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Outer-Connected 2-Resolving Hop Domination in Graphs

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Abstract. Let G be a connected graph. A set $S \subseteq V(G)$ is an outer-connected 2-resolving hop dominating set of G if S is a 2-resolving hop dominating set of G and $S = V(G)$ or the subgraph $\langle V(G)\backslash S\rangle$ induced by $V(G)\backslash S$ is connected. The outer-connected 2-resolving hop domination number of G, denoted by $\widetilde{\gamma_{c2Rh}}(G)$ is the smallest cardinality of an outer-connected 2-resolving hop dominating set of G. This study aims to combine the concept of outer-connected hop domination with the 2-resolving hop dominating sets of graphs. The main results generated in this study include the characterization of outer-connected 2-resolving hop dominating sets in the join, corona, edge corona and lexicographic product of graphs, as well as their corresponding bounds or exact values.

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Key Words and Phrases: Outer-connected 2-resolving hop dominating set, outer-connected 2-resolving hop domination number, join, corona, edge corona, lexicographic product

1. Introduction

The concept of domination in graphs is one of the most studied problems and one of the fastest growing areas in graph theory. This was formally studied by Claude Berge [1] in 1958 and Oystein Ore in 1962. In 2007, outer-connected domination, a variation of domination, was first introduced by Cyman [10]. In 2015, Natarajan and Ayyaswamy introduced and studied the concept of hop domination [16]. In 2022, Canoy and Saromines studied and published the outer-connect hop dominating sets in graphs [9].

On the other hand, in 1975 the term locating set, the concept of resolving sets for a connected graph was first introduced by Slater [19]. These concepts were studied much earlier in the context of the coin-weighing problem. Later that year, Harary and Melter introduced independently these concepts, but with different terminologies [11]. The term

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metric dimension was used by Harary and Melter instead of locating number.

Recently, 2-resolving hop dominating sets in graphs was studied in [12]. Moreover, other variations of resolving sets and hop dominating sets in graphs were also studied in [4–6, 8, 13–15], respectively.

Motivated by the 2-resolving hop domination concept and the introduction of the outer-connected hop domination concept by S.R. Canoy and C.J. Saromines [9], here authors introduced and studied the concept of outer-connected 2-resolving hop domination in graphs.

2. Terminology and Notation

In this study, we consider finite, simple, connected, undirected graphs. For basic graph-theoretic concepts, we then refer readers to [2] and [3]. The following concepts are found in [2], [16] and [18].

Let G be a connected graph. A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u,v) = 2$. The set $N_G(u,2) = \{v \in V(G) : d_G(v,u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u,2] = N_G(u,2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X,2) = \bigcup_{u \in X} N_G(u,2)$. The *closed hop neighborhood* of X in G is the set $N_G[X,2] = N_G(X,2) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if $N_G[S,2] = V(G)$, that is, for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

For an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v in G, we refer to the k-vector (ordered k-tuple)

$$r_G(v/W) = (d_G(v,w_1), d_G(v,w_2), ..., d_G(v,w_k))$$

as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representations with respect to W. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r_G(v/W) : v \in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality is called a *minimum resolving set* or a *basis*, and the cardinality of a basis for G is the *dimension* $\dim(G)$ of G. An ordered set of vertices $W = \{w_1, ..., w_k\}$ is a *k-resolving set* for G if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r_G(u/W)$ and $r_G(v/W)$ of u and v, respectively, differ in at least k positions. If $k = 1$, then the k-resolving set is called a *resolving set* for G. If $k = 2$, then the k-resolving set is called a 2-resolving set for G. If G has a k-resolving set, the minimum cardinality $\dim_k(G)$ of a k-resolving set is called the *k-metric dimension* of G.

A set $S \subseteq V(G)$ is an *outer-connected 2-resolving hop dominating set* of G if S is a 2-resolving hop dominating set of G and $S = V(G)$ or the subgraph $\langle V(G)\backslash S\rangle$ induced by $V(G)\backslash S$ is connected. The outer-connected 2-resolving hop domination number of G, denoted by $\widetilde{\gamma_{c2Rh}}(G)$ is the smallest cardinality of a outer-connected 2-resolving hop dominating set of G.

Definition 1. [6] Let G be any nontrivial connected graph and $S \subseteq V(G)$. A set $S \subset V(G)$ is a 2-*locating set* of G if it satisfies the following conditions:

(i) $\left|[\big(N_G(x)\backslash N_G(y)\big)\cap S]\cup[\big(N_G(y)\backslash N_G(x)\big)\cap S]\right| \geq 2$, for all $x, y \in V(G)\backslash S$ with $x \neq y$.

(ii) $\big(N_G(v)\backslash N_G(w)\big) \cap S \neq \varnothing$ or $\big(N_G(w)\backslash N_G[v]\big) \cap S \neq \varnothing$, for all $v \in S$ and for all $w \in V(G)\backslash S$.

The 2-*locating number* of G, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of G. A 2-locating set of G of cardinality $ln_2(G)$ is referred to as an ln_2-set of G.

Definition 2. [17] A set $D \subseteq V(G)$ is a point-wise non-dominating set of G if for each $v \in V(G)\backslash D$, there exists $u \in D$ such that $v \notin N_G(u)$. The smallest cardinality of a point-wise non-dominating set of G, denoted by $pnd(G)$, is called the point-wise non-domination number of G. Any point-wise non-dominating set D of G with $|D| = pnd(G)$, is called a pnd-set of$G$. A dominating set D which is also a point-wise non-dominating set of G is called a dominating pointwise non-dominating set of G. The smallest cardinality of a dominating point-wise non-dominating set of G will be denoted by $\gamma_{pnd}(G)$. Any dominating point-wise non-dominating set D of G with $|D| = \gamma_{pnd}(G)$, is called a γ_{pnd}-set of G.

Definition 3. [12] A 2-locating set $S \subseteq V(G)$ which is point-wise non-dominating is called a 2-*locating point-wise non-dominating set* in G. The minimum cardinality of a 2-locating point-wise non-dominating set in G, denoted by $ln_2^{pnd}(G)$ is called the 2-*locating point-wise non-domination number* of G. Any 2-locating point-wise non-dominating set of cardinality $ln_2^{pnd}(G)$ is then referred to as a ln_2^{pnd}-set in G.

Definition 4. A set $S \subseteq V(G)$ is an *outer-connected* 2-*locating point-wise non-dominating set* in G if S is a 2-locating point-wise non-dominating set in G and $S = V(G)$ or the subgraph $\langle V(G)\backslash S\rangle$ induced by $V(G)\backslash S$ is connected. The *outer-connected* 2-*locating point-wise non-dominating number* of G, denoted by $\widetilde{ln_2^{pnd}}(G)$, is the smallest cardinality of an outer-connected 2-locating point-wise non-dominating set in G. An outer-connected 2-locating point-wise non-dominating set of cardinality $\widetilde{ln_2^{pnd}}(G)$ is then referred to as an $\widetilde{ln_2^{pnd}}$-set in G.

Definition 5. [6] Let G be any nontrivial connected graph and $S \subseteq V(G)$. S is a $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *set* in G if S is 2-locating and $|N_G(y) \cap S| \leq |S| - 2$ ($|N_G(y) \cap S| \leq |S| - 1$, respectively), for all $y \in V(G)$. The $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *number* of G, denoted by $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively), is the smallest cardinality of a $(2,2)$-locating ($(2,1)$-locating, respectively) set in G. A $(2,2)$-locating ($(2,1)$-locating, respectively) set in G of cardinality $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively) is referred to as an $ln_{(2,2)}$-*set* ($ln_{(2,1)}$-*set*, respectively) in G.

Definition 6. [12] A (2,2)-locating ((2,1)-locating, respectively) set $S \subseteq V(G)$ which is a point-wise non-dominating is called a (2,2)-*locating point-wise non-dominating* ((2,1)-*locating point-wise non-dominating*, respectively) *set* in G. The minimum cardinality of a (2,2)-locating point-wise non-dominating ((2,1)-locating point-wise non-dominating, respectively) set in G, denoted by $ln^{pnd}_{(2,2)}(G)$ ($ln^{pnd}_{(2,1)}(G)$,respectively) is called the (2,2)-*locating point-wise non-domination* ((2,1)-*locating point-wise non-domination*) *number* of G. Any (2,2)-*locating point-wise non-dominating* ((2,1)-*locating point-wise non-dominating*, respectively) set of cardinality $ln^{pnd}_{(2,2)}(G)$ ($ln^{pnd}_{(2,1)}(G)$, respectively) is then referred to as a $ln^{pnd}_{(2,2)}$-set ($ln^{pnd}_{(2,1)}$-set) in G.

Definition 7. A set $S \subseteq V(G)$ is an *outer-connected* $(2,2)$-*locating point-wise non-dominating* ($(2,1)$-*locating point-wise non-dominating*, respectively) *set* in G if S is a $(2,2)$-locating point-wise non-dominating ($(2,1)$-*locating point-wise non-dominating*, respectively) set in G and $S = V(G)$ or the subgraph $\langle V(G)\backslash S\rangle$ induced by $V(G)\backslash S$ is connected. The *outer-connected* $(2,2)$-*locating point-wise non-domination* ($(2,1)$-*locating point-wise non-domination*, respectively) *number* of G, denoted by $\widetilde{ln^{pnd}_{(2,2)}}(G)$ ($\widetilde{ln^{pnd}_{(2,1)}}(G)$, respectively), is the smallest cardinality of an outer-connected $(2,2)$-locating point-wise non-dominating ($(2,1)$-locating point-wise non-dominating, respectively) set in G. An outer-connected $(2,2)$-locating point-wise non-dominating ($(2,1)$-locating point-wise non-dominating, respectively) set of cardinality $\widetilde{ln^{pnd}_{(2,2)}}(G)$ ($\widetilde{ln^{pnd}_{(2,1)}}(G)$, respectively) is then referred to as an $\widetilde{ln^{pnd}_{(2,2)}}$-set ($\widetilde{ln^{pnd}_{(2,1)}}$- set) in G.

3. Preliminary Results

Every nontrivial connected graph G admits an outer-connected 2-resolving hop dominating set. Indeed, the vertex set $V(G)$ of G is an outer-connected 2-resolving hop dominating set.

Remark 1. For any connected graph G of order $n \geq 2$, $2 \leq \widetilde{\gamma_{c2Rh}}(G) \leq n$. Moreover, $\widetilde{\gamma_{c2Rh}}(P_2) = 2$ and $\widetilde{\gamma_{c2Rh}}(K_n) = n$.

Proposition 1. (i) For a path P_n on n vertices

$$\widetilde{\gamma_{c2Rh}}(P_n) = \begin{cases} n, & \text{if } n = 2, 3; \\ n-2, & \text{if } n = 4, 5, 6; \\ n-3, & \text{if } n = 7; \\ n-4, & \text{if } n \geq 8. \end{cases}$$

(ii) For a cycle C_n on n vertices

$$\widetilde{\gamma_{c2Rh}}(C_n) = \begin{cases} n, & \text{if } n = 3, 4; \\ n-2, & \text{if } n = 5; \\ n-3, & \text{if } n = 6; \\ n-4, & \text{if } n \geq 7. \end{cases}$$

Now, consider the following results of outer-connected 2-locating point-wise non-dominating sets which are used in characterizing the outer-connected 2-resolving hop dominating sets in the join of two graphs.

Proposition 2. *Let G be any nontrivial connected graph. Then for any positive integers n, we have*

$$(i)\ \widetilde{ln_2^{pnd}}(P_n) = \begin{cases} n, & \text{if } n = 2, 3; \\ n-1, & \text{if } 4 \leq n \leq 7; \\ n-2, & \text{if } n \geq 8. \end{cases}$$

$$(ii)\ \widetilde{ln_2^{pnd}}(C_n) = \begin{cases} n, & \text{if } n = 3, 4; \\ n-2, & \text{if } n \geq 5. \end{cases}$$

$$(iii)\ \text{For all } n \geq 5,\ \widetilde{ln_{(2,2)}^{pnd}}(P_n) = \begin{cases} n-1, & \text{if } 5 \leq n \leq 7; \\ n-2, & \text{if } n \geq 8; \end{cases}$$

$$\text{For all } n \geq 6,\ \widetilde{ln_{(2,2)}^{pnd}}(C_n) = n-2.$$

$$(iv)\ \text{For all } n \geq 4,\ \widetilde{ln_{(2,1)}^{pnd}}(P_n) = \begin{cases} n-1, & \text{if } 4 \leq n \leq 7; \\ n-2, & \text{if } n \geq 8; \end{cases}$$

$$\text{For all } n \geq 4,\ \widetilde{ln_{(2,1)}^{pnd}}(C_n) = \begin{cases} n, & \text{if } n = 4; \\ n-2, & \text{if } n \geq 5. \end{cases}$$

Proof. (i) Let $P_n = [v_1, v_2, v_3, \ldots, v_n]$. Clearly, $\widetilde{ln_2^{pnd}}(P_n) = n$ for $n = 2, 3$. Let $n \geq 4$ and let S be an $\widetilde{ln_2^{pnd}}$-set in P_n. Since $\langle V(P_n) \setminus S \rangle$ is connected and S is a 2-locating point-wise non-dominating set, $1 \leq |V(P_n) \setminus S| \leq 2$. Clearly, at least one of v_1 and v_n is in S. Suppose that $v_1 \in S$. Suppose further that $|V(P_n) \setminus S| = 1$. Then $4 \leq n \leq 7$. Hence, $\widetilde{ln_2^{pnd}}(P_n) = n - 1$ for $4 \leq n \leq 7$. Next, suppose that $|V(P_n) \setminus S| = 2$. If p is the smallest integer such that $v_p \notin S$, then $p \notin \{1, 2, 3\}$. It follows that $v_1, v_2, v_3 \in S$. In this case, for $n \geq 8$, the set $S' = V(P_n) \setminus \{v_4, v_5\}$ is clearly an outer-connected 2-locating point-wise non-dominating set. Thus, $\widetilde{ln_2^{pnd}}(P_n) = n - 2$ for all $n \geq 8$.

(ii) Let $C_n = [v_1, v_2, v_3, \ldots, v_n]$. Clearly, $\widetilde{ln_2^{pnd}}(C_n) = n$ for $n = 3, 4$. Let $n \geq 5$ and let S be an $\widetilde{ln_2^{pnd}}$-set of C_n. Since $\langle V(C_n) \setminus S \rangle$ is connected and S is a 2-locating point-wise non-dominating set, $|V(C_n) \setminus S| = 2$. Therefore, $\widetilde{ln_2^{pnd}}(C_n) = n - 2$ for all $n \geq 5$.

The proofs of (iii) and (iv) are similar to (i) and (ii). □

Next, we show that every pair of positive integers are realizable as 2-resolving hop domination number and outer-connected 2-resolving hop domination number.

Remark 2. Every outer-connected 2-resolving hop dominating set of G is a 2-resolving hop dominating set of G. Thus, $\gamma_{2Rh}(G) \leq \widetilde{\gamma_{c2Rh}}(G)$.

Theorem 1. Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a nontrivial connected graph G such that $\gamma_{2Rh}(G) = a$ and $\widetilde{\gamma_{c2Rh}}(G) = b$.

Proof. Suppose $2 \leq a = b$. Consider Figure 1. Then $S = \{u_1, u_2, u_3, u_4, \ldots u_n\}$ is both a γ_{2Rh}-set and $\widetilde{\gamma_{c2Rh}}$-set of G_1. Hence, $2 \leq \gamma_{2Rh}(G_1) = \widetilde{\gamma_{c2Rh}}(G_1) = a = b$.

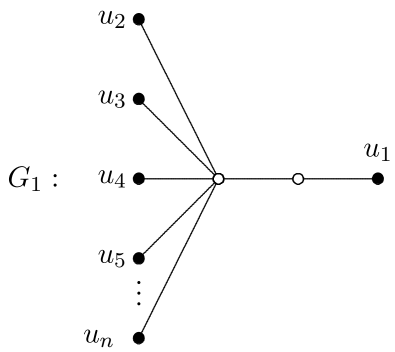

Figure 1

Suppose $2 < a < b$. Consider the graph G_2 in Figure 2. Then $S = \{x_1, x_2, \ldots, x_a\}$ is a γ_{2Rh}-set of G_2 and $X = S \cup \{y_1, y_2, \ldots, y_{b-a}\}$ is a $\widetilde{\gamma_{c2Rh}}$-set of G_2. Hence $\gamma_{2Rh}(G_2) = a$ and $\widetilde{\gamma_{c2Rh}}(G_2) = |X| = |S| + (b - a) = a + b - a = b$.

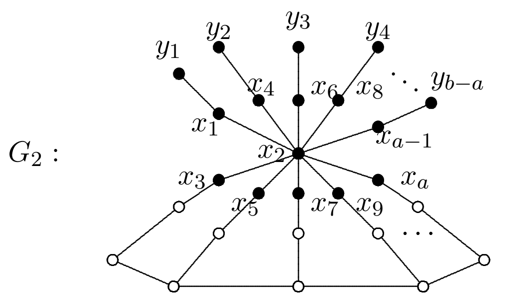

Figure 2

Corollary 1. For each positive integer n, there exists a connected graph G such that $\widetilde{\gamma_{c2Rh}}(G) - \gamma_{2Rh}(G) = n$, that is, $\widetilde{\gamma_{c2Rh}} - \gamma_{2Rh}$ can be made arbitrarily large.

We now characterize the outer-connected 2-resolving hop dominating sets in some graphs under some binary operations.

4. Join of Graphs

This section presents characterizations in the outer-connected 2-resolving hop dominating sets in the join of graphs.

Theorem 2. [12] Let G be a connected graph and let $K_1 = \{x\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving hop dominating set in $K_1 + G$ if and only if $S = \{x\} \cup T$ where T is a $(2, 1)$-locating point-wise non-dominating set in G.

Theorem 3. Let G be a connected graph and let $K_1 = \{x\}$. Then $S \subseteq V(K_1 + G)$ is an outer-connected 2-resolving hop dominating set in $K_1 + G$ if and only if $S = \{x\} \cup T$ where T is an outer-connected $(2, 1)$-locating point-wise non-dominating set in G.

Proof. Let $S \subseteq V(K_1 + G)$ be an outer-connected 2-resolving hop dominating set in $K_1 + G$. Then S is a 2-resolving hop dominating set in $K_1 + G$. Then by Theorem 2, $S = \{x\} \cup\ T$ where T is a (2,1)-locating point-wise non-dominating set in G. Now, since S is an outer-connected 2-resolving hop dominating set in $K_1 + G$, it follows that $S = V(K_1 + G)$ or $\langle V(K_1 + G)\backslash S\rangle = \langle V(G)\backslash T\rangle$ is connected. Thus, $T = V(G)$ or the subgraph $\langle V(G)\backslash T\rangle$ induced by $V(G)\backslash T$ is connected. Therefore, T is an outer-connected $(2, 1)$-locating point-wise non-dominating set in G.

Conversely, assume that $S = \{x\} \cup T$, where T is an outer-connected (2,1)-locating point-wise non-dominating set in G. By Theorem 2, S is a 2-resolving hop dominating set in $K_1 + G$. Next, since $\langle V(K_1 + G)\backslash S\rangle = \langle V(G)\backslash T\rangle$ and T is an outer-connected (2,1)-locating point-wise non-dominating set in G, it follows that S is a outer-connected 2-resolving hop dominating set in $K_1 + G$. □

Corollary 2. Let G be connected nontrivial graph. Then $\widetilde{\gamma_{c2Rh}}(K_1+G) = \widetilde{ln^{pnd}_{(2,1)}}(G)+1$.

Example 1. For a fan $F_n = P_n + 1$ on $n + 1$ vertices

$$\widetilde{\gamma_{c2Rh}}(F_n) = \widetilde{ln^{pnd}_{(2,1)}}(P_n) + 1 = \begin{cases} n, & \text{if } 4 \leq n \leq 7; \\ n-1, & \text{if } n \geq 8. \end{cases}$$

Example 2. For a wheel $W_n = C_n + 1$ on $n + 1$ vertices

$$\widetilde{\gamma_{c2Rh}}(W_n) = \widetilde{ln^{pnd}_{(2,1)}}(C_n) + 1 = \begin{cases} n+1, & \text{if } n = 4; \\ n-1, & \text{if } n \geq 5. \end{cases}$$

Theorem 4. [12] Let G and H be any two graphs. A set $S \subseteq V(G + H)$ is a 2-resolving hop dominating set in $G + H$ if and only if $S = S_G \cup S_H$ where $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating point-wise non-dominating sets in G and H, respectively, where S_G or S_H is a $(2, 2)$-locating point-wise non-dominating set or S_G and S_H are $(2, 1)$-locating point-wise non-dominating sets of G and H, respectively.

Theorem 5. [9] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is an outer-connected hop dominating set in $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating subsets of G and H, respectively, such that

(*i*) $\langle V(H)\backslash S_H \rangle$ is connected whenever $S_H \neq V(H)$ and $S_G = V(G)$ and

(*ii*) $\langle V(G)\backslash S_G \rangle$ is connected whenever $S_G \neq V(G)$ and $S_H = V(H)$.

Theorem 6. Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is an outer-connected 2-resolving hop dominating set in $G+H$ if and only if $S = S_G \cup S_H$ where $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating point-wise non-dominating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H are $(2,1)$-locating point-wise non-dominating sets of G and H, respectively, such that

(*i*) $\langle V(H)\backslash S_H \rangle$ is connected whenever $S_H \neq V(H)$ and $S_G = V(G)$ and

(*ii*) $\langle V(G)\backslash S_G \rangle$ is connected whenever $S_G \neq V(G)$ and $S_H = V(H)$.

Proof. Suppose that $S \subseteq V(G+H)$ is an outer-connected 2-resolving hop dominating set in $G+H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ then $S = S_G \cup S_H$. Now, since S is a 2-resolving hop dominating set, by Theorem 4, S_G and S_H are 2-locating point-wise non-dominating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating point-wise non-dominating set or S_G and S_H are $(2,1)$-locating point-wise non-dominating sets. Suppose $S_G = V(G)$ and $S_H \neq V(H)$. Since S is an outer-connected hop dominating set, by Theorem 5, $\langle V(H)\backslash S_H \rangle$ is connected. Hence, (*i*) holds. Similarly, suppose that $S_G \neq V(G)$ and $S_H = V(H)$. By Theorem 5, $\langle V(G)\backslash S_G \rangle$ is connected and so (*ii*) holds.

Conversely, suppose that $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ are sets as described and satisfying (*i*) and (*ii*). By Theorem 4, S is a 2-resolving hop dominating set of $G+H$. If $S_G = V(G)$ and $S_H = V(H)$, then $S = V(G+H)$ is an outer-connected 2-resolving hop dominating set. Suppose, $S \neq V(G+H)$. Consider the following cases:

Case 1: $S_G \neq V(G)$ and $S_H \neq V(H)$

Then $\langle V(G+H)\backslash S \rangle = \langle V(G)\backslash S_G \rangle + \langle V(H)\backslash S_H \rangle$ is connected.

Case 2: $S_G = V(G)$ and $S_H \neq V(H)$

Then $\langle V(G+H)\backslash S \rangle = \langle V(H)\backslash S_H \rangle$ is connected by (*i*).

Case 3: $S_H = V(H)$ and $S_G \neq V(G)$

Then $\langle V(G+H)\backslash S \rangle = \langle V(G)\backslash S_G \rangle$ is connected by (*ii*).

Accordingly, S is an outer-connected 2-resolving hop dominating set of $G+H$. □

As a consequence of Theorem 6 the next result follows.

Corollary 3. Let G and H be nontrivial connected graphs. Then

$$\widetilde{\gamma_{c2Rh}}(G+H) = \min\{ln^{pnd}_{(2,2)}(G) + ln^{pnd}_{2}(H), ln^{pnd}_{2}(G) + ln^{pnd}_{(2,2)}(H), ln^{pnd}_{(2,1)}(G) + ln^{pnd}_{(2,1)}(H)\},$$

5. Corona of Graphs

This section presents characterizations in the outer-connected 2-resolving hop dominating sets in the corona of graphs.

Remark 3. [7] Let $v \in V(G)$. For every $x, y \in V(H^v), d_{G\circ H}(x,w) = d_{G\circ H}(y,w)$ and $d_{G\circ H}(v,w) + 1 = d_{G\circ H}(x,w)$ for every $w \in V(G \circ H)\backslash V(H^v)$.

Theorem 7. [12] Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a 2-resolving hop dominating set of $G \circ H$ if and only if

$$S = A \cup \left(\bigcup_{v \in V(G)\cap N_G(A)} S_v \right) \cup \left(\bigcup_{w \in V(G)\backslash N_G(A)} D_w \right)$$

where

(*i*) $A \subseteq V(G)$ such that for each $w \in V(G)\backslash A$, there exists $x \in A$ with $d_G(w,x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H^y) \cap S \neq \varnothing$;

(*ii*) $S_v \subseteq V(H^v)$ is a 2-locating set of H^v for all $v \in V(G) \cap N_G(A)$; and

(*iii*) $D_w \subseteq V(H^w)$ is a 2-locating point-wise non-dominating set of H^w for all $w \in V(G)\backslash N_G(A)$.

Theorem 8. [9] Let G be a connected graph and let H be any graph. Then a subset C of $V(G \circ H)$ is an outer-connected hop dominating set of $G \circ H$ if and only if

$$C = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$$

where $S_v \subseteq V(H^v)$ for each $v \in V(G)$ and satisfies each of the following statements:

(*i*) $A = V(G)$ or $\langle V(G)\backslash A\rangle$ is connected;

(*ii*) If $A = V(G)$, then $\langle V(H^v)\backslash S_v\rangle$ is a connected proper subgraph of H^v for at most one vertex $v \in A$. Otherwise, $S_v = V(H^v)$ for all $v \in A$.

(*iii*) For all $v \in (V(G)\backslash N_G[A,2]$, there exists $w \in N_G(v)$ such that $S_w \neq \varnothing$;

(*iv*) S_v is a point-wise non-dominating set of H^v for all $v \in (V(G)\backslash N_G[A])$.

Theorem 9. Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is an outer-connected 2-resolving hop dominating set of $G \circ H$ if and only if

$$S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$$

where $S_v \subseteq V(H^v)$ for each $v \in V(G)$ and satisfies each of the following statements:

(i) $A = V(G)$ or $\langle V(G)\backslash A\rangle$ is connected;

(ii) If $A = V(G)$, then $\langle V(H^v)\backslash S_v\rangle$ is a connected proper subgraph of H^v for at most one vertex $v \in A$. Otherwise, $S_v = V(H^v)$ for all $v \in A$;

(iii) S_v is a 2-locating set for all $v \in V(G)$ where S_v is a (2-locating) point-wise non-dominating set of H^v if $v \in (V(G)\backslash N_G[A])$.

Proof. Suppose $S \subseteq V(G \circ H)$ is an outer-connected 2-resolving hop dominating set of $G \circ H$. Let $A = S \cap V(G)$, $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = A \cup \left(\bigcup_{v\in V(G)} S_v\right)$ Since S is an outer-connected hop dominating set, (i) and (ii) follow immediately from Theorem 8. Now, since S is a 2-resolving hop dominating set, by Theorem 7, (iii) holds.

Conversely, let S be the set as described and satisfies the given conditions. By Theorem 7, S is 2-resolving hop dominating set. Furthermore, because (i) and (ii) hold, S is an outer-connected hop dominating set. Accordingly, S is an outer-connected 2-resolving hop dominating set in $G \circ H$. □

Corollary 4. Let G and H be connected graphs of orders n and m, respectively. Then

$$\widetilde{\gamma_{c2Rh}}(G \circ H) \leq \min\{\widetilde{\gamma_c}(G)(m+1) + (n - \widetilde{\gamma_c}(G))ln_2(H), nln_2^{pnd}\}.$$

Proof. Let A be a $\widetilde{\gamma_c}$-set of G and S_v be an ln_2-set of H^v for each $v \in V(G) \setminus A$. Thus, by Theorem 9 $S = A \cup \left(\bigcup_{v\in V(G)} V(H^v)\right) \cup \left(\bigcup_{v\in V(G)\setminus A} S_v\right)$ is an outer-connected 2-resolving hop dominating set. Hence,

$$\begin{aligned}\widetilde{\gamma_{c2Rh}}(G \circ H) &\leq |S| = |A| + \sum_{v\in V(G)} |V(H^v)| + \sum_{v\in V(G)\setminus A} |S_v| \\ &= \widetilde{\gamma_c}(G)(m+1) + (n - \widetilde{\gamma_c}(G))ln_2(H).\end{aligned}$$

Let $A = \varnothing$, S_w be a ln_2^{pnd}-set of H^w. Then $S = A \cup \left(\bigcup_{w\in V(G)} S_w\right)$ is an outer-connected 2-resolving hop dominating set in $G \circ H$ by Theorem 9. Hence,

$$\widetilde{\gamma_{c2Rh}}(G \circ H) \leq |S| = |A| + \sum_{w\in V(G)} |S_w| = |V(G)| \cdot |S_w| = n(ln_2^{pnd}(H)).$$

Accordingly, $\widetilde{\gamma_{c2Rh}}(G \circ H) \leq \min\{\widetilde{\gamma_c}(G)(m+1) + (n - \widetilde{\gamma_c}(G))ln_2(H), nln_2^{pnd}\}$. □

6. Edge Corona of Graphs

This section presents characterizations in the outer-connected 2-resolving hop dominating sets in the edge corona of graphs.

Remark 4. [12] Let $uv \in E(G)$. For every $x, y \in V(H^{uv})$, $d_{G\diamond H}(x, w) = d_{G\diamond H}(y, w)$, $d_{G\diamond H}(u, w) = d_{G\diamond H}(x, w)$, and $d_{G\diamond H}(v, w)+1 = d_{G\diamond H}(x, w)$ for every $w \in V(G\diamond H)\backslash V(H^{uv})$.

Remark 5. [12] Let G and H be nontrivial connected graphs, $C \subseteq V(G \diamond H)$ and $S_{uv} = V(H^{uv}) \cap C$ where $uv \in E(G)$. For each $x \in V(H^{uv})\backslash S_{uv}$ and $z \in S_{uv}$,

$$d_{G\diamond H}(x, z) = \begin{cases} 1 & \text{if } z \in N_{H^{uv}}(x) \\ 2 & \text{otherwise.} \end{cases}$$

Definition 8. A leaf $l(G)$ of a graph G is a set of vertices v in G with $deg_G(v) = 1$.

Theorem 10. [12] Let $\gamma(G) \neq 1$ and H be any nontrivial connected graphs. A set $C \subseteq V(G \diamond H)$ is a 2-resolving hop dominating set of $G \diamond H$ if and only if

$$C = A \cup \left(\bigcup_{uv\in E(G)} S_{uv} \right)$$

where

(*i*) $A \subseteq V(G)$;

(*ii*) $S_{uv} \subseteq V(H^{uv})$ is a 2-locating set of H^{uv} for all $uv \in E(G)$ or if uv is a pendant edge, then S_{uv} is a $(2, 1)$-locating set of H^{uv} whenever $l(\langle\{u, v\}\rangle) \subseteq A$ and S_{uv} is a $(2, 2)$-locating set of H^{uv} otherwise.

Theorem 11. Let $\gamma(G) \neq 1$ and H be any nontrivial connected graphs. A set $S \subseteq V(G \diamond H)$ is an outer-connected 2-resolving hop dominating set of $G \diamond H$ if and only if

$$C = A \cup \left(\bigcup_{uv\in E(G)} S_{uv} \right)$$

where $S_{uv} \subseteq V(H^{uv})$ for each $uv \in E(G)$ and satisfies each of the following statements:

(*i*) $S_{uv} \subseteq V(H^{uv})$ is a 2-locating set of H^{uv} for all $uv \in E(G)$ or if uv is a pendant edge, then S_{uv} is a $(2, 1)$-locating set of H^{uv} whenever $l(\langle\{u, v\}\rangle) \subseteq A$ and S_{uv} is a $(2, 2)$-locating set of H^{uv} otherwise.

(*ii*) $A = V(G)$ or $\langle V(G)\backslash A\rangle$ is connected;

(*iii*) If $A = V(G)$, then $\langle V(H^{uv})\backslash S_{uv}\rangle$ is a connected proper subgraph of H^{uv} for at most one edge $uv \in E(G)$. Otherwise, $S_{uv} = V(H^{uv})$ for all $uv \in E(G)$;

Proof. Suppose C is an outer-connected 2-resolving hop dominating set in $G \diamond H$. Let $A = V(G) \cap C$ and $S_{uv} = C \cap V(H^{uv})$ for all $uv \in E(G)$. Then $C = A \cup \left(\bigcup_{uv \in E(G)} S_{uv} \right)$ where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv})$ for each $uv \in E(G)$. Then C is a 2-resolving hop dominating set in $G \diamond H$. By Theorem 10, (i) holds. Now, suppose $A \neq V(G)$. Then $C \neq V(G \diamond H)$. Since C is an outer-connected 2-resolving hop dominating set, it follows that

$$\langle V(G \diamond H) \backslash C \rangle = \langle V(H^{uv}) \backslash S_{uv} \rangle \cup \langle V(G) \backslash A \rangle$$

is connected. Hence, $\langle V(G) \backslash A \rangle$ is connected. Hence, (ii) holds. Suppose $A = V(G)$. If $V(G \diamond H) \neq C$, then $\langle V(G \diamond H) \backslash C \rangle = \langle V(H^{uv}) \backslash S_{uv} \rangle$. Since C is outer-connected 2-resolving hop dominating set, $\langle V(H^{uv}) \backslash S_{uv} \rangle$ is a connected proper subgraph of H^{uv} for at most one edge $uv \in E(G)$. Otherwise, if $V(G \diamond H) = C$, then $S_{uv} = V(H^{uv})$ for all $uv \in E(G)$. Hence, (iii) holds.

Conversely, let C be a set as described and satisfies the given conditions. By (i), C is a 2-resolving hop dominating set. If $V(G \diamond H) = C$, then we are done. Now, if $V(G \diamond H) \neq C$. Consider the following cases:

Case 1: $A = V(G)$

Then $\langle V(G \diamond H) \backslash C \rangle = \langle V(H^{uv}) \backslash S_{uv} \rangle$ and by (iii), $\langle V(H^{uv}) \backslash S_{uv} \rangle$ is a connected proper subgraph of H^{uv} for at most one edge $uv \in E(G)$. Thus, $\langle V(G \diamond H) \backslash C \rangle$ is connected.

Case 2: $A \neq V(G)$

Then $V(H^{uv}) = S_{uv}$ for all $uv \in E(G)$. Hence,

$$\langle V(G \diamond H) \backslash C \rangle = \langle V(H^{uv}) \backslash S_{uv} \rangle \cup \langle V(G) \backslash A \rangle = \langle V(G) \backslash A \rangle.$$

Thus, $\langle V(G \diamond H) \backslash C \rangle$ is connected since $\langle V(G) \backslash A \rangle$ is connected by (ii).
Accordingly, C is an outer-connected 2-resolving hop dominating set in $G \diamond H$. □

Corollary 5. Let $\gamma(G) \neq 1$ be any nontrivial connected graph of size m and H a nontrivial connected graph. Then the following statements hold.

(i) If G is a graph with no pendant edges, then $\widetilde{\gamma_{c2Rh}}(G \diamond H) = m \cdot ln_2(H)$.

(ii) If G is a graph with $k \geq 1$ pendant edges, then

$$\widetilde{\gamma_{c2Rh}}(G \diamond H) = \min\big\{ \big(m-k\big) ln_2(H) + k \cdot ln_{(2,1)}(H) + k, \big(m-k\big) ln_2(H) + k \cdot ln_{(2,2)}(H) \big\}$$

and $\widetilde{\gamma_{c2Rh}}(G \diamond H) = \big(m-k\big) ln_2(G) + k \cdot ln_{(2,2)}(G)$ whenever $ln_{(2,2)}(H) = ln_{(2,1)}(H)$.

7. Lexicographic Product of Graphs

This section presents characterizations on the outer-connected 2-resolving hop dominating sets in the lexicographic product of graphs.

Theorem 12. [12] Let G and H be nontrivial connected graphs. Then $W = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving hop dominating set in $G[H]$ if and only if

(*i*) $S = V(G)$;

(*ii*) T_x is a 2-locating set in H for every $x \in V(G)$;

(*iii*) T_x or T_y is a $(2, 1)$-locating set or one of T_x and T_y is a $(2, 2)$-locating set in H whenever $x, y \in EQ_1(G)$;

(*iv*) T_x and T_y are $(2 - locating)$ dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in EQ_2(G)$.

(*v*) T_x is a 2-locating point-wise non-dominating set in H for every $x \in S$ with $|N_G(x, 2)\cap S| = 0$.

Theorem 13. [9] Let G and H be connected nontrivial graphs. A subset $C = \bigcup_{x\in S}[\{x\}\times T_x]$ of $V(G[H])$ is an outer-connected hop dominating set of $G[H]$ if and only if

(*i*) S is a hop dominating set of G; and

(*ii*) T_x is a point-wise non-dominating set of H for every $x \in S$ with $|N_G(x, 2) \cap S| = 0$;

(*iii*) $\langle (V(G)\backslash S) \cup \{v \in S : T_v \neq V(H)\}\rangle$ is a connected graph in G.

Theorem 14. Let G and H be nontrivial connected graphs with $\Delta(H) \leq |V(H)| - 3$. Then $W = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is an outer-connected 2-resolving hop dominating set in $G[H]$ if and only if

(*i*) $S = V(G)$;

(*ii*) T_x is a 2-locating set of H for every $x \in V(G)$;

(*iii*) T_x and T_y are $(2, 1)$-locating set or one of T_x and T_y is a $(2, 2)$-locating set of H whenever $x, y \in EQ_1(G)$;

(*iv*) T_x and T_y are $(2 - locating)$ dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in EQ_2(G)$.

(*v*) T_x is a 2-locating point-wise non-dominating set of H for every $x \in S$ with $|N_G(x, 2) \cap S| = 0$.

(*vi*) $\langle \cup\{v \in V(G) : T_v \neq V(H)\}\rangle$ is a connected graph in G.

Proof. Let $W = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, be an outer-connected 2-resolving hop dominating set in $G[H]$. Then W is a 2-resolving hop dominating set in $G[H]$. Since W is an outer-connected hop dominating set and $S = V(G)$, by Theorem 13 (iii), $\langle\bigcup\{v \in V(G) : T_v \neq V(H)\}\rangle$ is a connected graph in G.

For the converse, let W be a 2-resolving hop dominating set and satisfies the given condition. If $V(G[H]) = W$, then we are done. On the other hand, suppose $V(G[H]) \neq W$. Since $S = V(G)$,

$$\langle(V(G)\backslash S) \cup \{v \in S : T_v \neq V(H)\}\rangle = \langle\cup\{v \in V(G) : T_v \neq V(H)\}\rangle$$

which is connected. By Theorem 13, Theorem 12(i), and by Theorem 12 (iii) Therefore, W is an outer-connected hop dominating set in $G[H]$. Accordingly, W is an outer-connected 2-resolving hop dominating set in $G[H]$. □

Corollary 6. Let G and H be any nontrivial connected graph with $\gamma(G) \neq 1$ and G is a free-equidistant. Then

$$\widetilde{\gamma_{c2Rh}}(G[H]) = |V(G)| \cdot ln_2(H).$$

Proof. Let $S = V(G)$ and let R_x be an ln_2-set of H for each $x \in S$. By Theorem 14, $W = \bigcup_{x\in S}[\{x\} \times R_x]$ is an outer-connected 2-resolving hop dominating set in $G[H]$. Thus,

$$\widetilde{\gamma_{c2Rh}}(G[H]) \leq |W| = |V(G)||R_x| = |V(G)|ln_2(H).$$

If $W_0 = \bigcup_{x\in S}(\{x\} \times T)$ is a $\widetilde{\gamma_{c2Rh}}$ -set of $G[H]$, then $S_0 = V(G)$ and T_x is a 2-locating set in H for each $x \in V(G)$ by Theorem 14. Hence,

$$\widetilde{\gamma_{c2Rh}}(G[H]) = |W_0| = |V(G)||T_x| \geq |V(G)| \cdot ln_2(H).$$

Therefore, $\widetilde{\gamma_{c2Rh}}(G[H]) = n \cdot ln_2(H)$. □

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