure 3: Chaotic attractor of Eq. (16), when $\alpha = 0.99$.

In Figure 2 and Figure 3, we plot the numerical solutions of Eq. (16) at the values $(a, b, c) = (7.5, 1, 5)$, and $(x_0, y_0, z_0) = (-5, -1, -1)$. In these figures, we display the Eq. (16) attractors obtained using the novel scheme for the fractional derivative when $\alpha = 0.99$ and $\alpha = 1$. This phenomena is known as the chaos and it is characterized by complex non-linear behaviors such as a periodic long-term behavior, erratic responses [9, 26, 28, 43].

5. Conclusions

A unique numerical approach was developed to solve Bernoulli equation and Chen system based on ABC operator. The shortcomings of the well-known predictor-corrector approach are addressed by a numerical methodology. It is based on the Lagrange polynomial and the fundamental theorem of fractional calculus. Rapid convergence, high efficiency and accuracy, and user-friendliness are the distinguishing features of this approach. The method was used to solve a fractional equation and system for which there exist solutions as well as a nonlinear system. We advise wider use of the method to address physics and engineering challenges that are becoming ever more complicated. In the future, we intend solve some new fractional models, such as in [35–37] and make comparisons with other numerical methods [3, 13, 14].

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