EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 1290-1301
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Hankel Determinant and Toeplitz Determinant on the Class of Bazilevič Functions Related to the Bernoulli Lemniscate

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Abstract. In this papers, we investigate the Hankel determinant and Toeplitz determinant for the class Bazilevič Function $\mathcal{B}_1(\alpha, \delta)$ related to the Bernoulli Lemniscate function on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and obtain the upper bounds of the determinant $H_2(1)$, $H_2(2)$, $T_2(1)$, and investigate $H_2(1)$ using coefficients invers function. We used lemma from Charateodory-Toeplitz and Libera about sharp inequalities for functions with positive real part.

2020 Mathematics Subject Classifications: 30C45, 30C50, 30C55, 30C80
Key Words and Phrases: Coefficients, Bazilevič functions, Bernoulli Lemniscate, Subordination, Hankel determinant, Toeplitz determinant.

1. Introduction

Let S denotes the class of analytic univalent function f defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$, and normalized by $f(0) = 0$ and $f'(0) = 1$, given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{P}$ denotes the class of analytic p and satisfies the condition $Re(p(z)) > 0$ for $z \in \mathbb{D} = \{z : |z| < 1\}$, $p \in \mathcal{P}$ gives,

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,\ n = 1, 2, 3, .. \tag{2}$$

where p_n is the positive real part [1].

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DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4772

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Definition 1. *Let $f \in S$ and satisfying the condition $f(0) = 1$ and $f'(0) = 0$. The function $f \in B_1(\alpha, \delta)$ for $\alpha \geq 0$ and $\delta > 0$ if and only if,*

$$\left[\frac{f'(z)\; f(z)^{\alpha-1}}{z^{\alpha-1}}\right] \prec \sqrt{1+z} =: \xi(z), for\ z \in \mathbb{D}\ and\ \xi(0) = 1, \tag{3}$$

where the branch of the square root is chosen to be $\xi(0) = 1$, the set $\xi(\mathbb{D})$ lies in the region bounded the right loop of the Bernoulli Lemniscate function is $(x^2+y^2)^2 - a^2(x^2-y^2)) = 0$, see [2], [14]. We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathbb{D}$, such that $\omega(0) = 0, |\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$, where $\omega(z) = \dfrac{\delta p(z) - 1}{\delta p(z) + 1}$. The form of the Lemniscate Bernoulli will be depended on the value of positive real δ. The following picture shows the Bazilevič function $B_1(\alpha, \delta)$ related to Bernoulli Lemniscate function.

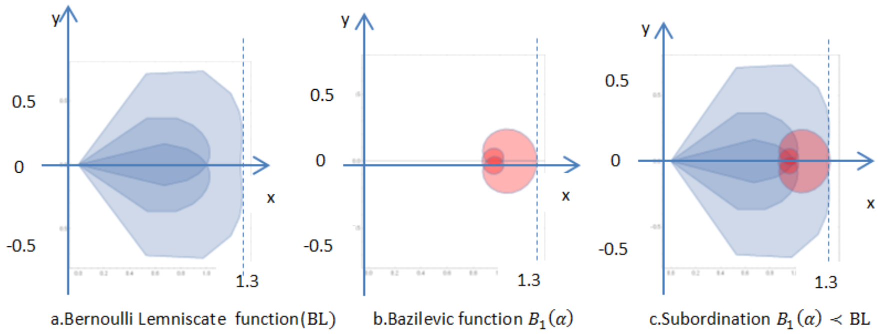

Figure 1: Bazilevič $\mathcal{B}_1(\alpha, \delta)$ subordination Bernoulli Lemniscate

From (3) we obtain initial coefficients which are used to determine the Hankel determinant and Toeplitz determinant for the sharp boundaries. The *q-th* Hankel determinant is denoted by $H_q(n)$, where $q \geq 1$ and $n \geq 1$ of functions f was stated by Noonan and Thomas [12] as,

$$H_q(n) = \begin{vmatrix} a_n & a_n+1 & .... & a_{n+q+1} \\ a_n+1 & a_n+2 & \cdots & a_n+q \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & a_n+q & \cdots & a_{n+2q-2} \end{vmatrix} \tag{4}$$

Since $f \in S$, $a_1 = 1$, in particular we have $H_2(1)$ as follow,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = (a_1a_3 - a_2^2).$$

Hankel determinant $H_2(1) = |a_3 - a_2^2|$ is well known as Fekete Szegö function. Previous research about Hankel determinant on Starlike function related to Bernoulli Lemniscate function in [4] obtained one of them is Hankel determinant $H_2(2)$. The other researches on Third Hankel determinant are studied in [5], [9]. Research by Thomas and Halim [15] defined the symmetric Toeplitz determinant $T_q(n)$ for $q \geq 1$ and $n \geq 1$ gives,

$$T_q(n) = \begin{vmatrix} a_n & a_n+1 & .... & a_{n+q-1} \\ a_n+1 & a_n & ... & a_{n+q-2} \\ ... & ... & ... & ... \\ a_{n+q-1} & a_{n+q-2} & ... & a_n \end{vmatrix} \tag{5}$$

An example of second order of Toeplitz determinant is $T_2(1)$ with $a_1 = 1$, is given by

$$T_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_1 \end{vmatrix} = (a_1^2 - a_2^2).$$

The well known research about the contruction of Toeplitz matrices has previously studied by (see [13] for more detail). In his work whose element are the coefficient f univalent functions assosiated with q-derivative operator.

2. Preliminaries

We have some lemmas used to determine sharp inequalities boundaries of Hankel determinant and Toeplitz determinant.

Lemma 1. *[1], [3]. If $p \in \mathcal{P}$ analityc in $\mathbb{D}$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ for $n \geq 1$ than*

$$|p_n| \leq 2 \tag{6}$$

For the $p(z) = (1+z)/(1-z)$, this lemmas an know as inequlity Caratheodory Toeplitz.

Lemma 2. *[7]. If $p \in \mathcal{P}$ analityc in $\mathbb{D}$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ then for some complex values x with $|x| \leq 1$ and some complex values ρ with $|\rho| \leq 1$,*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{7}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\rho \tag{8}$$

3. Results

Now, we state and prove the results from Hankel determinant and Toeplitz determinant of our investigation.

Theorem 1. *If $f \in B_1(\alpha, \delta)$ for $0 \leq \alpha \leq 1$ and $0 < \delta \leq 1$ then*

$$H_2(1) \leq \frac{3\sqrt{2}\sqrt{\delta} + 2(1+\alpha)(2+\alpha)\delta\sqrt{1+\delta} + 3\sqrt{2}\delta^{3/2}((3+2\alpha))}{2(2+\alpha)(1+\delta)^{7/2}},$$

and the inequality is sharp.

Proof. First consider from (3), we have initial coefficients a_1, a_2 and a_3 by [10], with $a_1 = 1$, a_2 and a_3 gives,

$$a_2 = \frac{p_1\sqrt{\alpha}}{\sqrt{2}(1+\delta)^{3/2}}, \tag{9}$$

$$a_3 = \frac{\delta}{8(2+\alpha)(1+\delta)^{7/8}}\Bigg(4\sqrt{2}p_2(1+\delta)^2 - p_1^2(\sqrt{2}+5\sqrt{2})\delta + 4\sqrt{2}\delta^2 - 2(-2+\alpha+\alpha^2\sqrt{\delta}\sqrt{1+\delta})\Bigg). \tag{10}$$

From (3) and (4), we can write Hankel determinant $H_2(1)$ gives,

$$\begin{aligned} H_2(1) &= \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_1a_3 - a_2^2| \\ &= \Bigg|\frac{p_2\sqrt{\delta}}{\sqrt{2}(2+\alpha)(1+\delta)^{7/2}} \\ &\quad + \frac{p_1^2\sqrt{\delta}(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(2+3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{3/2}}\Bigg|. \end{aligned} \tag{11}$$

Next, applying Lemma (2) to (11), gives

$$\begin{aligned} H_2(1) &= \Bigg|\frac{(p_1^2+(4-p_1^2)x)\sqrt{\delta}}{\sqrt{2}(2+\alpha)(1+\delta)^{7/2}} \\ &\quad + \frac{p_1^2\sqrt{\delta}(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(2+3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{3/2}}\Bigg|. \end{aligned} \tag{12}$$

By taing $p_1 = p$ and $0 \leq p \leq 2$ and applying them to (12) it follows that,

$$\begin{aligned} H_2(1) &\leq \frac{(p^2+(4-p^2)|x|)\sqrt{\delta}}{2\sqrt{2}(2+\alpha)(1+\delta)^{3/2}} \\ &\quad + \frac{p^2\sqrt{\delta}(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(2+3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{3/2}} \\ &:= \varphi_1(\alpha,\delta,p,|x|) \end{aligned} \tag{13}$$

From (13) then taking $|x| \leq 1$ gives,

$$\begin{aligned} H_2(1) &\leq \frac{p^2+(4-p^2)\sqrt{\delta}}{2\sqrt{2}(2+\alpha)(1+\delta)^{3/2}} \\ &\quad + \frac{p^2(\sqrt{\delta}(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(2+3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}} \end{aligned}$$

$$\begin{aligned} &= \frac{\sqrt{\delta}(8\sqrt{2}(1+\delta)^2}{8(2+\alpha)(1+\delta)^{3/2}} \\ &\quad + \frac{p^2\sqrt{\delta}(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(2+3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}} \\ &:= \varphi_1(\alpha,\delta,p) \end{aligned} \tag{14}$$

Next, we determine the derivative of $\varphi(\alpha,\delta,p)$ with respect to p from (14)are we obtain,

$$\varphi_1'(\alpha,\delta,p) = \frac{2p\sqrt{\delta}(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(2+3\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}}. \tag{15}$$

Let the derivative of $\varphi_1(\alpha,\delta,p)$ with respect to p is $\varphi_1'(\alpha,\delta,p)$. Then, from (15), we can show that $\varphi_1' > 0$ for $0 \le p \le 2$. Hence, φ_1 is an increasing monoton function. From which we obtain

$$H_2(1) \le \varphi_1(\alpha,\delta,2) = \frac{3\sqrt{2}\sqrt{\delta}+2(1+\alpha)(2+\alpha)\delta\sqrt{1+\delta}+3\sqrt{2}\delta^{3/2}((3+2\alpha))}{2(2+\alpha)(1+\delta)^{7/2}}.$$

The inequality is sharp when $p_1 = p_2 = 2$. The proof is completed.

Theorem 2. *If $f \in B_1(\alpha,\delta)$ for $\alpha_1 \le \alpha \le 1$ and $0 < \delta \le 1$ then*

$$\begin{aligned} H_2(2) \le\ & \left(\frac{\delta(1+\delta)^{3/2})}{6(2+\alpha)^2(1+\delta)^8}\right)\Big[(6\sqrt{2}(-1+\delta)(2+\alpha)^2\sqrt{\delta}+30\sqrt{2}(-1+\alpha) \\ & (2+\alpha)^2\delta^{3/2}+24\sqrt{2}(-1+\alpha)(2+\alpha)^2\delta^{5/2}+3(7+8\alpha+2\alpha^2) \\ & \sqrt{1+\delta}+(117+64\alpha-20\alpha^2+36\alpha^3+38\alpha^4+8\alpha^5)\delta\sqrt{1+\delta} \\ & +72(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}+48(3+4\alpha+\alpha^2)\delta^3\sqrt{1+\delta}) \\ & -12(1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4)-(2+\alpha)^2(1+\delta)^5\Big], \\ & \textit{with } \alpha_1 = 0,205 \textit{ is real root of the equation } x^3+4x^2+4x-1=0, \end{aligned}$$

and the inequality is sharp.

Proof. Based on equation (3), we have initial coefficients a_2, and a_3 in equation (9) and (10) respectively while a_4 is,

$$\begin{aligned} a_4 =\ & \left(\frac{1}{48(2+\alpha)(1+\alpha)^{9/2}}\right)\Big[\sqrt{\alpha}(24\sqrt{2}p_3(2+\alpha)(1+\delta)^3-12p_1p_2(1+\delta) \\ & (2\alpha^2\sqrt{\delta}\sqrt{1+\delta}+2(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2-3\sqrt{\delta}\sqrt{1+\delta})+\alpha(\sqrt{2}+5\sqrt{2}\delta \\ & +4\sqrt{2}\delta^2-4\sqrt{\delta}\sqrt{1+\delta}))+p_1^3(14\sqrt{2}\alpha^3\delta+4\sqrt{2}\alpha^4\delta+6(\sqrt{2}+7\sqrt{2}\delta \end{aligned}$$

$$+12\sqrt{2}\delta^2+8\sqrt{2}\delta^3-3\sqrt{\delta}\sqrt{1+\delta}-12\delta^{3/2}\sqrt{1+\delta})+\delta^2(-4\sqrt{2}\delta$$
$$+6\sqrt{\delta}\sqrt{1+\delta}+24\delta^{3/2}\sqrt{1+\delta})+\alpha(3\sqrt{2}-11\sqrt{2}\delta+36\sqrt{2}\delta^2$$
$$+24\sqrt{2}\delta^3+12\delta\sqrt{1+\delta}+48\delta^{3/2}\sqrt{1+\delta})))\Bigg]. \tag{16}$$

We can write Hankel determinant $H_2(2)$ as,

$$\begin{aligned}
H_2(2) &= \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4-a_3^2| \\
&= \Bigg|\left(\frac{p_1p_3\delta}{2(1+\delta)^3}\right)-\left(\frac{p_2^2\delta(1+5\alpha+10\alpha^2+16\alpha^3+5\alpha^4)}{2(2+\alpha)^2(1+\delta)^8}\right) \\
&\quad +\left(\frac{p_1^4\delta}{96(2+\alpha)^2(1+\delta)^{13/2}}\right)\Bigg[(6\sqrt{2}(-1+\delta)(2+\alpha)^2\sqrt{\delta}+30\sqrt{2} \\
&\quad (-1+\alpha)(2+\alpha)^2\delta^{3/2}+24\sqrt{2}(-1+\alpha)(2+\alpha)^2\delta^{5/2}+3(7+8\alpha+2\alpha^2) \\
&\quad \sqrt{1+\delta}+(117+64\alpha-20\alpha^2+36\alpha^3+38\alpha^4+8\alpha^5)\delta\sqrt{1+\delta} \\
&\quad +72(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}+48(3+4\alpha+\alpha^2)\delta^3\sqrt{1+\delta})\Bigg] \\
&\quad -\left(\frac{p_2\delta}{4(2+\alpha)^2(1+\delta)^{17/2}}\right)\Bigg[(2p_2\delta^5\sqrt{1+\delta}+p_1^2\sqrt{2}(-1+\alpha)(2+\alpha)^2 \\
&\quad \sqrt{\delta}(1+\delta)^4+(3+4\alpha+\alpha^2)(1+\delta)^{9/2}+4(3+4\alpha+\alpha^2)(1+\delta)^{9/2})\Bigg]\Bigg|.
\end{aligned} \tag{17}$$

Applying (17), Lemma 2 and taking $p_1 = p$ so that $0 \le p \le 2$ gives,

$$\begin{aligned}
H_2(2) &= \Bigg|\frac{\delta(4-p^2)x^2}{8(2+\alpha)^2(1+\delta)^3}+\frac{2p\delta(4-p^2)(1-x^2)\rho}{8(1+\delta)^3+(3+4\alpha+\alpha^2} \\
&\quad +\left(\frac{p^2}{8(2+\alpha)^2(1+\delta)^{17/2}}\right)\Bigg[(4-p^2)x\Bigg(\sqrt{2}(-1+\alpha)(2+\alpha)^2\sqrt{\delta}) \\
&\quad \sqrt{1+\delta}+2(3+4\alpha+\alpha^2)\delta\sqrt{1+\delta}-2(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}\Bigg)\Bigg] \\
&\quad +\left(\frac{p^4\delta}{96(2+\alpha)^2(1+\delta)^{13/2}}\right)\Bigg[(6\sqrt{2}(-1+\delta)(2+\alpha)^2\sqrt{\delta}+30\sqrt{2}(-1+\alpha) \\
&\quad (2+\alpha)^2\delta^{3/2}+24\sqrt{2}(-1+\alpha)(2+\alpha)^2\delta^{5/2}+3(7+8\alpha+2\alpha^2) \\
&\quad \sqrt{1+\delta}+(117+64\alpha-20\alpha^2+36\alpha^3+38\alpha^4+8\alpha^5)\delta\sqrt{1+\delta} \\
&\quad +72(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}+48(3+4\alpha+\alpha^2)\delta^3\sqrt{1+\delta})
\end{aligned}$$

$$
\begin{aligned}
& \left. \left. -12(1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4)-(2+\alpha)^2(1+\delta)^5 \right] \right| \\
:= \;& \varphi_1(\alpha,\delta,p,x,\rho). \qquad (18)
\end{aligned}
$$

From (18), then for some $|\rho| \leq 1$ gives

$$
\begin{aligned}
H_2(2) \;\leq\; & \frac{\delta(4-p^2)|x|^2}{8(2+\alpha)^2(1+\delta)^3}+\frac{2p\delta(4-p^2)(1-|x|^2)}{8(1+\delta)^3} \\
& +\left(\frac{p^2}{8(2+\alpha)^2(1+\delta)^{17/2}}\right)\Bigg[(4-p^2)|x|\Big(\sqrt{2}(-1+\alpha)(2+\alpha)^2\sqrt{\delta} \\
& +(3+4\alpha+\alpha^2)\sqrt{1+\delta}+2(3+4\alpha+\alpha^2)\delta\sqrt{1+\delta} \\
& -2(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}\Big)\Bigg] \\
& +\left(\frac{p^4\delta}{96(2+\alpha)^2(1+\delta)^8}\right)\Bigg[(6\sqrt{2}(-1+\delta)(2+\alpha)^2\sqrt{\delta}+30\sqrt{2}(-1+\alpha) \\
& (2+\alpha)^2\delta^{3/2}+24\sqrt{2}(-1+\alpha)(2+\alpha)^2\delta^{5/2}+3(7+8\alpha+2\alpha^2) \\
& \sqrt{1+\delta}+(117+64\alpha-20\alpha^2+36\alpha^3+38\alpha^4+8\alpha^5)\delta\sqrt{1+\delta} \\
& +72(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}+48(3+4\alpha+\alpha^2)\delta^3\sqrt{1+\delta}) \\
& -12(1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4)-(2+\alpha)^2(1+\delta)^5\Bigg] \\
:= \;& \varphi_1(\alpha,\delta,p,|x|). \qquad (19)
\end{aligned}
$$

Now we check the derivative of $\varphi_1(\alpha,\delta,p,|x|)$ with respect to $|x|$ from (18),

$$
\begin{aligned}
\varphi_1^{'}(\alpha,\delta,p,|x|) \;=\; & \frac{\delta(4-p^2)^2|x|}{4(2+\alpha)^2(1+\delta)^3}-\frac{p\delta(4-p^2)|x|}{2(1+\delta)^3} \\
& +\left(\frac{p^2}{8(2+\alpha)^2(1+\delta)^{17/2}}\right)\Bigg[(4-p^2)\Big(\sqrt{2}(-1+\alpha)(2+\alpha)^2\sqrt{\delta} \\
& +(3+4\alpha+\alpha^2)\sqrt{1+\delta}+2(3+4\alpha+\alpha^2)\delta\sqrt{1+\delta} \\
& -2(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}\Big)\Bigg] \qquad (20)
\end{aligned}
$$

Since $\varphi_1^{'}(\alpha,\delta,p,|x|) \geq 0$ when $\alpha_1 \leq \alpha \leq 1$ and $0 < \delta \leq 1$, then φ_1 is increasing monoton function. So that the maximum value of $\varphi_1(\alpha,\delta,p,|x|)$ is provided when $|x| = 1$ or

$$
H_2(2) \;\leq\; \frac{\delta(4-p^2)}{8(2+\alpha)^2(1+\delta)^3}+\left(\frac{p^2}{8(2+\alpha)^2(1+\delta)^{17/2}}\right)\Bigg[(4-p^2)\Big(\sqrt{2}(-1+\alpha)
$$

$$
\begin{aligned}
&(2+\alpha)^2\sqrt{\delta}+(3+4\alpha+\alpha^2)\sqrt{1+\delta}+2(3+4\alpha+\alpha^2)\delta\sqrt{1+\delta}\\
&-2(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}\Big)\Big]\\
&+\left(\frac{p^4\delta}{96(2+\alpha)^2(1+\delta)^8}\right)\Big[(6\sqrt{2}(-1+\delta)(2+\alpha)^2\sqrt{\delta}+30\sqrt{2}(-1+\alpha)\\
&(2+\alpha)^2\delta^{3/2}+24\sqrt{2}(-1+\alpha)(2+\alpha)^2\delta^{5/2}+3(7+8\alpha+2\alpha^2)\\
&\sqrt{1+\delta}+(117+64\alpha-20\alpha^2+36\alpha^3+38\alpha^4+8\alpha^5)\delta\sqrt{1+\delta}\\
&+72(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}+48(3+4\alpha+\alpha^2)\delta^3\sqrt{1+\delta})\\
&-12(1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4)-(2+\alpha)^2(1+\delta)^5\Big]\\
:=\;& \varphi_1(\alpha,\delta,p). \qquad (21)
\end{aligned}
$$

Next, the derivative of $\varphi_1(\alpha,\delta,p)$ with respect to p from (21) is,

$$
\begin{aligned}
\varphi_1'(\alpha,\delta,p) \;=\;& -\frac{p\delta(4-p^2)}{2(2+\alpha)^2(1+\delta)^3}-\left(\frac{p^3}{4(2+\alpha)^2(1+\delta)^{17/2}}\right)\Big(\sqrt{2}(-1+\alpha)\\
&(2+\alpha)^2\sqrt{\delta}+(3+4\alpha+\alpha^2)\sqrt{1+\delta}+2(3+4\alpha+\alpha^2)\delta\sqrt{1+\delta}\\
&-2(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}\Big)+\left(\frac{p(4-p^2)}{4(2+\alpha)^2(1+\delta)^{17/2}}\right)\Big(\sqrt{2}(-1+\alpha)\\
&(2+\alpha)^2\sqrt{\delta}+(3+4\alpha+\alpha^2)\sqrt{1+\delta}+2(3+4\alpha+\alpha^2)\delta\sqrt{1+\delta}\\
&-2(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}\Big)+\left(\frac{p^3\delta(1+\delta)^{3/2})}{24(2+\alpha)^2(1+\delta)^8}\right)\Big[(6\sqrt{2}(-1+\delta)\\
&(2+\alpha)^2\sqrt{\delta}+30\sqrt{2}(-1+\alpha)(2+\alpha)^2\delta^{3/2}+24\sqrt{2}(-1+\alpha)\\
&(2+\alpha)^2\delta^{5/2}+3(7+8\alpha+2\alpha^2)\sqrt{1+\delta}+(117+64\alpha-20\alpha^2\\
&+36\alpha^3+38\alpha^4+8\alpha^5)\delta\sqrt{1+\delta}+72(3+4\alpha+\alpha^2)\delta^2\sqrt{1+\delta}\\
&+48(3+4\alpha+\alpha^2)\delta^3\sqrt{1+\delta})-12(1+5\alpha+10\alpha^2\\
&+10\alpha^3+5\alpha^4)-(2+\alpha)^2(1+\delta)^5\Big] \qquad (22)
\end{aligned}
$$

From (22) we find the maximum value of $\varphi_1(\alpha,\delta,p)$ when $0\le p\le 2$. With elementary calculus, we can show that ${\varphi_1}'(\alpha,\delta,p)=0$ has three values of p but the only valid value is $p=0$ while the thers are not valid. Since $\varphi_1(\alpha,\delta,0)\le\varphi_1(\alpha,\delta,2)$ for $\alpha_1\le\alpha\le 1$ and $0<\delta\le 1$, then $H_2(2)\le\varphi_1(\alpha,\delta,2)$. The inequality is sharp when $p_1=p_2=p_3=2$. The proof is completed.

Theorem 3. *If $f\in B_1(\alpha,\delta)$, for $0\le\alpha\le 0$ and $0<\delta\le 1$ then*

$$T_2(1)=|a_1^2-a_2^2|\le 1, \qquad (23)$$

and the inequality is sharp.

Proof. Based on Definition 1, we have initial coefficients $a_1 = 1$ and a_2 (see (9)) and we can write Toeplitz determinant $T_2(1)$ as

$$T_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_1 \end{vmatrix} = |a_1^2 - a_2^2| = |1 - \frac{p_1^2\sqrt{\delta}}{(2+\delta)^3}|$$

Since $\left(1 - \frac{p_1^2\sqrt{\delta}}{(2+\delta)^3}\right) \geq 0$, if $\delta \geq 0$ and $0 < p_1 \leq 2$, we have,

$$T_2(1) = 1 - \frac{p_1^2\sqrt{\delta}}{(2+\delta)^3} := \varphi(p_1) \tag{24}$$

The derivative of (24) is,

$$\varphi'(p_1) = \frac{-2p_1\sqrt{\delta}}{(2+\delta)^3} \leq 0 \tag{25}$$

for all $p_1 \in [0, 2]$ and $\delta > 0$. According to (25), $\varphi(p_1)$ is monoton decreasing function, so the maximum value of $\varphi(0) = 1$. The inequality boundary is sharp for $p_1 = 0$. The proof is completed.

This research, we also obtain the upper bounds of the determinant Hankel $H_2(1)$ using coefficients invers function.

Theorem 4. *If $f \in B_1(\alpha, \delta)$ for $0 \leq \alpha \leq 0$ and $0 < \delta \leq 1$ then*

$$H_2(1) = |A_1A_3 - A_2^2| \leq \frac{\sqrt{2}\sqrt{\delta}}{(2+\alpha)(1+\delta)^{3/2}}, \tag{26}$$

and the inequality is sharp.

Proof. Let the coefficients on the inverse function are A_1, A_2 and A_3 by [11] gives,

$$\begin{aligned} A_2 &= -\frac{p_1\sqrt{\alpha}}{\sqrt{2}(1+\delta)^{3/2}}, \quad (27) \\ A_3 &= \frac{1}{8(2+\alpha)(1+\delta)^{7/8}}\bigg(-\sqrt{\delta}(4\sqrt{2}p_2(1+\delta)^2 \\ &\quad + p_1^2(\sqrt{2}+5\sqrt{2})\delta + 4\sqrt{2}\delta^2 - 2(-2+\alpha+\alpha^2\sqrt{\delta}\sqrt{1+\delta})\bigg). \quad (28) \end{aligned}$$

From (4) we have $H_2(1)$,

$$\begin{aligned} H_2(1) &= \begin{vmatrix} A_1 & A_2 \\ A_2 & A_3 \end{vmatrix} = |A_1A_3 - A_2^2| \\ &= \bigg| -\frac{p_2\sqrt{\delta}}{\sqrt{2}(2+\alpha)(1+\delta)^{3/2}} \end{aligned}$$

$$+\frac{p_1^2\sqrt{\delta}(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(-14-5\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}}\Bigg|. \tag{29}$$

Applying (29), lemma 2, and taking $p_1 = p$ and $0 \le p \le 2$ gives,

$$\begin{aligned} H_2(1) &= \Bigg| -\frac{(p^2+(4-p^2)x)\sqrt{\delta}}{2\sqrt{2}(2+\alpha)(1+\delta)^{3/2}} \\ &\quad +\frac{p^2\sqrt{\delta}(\sqrt{2}+5\sqrt{2}\delta+4\sqrt{2}\delta^2+2(-14-5\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}}\Bigg| \\ &= \Bigg|\frac{(4-p^2)x\sqrt{\delta}}{2\sqrt{2}(2+\alpha)(1+\delta)^{3/2}} \\ &\quad +\frac{p^2\sqrt{\delta}(3\sqrt{2}+9\sqrt{2}\delta+6\sqrt{2}\delta^2+2(-14-5\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}}\Bigg| \end{aligned} \tag{30}$$

Case 1. When $0 \le \alpha \le 1$ and $0 < \delta < \delta_1(\alpha)$, with $\delta_1(\alpha)$ is real number root of the equation

$$1+(-383-280\alpha+6\alpha^2+20\alpha^3-2\alpha^4)x+24x^2+16x^3=0.$$

From (30), if $|x| \le 1$ then,

$$\begin{aligned} H_2(1) &\le \frac{(4-p^2)|x|\sqrt{\delta}}{2\sqrt{2}(2+\alpha)(1+\delta)^{3/2}} \\ &\quad + \frac{p^2\sqrt{\delta}(3\sqrt{2}+9\sqrt{2}\delta+6\sqrt{2}\delta^2+2(-14-5\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}} \\ &\le \frac{2\sqrt{\delta}}{\sqrt{2}(2+\alpha)(1+\delta)^{3/2}} \\ &\quad + \frac{p^2\sqrt{\delta}(3\sqrt{2}+9\sqrt{2}\delta+6\sqrt{2}\delta^2+2(-14-5\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}} \\ &:= \varphi_1(\alpha,\delta,p). \end{aligned} \tag{31}$$

Let the derivative of $\varphi_1(\alpha,\delta,p)$ with respect to p is $\varphi_1'(\alpha,\delta,p)$. By solving $\varphi_1'(\alpha,\delta,p) = 0$, we obtain stationary point when $p = 0$. So we have two critical points $p = 0$ and $p = 2$.

Case 2. When $0 \le \alpha \le 1$ and $\delta_1(\alpha) \le \delta \le 1$.

From (30), if $|x| \le 1$ then,

$$H_2(1) \le \frac{(4-p^2)|x|\sqrt{\delta}}{2\sqrt{2}(2+\alpha)(1+\delta)^{3/2}}$$

$$
\begin{aligned}
& -\frac{p^2\sqrt{\delta}(3\sqrt{2}+9\sqrt{2}\delta+6\sqrt{2}\delta^2+2(-14-5\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}} \\
\leq\ & \frac{2\sqrt{\delta}}{\sqrt{2}(2+\alpha)(1+\delta)^{3/2}} \\
& -\frac{p^2\sqrt{\delta}(3\sqrt{2}+9\sqrt{2}\delta+6\sqrt{2}\delta^2+2(-14-5\alpha+\alpha^2)\sqrt{\delta}\sqrt{1+\delta})}{8(2+\alpha)(1+\delta)^{7/2}} \\
:=\ & \varphi_1(\alpha,\delta,p) \qquad (32)
\end{aligned}
$$

The same conclusion of case 1, let the derivative of $\varphi_1(\alpha,\delta,p)$ with respect to p is $\varphi_1'(\alpha,\delta,p)$. By solving $\varphi_1'(\alpha,\delta,p)=0$, we obtain stationary point when $p=0$. So we have two critical points $p=0$ and $p=2$.

Since $\varphi_1(\alpha,\delta,0)\geq\varphi_1(\alpha,\delta,2)$, then $H_2(1)\leq\varphi_1(\alpha,\delta,0)=\frac{\sqrt{2}\sqrt{\delta}}{(2+\alpha)(1+\delta)^{3/2}}$. The inequality is sharp when $p_1=0$ and $p_2=2$. The proof is completed.

Acknowledgements

Many thanks and appreciation for my supervisor Marjono and my co supervisor Sa‘adatul Fitri and Ratno Bagus Edy Wibowo at the Department of Mathematics and Natural Sciences Brawijaya University, for their support and guidance in completing this research and this paper.

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