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Adapting Integral Transforms to Create Solitary Solutions for Partial Differential Equations Via A New Approach

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Abstract. In this article, a new effective technique is implemented to solve families of nonlinear partial differential equations (NLPDEs). The proposed method combines the double ARA-Sumudu transform with the numerical iterative method to get the exact solutions of NLPDEs. The successive iterative method was used to find the solution of nonlinear terms of these equations. In order to show the efficiency and applicability of the presented method, some physical applications are analyzed and illustrated, and to defend our results, some numerical examples and figures are discussed.

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Key Words and Phrases: Laplace transform, Sumudu transform, Double ARA–Sumudu transform, New iterative method, Nonlinear partial differential equations.

1. Introduction

For Original Research Articles, Clinical Trial Articles, and Technology Reports the introduction should be succinct, with no subheadings. For Case Reports the Introduction should include symptoms at presentation, physical exams and lab results.
Partial differential equations are a common type of equations used to model complicated dynamical systems in the natural and engineering sciences. Typically, time and one or more space variables make up the independent variables in these equations. It is noteworthy that a large variety of scientifically and technologically significant phenomena are non-linear, and it is difficult to solve them analytically. Of further importance is the fact that several techniques exist to find special solutions to these problems and these particular solutions are required in order to analyze the relevant phenomena. Various effective mathematical

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methods have been introduced for obtaining exact and approximate analytic solutions of partial differential equations. Some of the classic analytic and numerical methods are the Adomian decomposition method [1, 11, 12, 17], the homotopy perturbation method [20], the reduced differential transform method [6, 13, 14] and the variational iteration method [28, 29], and others [15, 21].

There are several integral transforms in the literature, and they are widely employed in engineering, physics, and astronomy such as Laplace transform, Fourier transform, Mellin transform and others. It is also known that the integral transform method is an effective method to solve some certain differential equations. The Sumudu transform (ST) which is defined by the following formula

$$S\left[f\left(x\right)\right]=\frac{1}{u}\int_{0}^{\infty}e^{-\frac{x}{u}}f\left(x\right)dx,\quad u>0,$$

was introduced in [27] and applied to solve ordinary and partial differential equations. Recently, a new transform called the ARA transform (ARAT) was introduced in [22] and it is given by

$$\mathcal{G}_n\left[f\left(x\right)\right](s)=F\left(n,s\right)=s\int_{0}^{t}t^{n-1}e^{-sx}f\left(x\right)dx,\quad s>0,\quad n\in\mathbb{N},$$

$$\mathcal{G}_1\left[f\left(x\right)\right]=s\int_{0}^{\infty}e^{-sx}f\left(x\right)dx,\quad s>0.$$

On the other hand, recent research has focused mainly on the double integral transform method, which essentially involves applying the same transform twice to functions with two variables or using two distinct transforms on the same function. The literature contains a number of definitions of the double integral transform, like the double Laplace transform, the double Sumudu transform [9, 16, 26], the Laplace-Sumudu transform [2, 3, 5] and the double ARA-Sumudo transform (DARA-ST) [18, 23].

It should be noted that the DARA-ST have been applied effectively to get the solution of some partial differential equations [18]. A powerful formula for the DARA-ST of the Caputo fractional derivative was presented in [23] and used to create a series for some families of linear fractional differential equations. Regrettably, unlike other integral transforms, DARA cannot solve many complicated mathematical models or nonlinear problems. As a result, to solve a variety of nonlinear differential equations, some researchers have combined these integral transforms with other methods, including the differential transform method, the Adomian decomposition method, the variational iteration method, and the homotopy perturbation method [19, 25].

The main objective of this article is to find accurate solutions to nonlinear partial differential equations using the DARA-ST approach, including the new iterative method presented in [10, 24]. Numerous researchers have extensively employed the new iterative method to treat differential equations that are have integer and fractional orders [4, 7, 8]. The advantage of using the DARA-ST approach along with the iterative method is that, in contrast to other well-known methods [4, 7, 8]. The exact solution is quick convergent

without making any constrictive assumptions on the solution.
In this paper, we introduce the main properties of DARA-ST and some results related to partial derivatives, then the main idea of IDARA-ST approach, and finally we present some numerical applications.

2. Basic definitions and theorems

In this section we introduce the definitions of the DARA-ST with some properties that are essential to construct solution formula for some family of nonlinear partial differential equations.

Definition 1. *Let $\phi(x,t)$ be a continuous function of two variables $x > 0$ and $t > 0$. The DARA-ST of $\phi(x,t)$ which is a combination between ARAT of order one and the ST is given by*

$$\mathcal{G}_x S_t [\phi(x,t)] = \Phi(s,u) = \frac{s}{u}\int_0^\infty \int_0^\infty e^{-sx-\frac{t}{u}} \phi(x,t)\, dxdt, \quad s>0, \quad u>0, \tag{1}$$

provided the integral exists.
It is Clear that, the DARA-ST is linear, since

$$\mathcal{G}_x S_t [a\ \phi(x,t) + b\ \psi(x,t)] = a\ \mathcal{G}_x S_t [\ \phi(x,t)] + b\ \mathcal{G}_x S_t [\ \psi(x,t)], \tag{2}$$

where a and b are constants.

Definition 2. *The inverse DARA-ST is given by*

$$\mathcal{G}_x^{-1} S_t^{-1} [\Phi(s,u)] = \phi(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s} ds\ \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} \frac{e^{\frac{t}{u}}}{u} \Phi(s,u)\, du. \tag{3}$$

The DARA-ST for some functions of two variables are introduced in Table 1.

Table 1: DARA-ST for some functions [10].

$\phi(x,t)$	$\mathcal{G}_x S_t [\phi(x,t)] = \Phi(s,u)$
1	1
$x^a t^b$	$s^{-a}\Gamma(a+1)u\Gamma(b+1)$
e^{ax+bt}	$\frac{s}{(s-a)(1-bu)}$
$e^{I(ax+bt)}$	$\frac{Is}{(s-Ia)(bu+I)}$
$\sin(ax+bt)$	$\frac{s(a+bsu)}{(a^2+s^2)(b^2\ u^2+1)}$
$\cos(ax+bt)$	$\frac{s(s-abu)}{(a^2+s^2)(b^2u^2+1)}$
$\sinh(ax+bt)$	$\frac{s(a+bsu)}{(a^2-s)(b^2u^2-1)}$
$\cosh(ax+bt)$	$\frac{s(s+abu)}{(a^2-s^2)(b^2u^2-1)}$
$J_0\left(c\sqrt{xt}\right)$, J_0 is the zero order Bessel function	$\frac{4s}{4s+c^2u}$
$\phi(x-\delta,t-\epsilon)\ H(x-\delta,t-\epsilon)$	$e^{-s\delta-\frac{\epsilon}{y}}\Phi(s,u)$
$(\phi**\psi)(x,t)$	$\left(\frac{u}{s}\right)\Phi(s,u)\Psi(s,u)$

If function $\phi(x,t)$ is of exponential order c and d at $x \to \infty$ and $t \to \infty$, then there exists a non negative constant K such that $\forall x > X$ and $t > T$; we have the following

$$|\phi(x,t)| \leq K e^{cx+dt},$$

and we write the following

$$\phi(x,t) = O\left(e^{cx+dt}\right),$$

as x and t tend to infinity or the following is obtained.

$$\lim_{\substack{x \to +\infty, \\ t \to +\infty}} e^{-vx-\frac{t}{\omega}} |\phi(x,t)| = K \lim_{\substack{x \to +\infty, \\ t \to +\infty}} e^{-(v-c)x-\left(\frac{1}{\omega}-d\right)t} = 0, \quad v > c, \quad \frac{1}{w} > d.$$

Then, ϕ is the exponential order as x and t tend to infinity.
In the following theorem, we introduce the sufficient condition for existence of DARA-ST.

Theorem 1. *Let $\phi(x,t)$ be a continuous function on the region $[0,X) \times [0,T)$. If $\phi(x,t)$ is exponential orders λ and γ, then DARA-ST of $\phi(x,t)$ exists, for Re $[s] > \lambda$ and Re $\left[\frac{1}{u}\right] > \gamma$.*

In the following theorem, we present some derivatives properties of DARA-ST.

Theorem 2. *[18]. (Derivative properties) If $\Phi(s,u) = \mathcal{G}_x S_t[\phi(x,t)]$, then*

(i)

$$\mathcal{G}_x S_t\left[\frac{\partial \phi(x,t)}{\partial x}\right] = s\Phi(s,u) - sS_t[\phi(0,t)].$$

(ii)

$$\mathcal{G}_x S_t\left[\frac{\partial \phi(x,t)}{\partial t}\right] = \frac{1}{u}\Phi(s,u) - \frac{1}{u}\mathcal{G}_x[\phi(x,0)].$$

(iii)

$$\mathcal{G}_x S_t\left[\frac{\partial^2 \phi(x,t)}{\partial x^2}\right] = s^2\Phi(s,u) - s^2 S_t[\phi(0,t)] - sS_t[\phi_x(0,t)].$$

(iv)

$$\mathcal{G}_x S_t\left[\frac{\partial^2 \phi(x,t)}{\partial t^2}\right] = \frac{1}{u}\Phi(s,u) - \frac{1}{u^2}\mathcal{G}_x[\phi(x,0)] - \frac{1}{u}\mathcal{G}_x[\phi_t(x,0)].$$

(v)

$$\mathcal{G}_x S_t\left[\frac{\partial^2 \phi(x,t)}{\partial x \partial t}\right] = \frac{s}{u}\left(\Phi(s,u) - S_t[\phi(0,t)] - \mathcal{G}_x[\phi(x,0)] + \phi(0,0)\right).$$

Theorem 3. *[23]. (Convolution theorem) If $\mathcal{G}_x S_t[\phi(x,t)] = \Phi(s,u)$ and $\mathcal{G}_x S_t[\psi(x,t)] = \Psi(s,u)$, then*

$$\mathcal{G}_x S_t[(\phi * * \psi)(x,t)] = \left(\frac{u}{s}\right)\Phi(s,u)\Psi(s,u),$$

where

$$(\phi * * \psi)(x,t) = \int_0^x \int_0^t \phi(x-\delta, t-\epsilon)\psi(\delta,\epsilon)\, d\delta d\epsilon.$$

3. Idea of IDARA-ST method

The basic idea of the IDARA-ST approach, is mainly based on taking DARA-ST on the target equation and simplify the obtained one, then using the iterative method, we get our analytical solution.
In this section we apply IDARA-ST method to solve the following problem.
Let us consider the following nonlinear partial differential equation

$$A\phi_{xx}(x,t)+B\phi_{tt}(x,t)+C\phi_{x}(x,t)+D\phi_{t}(x,t)+E\phi(x,t)+\mathcal{N}[\phi(x,t)]=u(x,t), \quad (4)$$

where $(x,t)\in\mathbb{R}^{+}\times\mathbb{R}^{+}$, with the conditions

$$\begin{aligned} \phi(x,0)&=h_1(x), \quad \phi_t(x,0)=h_2(x),\\ \phi(0,t)&=r_1(t), \quad \phi_x(0,t)=r_2(t). \end{aligned} \quad (5)$$

Where A, B, C, D and E are constants, $\mathcal{N}$ is a nonlinear operator and

$$u(x,t)=u_1(x,t)+u_2(x,t)$$

is a source function.
To get the solution of equation (4) an (5) by the proposed method, we firstly, apply DARA-ST to equation (4) to get

$$\begin{aligned} &A\mathcal{G}_xS_t[\phi_{xx}(x,t)]+B\mathcal{G}_xS_t[\phi_{tt}(x,t)]+C\mathcal{G}_xS_t[\phi_{x}(x,t)]+D\mathcal{G}_xS_t[\phi_{t}(x,t)]\\ &\quad+E\mathcal{G}_xS_t[\phi(x,t)]+\mathcal{G}_xS_t[\mathcal{N}[\phi(x,t)]]=\mathcal{G}_xS_t[u(x,t)]. \end{aligned} \quad (6)$$

The facts in Theorem 2 and the linearity property imply that

$$\begin{aligned} &A\big(s^2\Phi(s,u)-s^2S_t[\phi(0,t)]-sS_t[\phi_x(0,t)]\big)\\ &\quad+B\left(\frac{1}{u^2}\Phi(s,u)-\frac{1}{u^2}\mathcal{G}_x[\phi(x,0)]-\frac{1}{u}\mathcal{G}_x[\phi_t(x,0)]\right)\\ &\quad+C\left(s\Phi(s,u)-sS_t[\phi(0,t)]\right)\\ &\quad+D\left(\frac{1}{u}\Phi(s,u)-\frac{1}{u}\mathcal{G}_x[\phi(x,0)]\right)+E\Phi(s,u)\\ &\quad=\mathcal{G}_xS_t[u_1(x,t)]+\mathcal{G}_xS_t[u_2(x,t)]-\mathcal{G}_xS_t[\mathcal{N}[\phi(x,t)]]. \end{aligned} \quad (7)$$

After simple computations and using the condition (5) equation (7) become

$$\begin{aligned} &\left(As^2+\frac{B}{u^2}+Cs+\frac{D}{u}+E\right)\Phi(s,u))=As^2R_1(u)+AsR_2(u)\\ &\quad+\frac{B}{u^2}H_1(s)+\frac{B}{u}H_2(s)+CsR_1(u)+\frac{D}{u}H_1(s)+U_1(s,u)\\ &\quad+\mathcal{G}_xS_t[u_2(x,t)-\mathcal{N}[\phi(x,t)]]. \end{aligned} \quad (8)$$

where

$$\begin{aligned}
R_1(u) &= S_t[\phi(0,t)] = S_t[r_1(t)], \\
R_2(u) &= S_t[\phi_x(0,t)] = S_t[r_2(t)], \\
H_1(s) &= \mathcal{G}_x[\phi(x,0)] = \mathcal{G}_x[h_1(x)], \\
H_2(s) &= \mathcal{G}_x[\phi_t(x,0)] = \mathcal{G}_x[h_2(x)], \\
U_1(s,u) &= \mathcal{G}_x[u_1(x,t)]
\end{aligned}$$

and

$$U_2(s,u) = \mathcal{G}_x[u_2(x,t)].$$

Now, simplifying equation (8), we get

$$\begin{aligned}
\Phi(s,u) = &\left(As^2 + \frac{B}{u^2} + Cs + \frac{D}{u} + E\right)^{-1} \left(As^2 R_1(u) + AsR_2(u) + \frac{B}{u^2} H_1(s)\right. \\
&\left. + \frac{B}{u} H_2(s) + CsR_1(u) + \frac{D}{u} H_1(s) + U_1(s,u)\right) \\
&+ \left(As^2 + \frac{B}{u^2} + Cs + \frac{D}{u} + E\right)^{-1} \mathcal{G}_x S_t[u_2(x,t) - \mathcal{N}[\phi(x,t)]].
\end{aligned} \tag{9}$$

Operating with the transform $\mathcal{G}_x^{-1} S_t^{-1}$ to (9), to get

$$\begin{aligned}
\phi(x,t) = &\mathcal{G}_x^{-1} S_t^{-1} \left[\left(As^2 + \frac{B}{u^2} + Cs + \frac{D}{u} + E\right)^{-1} \left(As^2 R_1(u) + AsR_2(u)\right.\right. \\
&\left.\left. + \frac{B}{u^2} H_1(s) + \frac{B}{u} H_2(s) + CsR_1(u) + \frac{D}{u} H_1(s) + U_1(s,u)\right)\right] \\
&+ \mathcal{G}_x^{-1} S_t^{-1} \left[\left(As^2 + \frac{B}{u^2} + Cs + \frac{D}{u} + E\right)^{-1} \mathcal{G}_x S_t[u_2(x,t) - \mathcal{N}[\phi(x,t)]]\right].
\end{aligned} \tag{10}$$

Now, we use the iteration method, so define the series solution

$$\phi(x,t) = \sum_{i=0}^{\infty} \phi_i(x,t). \tag{11}$$

Substituting (11) in (10) and assuming that the nonlinear term $\mathcal{N}[\phi(x,t)]$ can be decomposed as

$$\mathcal{N}\left[\sum_{i=0}^{\infty} \phi_i(x,t)\right] = \mathcal{N}[\phi_0(x,t)] + \sum_{i=1}^{\infty} \left(\mathcal{N}\left[\sum_{k=0}^{i} \phi_k(x,t)\right] - \mathcal{N}\left[\sum_{i=0}^{i-1} \phi_k(x,t)\right]\right),$$

thus, we have

$$\sum_{i=0}^{\infty}\phi_i(x,t)=\mathcal{G}_x^{-1}S_t^{-1}\left[\left(As^2+\frac{B}{u^2}+Cs+\frac{D}{u}+E\right)^{-1}\left(As^2R_1(u)+AsR_2(u)\right.\right.$$
$$\left.\left.+\frac{B}{u^2}H_1(s)+\frac{B}{u}H_2(s)+CsR_1(u)+\frac{D}{u}H_1(s)+U_1(s,u)\right)\right]$$
$$+\mathcal{G}_x^{-1}S_t^{-1}\left[\left(As^2+\frac{B}{u^2}+Cs+\frac{D}{u}+E\right)^{-1}\mathcal{G}_xS_t\left[u_2(x,t)-\mathcal{N}\left[\phi_0(x,t)\right]\right]\right]$$
$$+\mathcal{G}_x^{-1}S_t^{-1}\left[\left(As^2+\frac{B}{u^2}+Cs+\frac{D}{u}+E\right)^{-1}\right.$$
$$\left.\mathcal{G}_xS_t\left[\sum_{i=1}^{\infty}\left(\mathcal{N}\left[\sum_{k=0}^{i}\phi_k(x,t)\right]-\mathcal{N}\left[\sum_{i=0}^{i-1}\phi_k(x,t)\right]\right)\right]\right].$$

As a result, we obtain the following relations for the solution

$$\phi_0(x,t)=\mathcal{G}_x^{-1}S_t^{-1}\left[\left(As^2+\frac{B}{u^2}+Cs+\frac{D}{u}+E\right)^{-1}\left(As^2R_1(u)+AsR_2(u)+\frac{B}{u^2}H_1(s)\right.\right.$$
$$\left.\left.+\frac{B}{u}H_2(s)+CsR_1(u)+\frac{D}{u}H_1(s)+U_1(s,u)\right)\right],$$
$$\phi_1(x,t)=\mathcal{G}_x^{-1}S_t^{-1}\left[\left(As^2+\frac{B}{u^2}+Cs+\frac{D}{u}+E\right)^{-1}\mathcal{G}_xS_t\left[u_2(x,t)-\mathcal{N}\left[\phi_0(x,t)\right]\right]\right],$$
$$\vdots$$
$$\phi_{r+1}\ (x,t)=-\mathcal{G}_x^{-1}S_t^{-1}\left[\left(As^2+\frac{B}{u^2}+Cs+\frac{D}{u}+E\right)^{-1}\right.$$
$$\left.\mathcal{G}_xS_t\left[\sum_{i=1}^{\infty}\left(\mathcal{N}\left[\sum_{k=0}^{i}\phi_k(x,t)\right]-\mathcal{N}\left[\sum_{i=0}^{i-1}\phi_k(x,t)\right]\right)\right]\right]. \tag{12}$$

Following that, we get the analytical series solution of equation (4) as

$$\phi(x,t)=\phi_0(x,t)+\phi_1(x,t)+\ldots.$$

4. Numerical Problems

Some problems of familiar nonlinear partial differential equations, are solved in this section, these equations present some physical applications, such as advection equation, wave equation, heat equation and KdV equation, all of these equations are important in physical science. We apply IDARA-ST to solve the proposed equations and we sketch the exact solutions.

Problem 1. *Consider the non-homogeneous advection problem*

$$\phi_t\left(x,t\right)+\phi\left(x,t\right)\phi_x\left(x,t\right)=2t+x+t^3+xt^2, \tag{13}$$

with initial condition

$$\phi\left(x,t\right)=0. \tag{14}$$

Solution. To apply IDARA-ST method, we operate DARA-ST to (13), thus we get

$$\mathcal{G}_xS_t\left[\phi_t\left(x,t\right)\right]+\mathcal{G}_xS_t\left[\phi\left(x,t\right)\phi_x\left(x,t\right)\right]=\mathcal{G}_xS_t\left[2t+x\right]+\mathcal{G}_xS_t\left[t^3+xt^2\right]. \tag{15}$$

Running DARA-ST on equation (15) and using condition (14) and the results in Theorem 2, we get after simple computations

$$\Phi\left(s,u\right)=2u^2+\frac{u}{s}+u\mathcal{G}_xS_t\left[t^3+xt^2-\phi\left(x,t\right)\phi_x\left(x,t\right)\right]. \tag{16}$$

Running the inverse transform $\mathcal{G}_x^{-1}S_t^{-1}$ on (16), we have

$$\phi\left(x,t\right)=t^2+xt+\mathcal{G}_x^{-1}S_t^{-1}\left[u\mathcal{G}_xS_t\left[t^3+xt^2-\phi\left(x,t\right)\phi_x\left(x,t\right)\right]\right]. \tag{17}$$

Now, applying the iterative method, we substitute (11) in (17), thus we use the equations in (12).
Following that, we get the components of the solution as

$$\begin{aligned}
\phi_0\left(x,t\right)&=t^2+xt,\\
\phi_1\left(x,t\right)&=\mathcal{G}_x^{-1}S_t^{-1}\left[u\mathcal{G}_xS_t\left[t^3+xt^2-\phi_0\left(x,t\right)\frac{\partial\phi_0\left(x,t\right)}{\partial x}\right]\right]=0,\\
\phi_2\left(x,t\right)&=\mathcal{G}_x^{-1}S_t^{-1}\left[u\mathcal{G}_xS_t\left[\left(\phi_0\left(x,t\right)+\phi_1\left(x,t\right)\right)\frac{\partial\left(\phi_0\left(x,t\right)+\phi_1\left(x,t\right)\right)}{\partial x}\right.\right.\\
&\quad\left.\left.-\ \phi_0\left(x,t\right)\frac{\partial\phi_0\left(x,t\right)}{\partial x}\right]\right]=0.
\end{aligned} \tag{18}$$

As a result, we have the solution of Equation (13) as follows

$$\phi\left(x,t\right)=t^2+xt. \tag{19}$$

Figure 1 below, shows the 3D graph of the exact solution of problem (13) and (14).

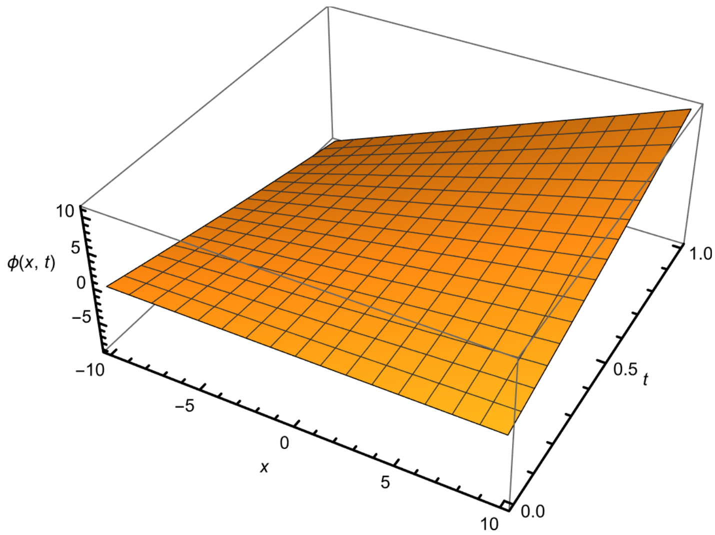

Figure 1: Exact solution $\phi(x,t)$ of Problem 1.

Problem 2. *Consider the nonlinear wave equation*

$$\phi_{tt}(x,t) - \phi_{xx}(x,t) + (\phi(x,t)\,\phi_x(x,t))_t = 2e^{-t}\sin x - 2e^{-2t}\sin x\cos x, \tag{20}$$

with the initial conditions

$$\phi(x,0) = \sin x, \quad \phi_t(x,t) = -\sin x, \tag{21}$$

and the boundary conditions

$$\phi(0,t) = 0, \quad \phi_x(0,t) = e^{-t}. \tag{22}$$

Solution. To apply IDARA-ST method, apply DARA-ST to equation (20) and ARAT to (21) and ST to (22), we get

$$\begin{aligned} &\mathcal{G}_x S_t[\phi_{tt}(x,t)] - \mathcal{G}_x S_t[\phi_{xx}(x,t)] + \mathcal{G}_x S_t[(\phi(x,t)\,\phi_x(x,t))_t] \\ &\quad = \frac{2}{(s^2+1)(1+u)} - \mathcal{G}_x S_t\left[2e^{-2t}\sin x\cos x\right], \end{aligned} \tag{23}$$

The results in Theorem 2, and the linearity property, imply that equation (23) becomes

$$\begin{aligned} &\frac{1}{u^2}\Phi(s,u) - \frac{1}{u^2}\mathcal{G}_x[\phi(x,0)] - \frac{1}{u}\mathcal{G}_x[\phi_t(x,0)] - s^2\Phi(s,u) + s^2 S_t[\phi(0,t)] \\ &\quad + sS_t[\phi_x(0,t)] + \mathcal{G}_x S_t[(\phi(x,t)\,\phi_x(x,t))_t] \\ &\quad = \mathcal{G}_x S_t\left[2e^{-t}\sin x\right] - \mathcal{G}_x S_t\left[2e^{-2t}\sin x\cos x\right]. \end{aligned} \tag{24}$$

Substituting the values of the conditions (21) and (22) and running DARA-ST on (24) with simple computations, one can get

$$\Phi(s,u) = \frac{s}{(s^2+1)(1+u)} - \frac{u^2}{1-s^2u^2}\mathcal{G}_x S_t\left[2e^{-2t}\sin x\cos x + (\phi(x,t)\,\phi_x(x,t))_t\right]. \tag{25}$$

Operating the inverse transform $\mathcal{G}_x^{-1}S_t^{-1}$ on (25), we have

$$\phi(x,t) = e^{-t}\sin x - \mathcal{G}_x^{-1}S_t^{-1}\left[\frac{u^2}{1-s^2u^2}\mathcal{G}_xS_t\left[2e^{-2t}\sin x\cos x + \left(\phi(x,t)\,\phi_x(x,t)\right)_t\right]\right]. \tag{26}$$

Now, we apply the iterative method, with same procedure of Problem 1, we get the following components of the solution

$$\begin{aligned}\phi_0(x,t) &= e^{-t}\sin x,\\ \phi_1(x,t) &= -\mathcal{G}_x^{-1}S_t^{-1}\left[\frac{u^2}{1-s^2u^2}\mathcal{G}_xS_t\left[2e^{-2t}\sin t\cos x + \left(\phi_0(x,t)\,\phi_{0x}(x,t)\right)_t\right]\right] = 0,\\ \phi_2(x,t) &= -\mathcal{G}_x^{-1}S_t^{-1}\left[\frac{u^2}{1-s^2u^2}\mathcal{G}_xS_t\left[\frac{\partial}{\partial t}\left((\phi_0(x,t) + \phi_1(x,t))\left(\phi_0(x,t)+\phi_1(x,t)\right)_t - \left(\phi_0(x,t)\,\phi_{0x}(x,t)\right)_t\right)\right]\right] = 0.\end{aligned} \tag{27}$$

Following that, we have the solution of Equation (27) as follows

$$\phi(x,t) = e^{-t}\sin x. \tag{28}$$

Figure 2 below, shows the 3D graph of the exact solution of the initial boundary value problem (20) (21) and (22).

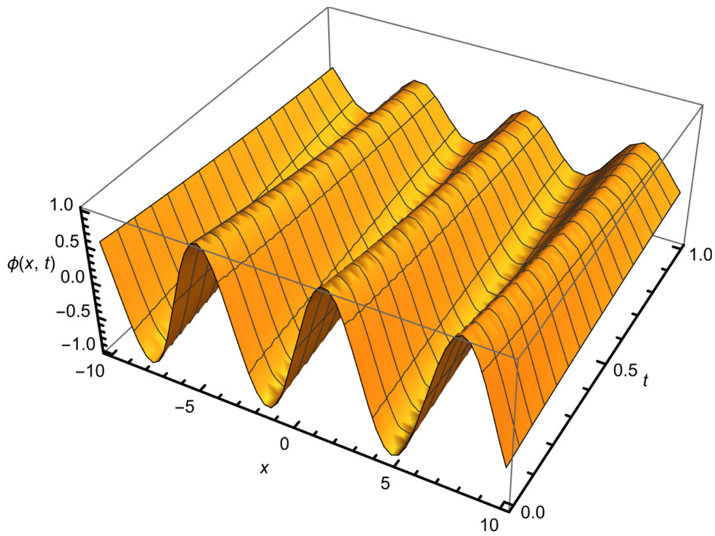

Figure 2: Exact solution $\phi(x,t)$ of Problem 2.

Problem 3. *Consider the nonhomogeneous KdV equation*

$$\phi_t(x,t) - \phi(x,t)\,\phi_x(x,t) + \phi_{xxx}(x,t) = -e^x(t+1) + te^x\left(-te^x+1\right), \tag{29}$$

with the initial condition

$$\phi(x,0)=1, \tag{30}$$

and the boundary conditions

$$\phi(0,t)=1-t, \quad \phi_x(0,t)=\phi_{xx}(0,t)=-t. \tag{31}$$

Solution. To apply IDARA-ST method, apply DARA-ST to equation (29), to get

$$\begin{aligned}\mathcal{G}_xS_t[\phi_t(x,t)]-\mathcal{G}_xS_t[\phi(x,t)\phi_x(x,t)]+\mathcal{G}_xS_t[\phi_{xxx}(x,t)]\\ =\mathcal{G}_xS_t[-e^x(t+1)]+\mathcal{G}_xS_t[te^x(-te^x+1)],\end{aligned} \tag{32}$$

Running DARA-ST on (32) and using the conditions (30) and (31), and the results in Theorem 2, we get after simple computations,

$$\Phi(s,u)=1-\frac{su}{s-1}+\frac{u}{1+us^3}\mathcal{G}_xS_t[te^x(-te^x+1)+\phi(x,t)\phi_x(x,t)]. \tag{33}$$

Operating the inverse transform $\mathcal{G}_x^{-1}S_t^{-1}$ on (33), we have

$$\phi(x,t)=1-te^x+\mathcal{G}_x^{-1}S_t^{-1}\left[\frac{u}{1+us^3}\mathcal{G}_xS_t[te^x(-te^x+1)+\phi(x,t)\phi_x(x,t)]\right]. \tag{34}$$

Applying the iterative method, we get the following components of the solution

$$\begin{aligned}\phi_0(x,t)&=1-te^x,\\ \phi_1(x,t)&=\mathcal{G}_x^{-1}S_t^{-1}\left[\frac{u}{1+us^3}\mathcal{G}_xS_t[te^x(-te^x+1)+\phi_0(x,t)\phi_{0x}(x,t)]\right]=0,\\ \phi_2(x,t)&=\mathcal{G}_x^{-1}S_t^{-1}\left[\frac{u}{1+us^3}\mathcal{G}_xS_t[(\phi_0(x,t)+\phi_1(x,t))(\phi_0(x,t)+\phi_1(x,t))_x\right.\\ &\quad\left.-\ \phi_0(x,t)\phi_{0x}(x,t)]\right]=0.\end{aligned} \tag{35}$$

Following that, we get the solution of Equation (29) as follows

$$\phi(x,t)=1-te^x. \tag{36}$$

Figure 3 below, shows the 3D graph of the exact solution of the initial boundary value problem (29) (30) and (31).

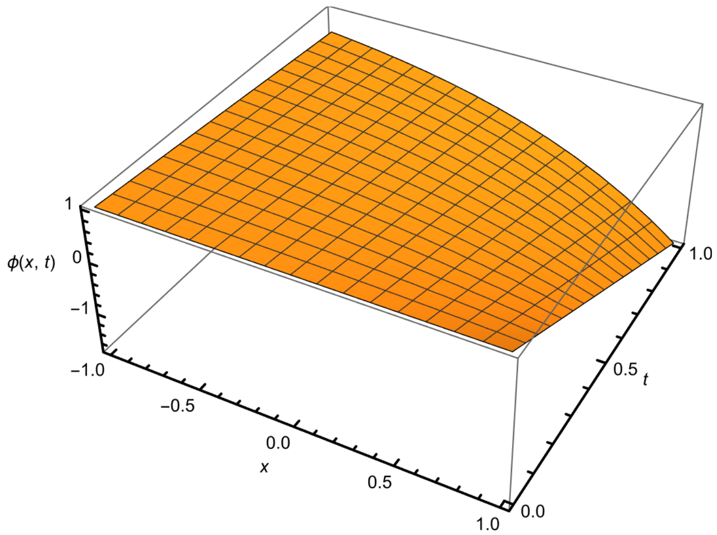

Figure 3: Exact solution $\phi(x,t)$ of Problem 3.

Problem 4. *Consider the following non-linear telegraph equation*

$$\phi_{xx}(x,t) = \phi_{tt}(x,t) + 2\phi_t(x,t) + \phi^2(x,t) + e^{x-2t} - e^{2x-4t}, \tag{37}$$

with the initial conditions

$$\phi(x,0) = e^x, \quad \phi_t(x,0) = -2e^x, \tag{38}$$

and the boundary conditions

$$\phi(0,t) = \phi_x(0,t) = e^{-2t}. \tag{39}$$

Solution. To apply IDARA-ST method, apply DARA-ST to equation (37) and ARAT to (37) and ST to (38), to get

$$\mathcal{G}_x S_t[\phi_{xx}(x,t)] = \mathcal{G}_x S_t[\phi_{tt}(x,t)] + \mathcal{G}_x S_t[2\phi_t(x,t)] + \mathcal{G}_x S_t\left[\phi^2(x,t) + e^{x-2t} - e^{2x-4t}\right], \tag{40}$$

Running DARA-ST on (40) and using conditions (38) and (39), and the results in Theorem 2, we get after simple computations,

$$\Phi(s,u) = \frac{s}{(s-1)(1+2u)} + \frac{u^2}{u^2s^2-2u-1}\mathcal{G}_x S_t\left[\phi^2(x,t) - e^{2x-4y}\right]. \tag{41}$$

Running the inverse transform $\mathcal{G}_x^{-1} S_t^{-1}$ on (41), we have

$$\phi(x,t) = e^{x-2t} + \mathcal{G}_x^{-1} S_t^{-1}\left[\frac{u^2}{u^2s^2-2u-1}\mathcal{G}_x S_t\left[\phi^2(x,t) - e^{2x-4t}\right]\right]. \tag{42}$$

Applying the iterative approach, we get the components of the solution as

$$\phi_0(x,t) = e^{x-2t},$$
$$\phi_1(x,t) = \mathcal{G}_x^{-1} S_t^{-1}\left[\frac{u^2}{u^2 s^2 - 2u - 1}\mathcal{G}_x S_t\left[\phi_0^2(x,t) - e^{2x-4t}\right]\right] = 0, \tag{43}$$
$$\phi_2(x,t) = \mathcal{G}_x^{-1} S_t^{-1}\left[\frac{u^2}{u^2 s^2 - 2u - 1}\mathcal{G}_x S_t\left[\left(\phi_0(x,t) + \phi_1(x,t)\right)^2 - \phi_0^2(x,t)\right]\right] = 0.$$

Thus, we get the solution of Equation (37) as follows.

$$\phi(x,t) = e^{x-2t}. \tag{44}$$

Figure 4 below, shows the 3D graph of the exact solution of the initial boundary value problem (37) (38) and (39).

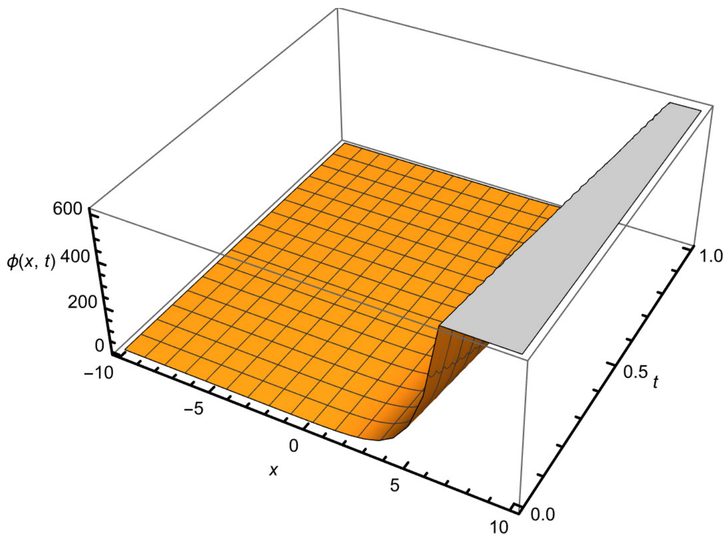

Figure 4: Exact solution $\phi(x,t)$ of Problem 4.

5. Conclusions

In this study, we introduced a novel approach in double transform to get the solutions of nonlinear partial differential equations with initial and boundary conditions. This new approach combines the double ARA-Sumudu transform with the numerical technique iterative method. To illustrate the applicability and efficiency of the technique, some numerical examples were considered. Our study led us to the conclusion that combining DARA-ST with the iterative technique produces effective analytical results with less computational efforts.

Conflict of Interest

The authors declare no conflicts of interest.

Author Contributions

All authors have equal contributions to the article and agreed on the submitted version.

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