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Coefficient Problems for Star-like Functions with Respect to Symmetric Conjugate Points Connected to the Sine Function

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Abstract. In this paper, we introduce the subclass of star-like functions with respect to symmetric conjugate points associated with the sine function. Some coefficient functionals for this class are considered. Bounds of Taylor coefficients, logarithmic coefficients, and the Hankel and Toeplitz determinants whose entries are logarithmic coefficients are provided.

2020 Mathematics Subject Classifications: 30C45, 30C50

Key Words and Phrases: Star-like functions with respect to symmetric conjugate points, sine function, coefficient estimates, logarithmic coefficients, Hankel and Toeplitz determinants of logarithmic coefficients, subordination

1. Introduction

Let A and S denote the classes of analytic and univalent functions, respectively. These classes are defined in the form of

$$\mathrm{A} = \left\{ f \in \mathrm{K}(E) : \ f(0) = f'(0) - 1 = 0, \ z \in E \right\}$$

and

$$S = \{ f \in \mathrm{A} : \ f \text{ is univalent in } E \},$$

where $\mathrm{K}(E)$ is the set of analytic functions in the open unit disk $E = \{ z \in \mathbb{C} : |z| < 1 \}$.

If $f \in \mathrm{A}$, then it can be expressed in the series representation of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in E. \tag{1}$$

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Let H denotes the class of Schwarz functions υ which are analytic in E given by

$$\upsilon(z) = \sum_{k=1}^{\infty} b_k z^k,\ z \in E$$

and satisfying $\upsilon(0) = 0$ and $|\upsilon(z)| < 1$. Given two functions $f, g \in \mathrm{A}$. We let $\prec$ to denote the subordination. The analytic function f is subordinate to another analytic function g if there exists a Schwarz function $\upsilon \in H$ such that $f(z) = g(\upsilon(z))$ for all $z \in E$. Furthermore, if g is univalent in E, then we have the following equivalence

$$f \prec g \Leftrightarrow f(0) = g(0)$$

and

$$f(E) = g(E).$$

Let $P(A, B)$ denotes the class of analytic functions defined in the form of

$$P(A,B) = \left\{ p \in \mathrm{A}\colon\ p(z) \prec \frac{1+Az}{1+Bz},\ \ -1 \leqslant B < A \leqslant 1,\ z \in E \right\},$$

where p has a series form given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,\ z \in E.$$

The class $P(A,B)$ is known as the class of Janowski and was introduced by Janowski [13]. If $P(1,-1)$, then it reduces to the class P, the well-known class of functions with positive real part consists of functions p that satisfy $\mathrm{Re}\, p(z) > 0$ and $p(0) = 1$.
If $p \in P$, then a Schwarz function $\upsilon \in H$ exists with $\upsilon(0) = 0$ and $|\upsilon(z)| < 1$ such that

$$p(z) = \frac{1+\upsilon(z)}{1-\upsilon(z)},\ z \in E.$$

We now introduce the subclass of star-like functions with respect to symmetric conjugate points connected to the sine function as follows:

Definition 1. *Let* $S^*_{SC}(\sin z)$ *be the class of functions defined by*

$$\frac{zf'(z)}{h(z)} \prec \varphi(z),\ z \in E, \tag{2}$$

where $h(z) = \frac{f(z)-\overline{f(-\overline{z})}}{2}$ and $\varphi(z) = 1 + \sin z$.
It is observed that the classes S^*_{SC} and $S^*_{SC}(A,B)$ consisting of star-like functions with respect to symmetric conjugate points defined by El-Ashwah and Thomas [11] and Ping and Janteng [26], respectively, are obtained if the right-hand side of (2) is changed to $\varphi(z) = \frac{1+z}{1-z}$ and $\varphi(z) = \frac{1+Az}{1+Bz}$, i.e.,

$$S^*_{SC} = \left\{ f \in \mathrm{A} : \mathrm{Re}\left(\frac{zf'(z)}{h(z)}\right) > 0,\ z \in E \right\}$$

and

$$S^*_{SC}(A,B)=\left\{f\in \mathrm{A}:\frac{zf'(z)}{h(z)}\prec\frac{1+Az}{1+Bz},\ -1\leqslant B<A\leqslant 1,\ z\in E\right\},$$

where $h(z)=\frac{f(z)-\overline{f(-\overline{z})}}{2}$. Besides that, some subclasses of star-like functions with respect to symmetric conjugate points are also studied from a different perspective by Halim [1] and Mohamad et al. [24]. These subclasses are defined as follows:

$$S^*_{SC}(\delta)=\left\{\mathrm{Re}\left(\frac{zf'(z)}{h(z)}\right)>\delta,\ 0\leqslant\delta<1,\ z\in E\right\} \tag{3}$$

and

$$S^*_{SC}(\alpha,\delta,A,B)=\left\{f\in\mathrm{A}:\left(e^{i\alpha}\frac{zf'(z)}{h(z)}-\delta-i\sin\alpha\right)\frac{1}{\tau_{\alpha\delta}}\prec\frac{1+Az}{1+Bz}\right\}, \tag{4}$$

where $h(z)=\frac{f(z)-\overline{f(-\overline{z})}}{2}$, $\tau_{\alpha\delta}=\cos\alpha-\delta,\ 0\leqslant\delta<1,$ and $|\alpha|<\frac{\pi}{2}$.

The problem of computing the bounds of the Hankel determinant has been studied in almost every subclass of A and has consistently piqued the interest of geometric functions theory researchers. The Hankel determinant of a function $f\in$ A whose elements are Taylor coefficients of $f\in$ A is defined as [27, 28]

$$H_{q,n}(f)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix},$$

where $q,n\in\mathbb{N}$ and $a_1=1$. It plays an important role in the study of singularities and power series with integral coefficients [7, 8]. On the other hand, it is known that the Toeplitz matrices are closely related to the Hankel matrices and one of the well-studied classes of structured matrices. Unlike the Hankel matrices, which have constant entries along the reverse diagonals, Toeplitz matrices have constant entries along the diagonals. Toeplitz matrices have a wide range of uses in both pure and applied mathematics, which has sparked some of the most important advances in research on the Toeplitz determinants, kernel, operators, and q-deformed Toeplitz matrices (see Ye and Lim [35]). Thomas and Halim [32] introduced the symmetric Toeplitz determinant of a function $f\in$ A whose elements are Taylor coefficients of $f\in$ A and it is defined as

$$T_{q,n}(f)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_n & \cdots & a_{n+q-2}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n\end{vmatrix},$$

where $q,n\in\mathbb{N}$ and $a_1=1$. It is worth noting that the exact bounds of the Hankel and Toeplitz determinants for some subclasses of S are still not sharp and are yet undiscovered. For recent work especially related to the class consisting of star-like functions with respect to other points, i.e., symmetric points, conjugate points, and symmetric conjugate points, see ([2, 16, 22, 24, 30, 31, 34] and reference therein).

Recently, the Hankel and Toeplitz determinants of a function $f \in$ A whose elements are logarithmic coefficients of $f \in$ A have been introduced by Kowalczyk and Lecko [17, 18] and Giri and Kumar [12], respectively, as follows:

$$H_{q,n}\left(F_f/2\right) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix},$$

and

$$T_{q,n}\left(\gamma_f\right) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_n & \cdots & \gamma_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q-2} & \cdots & \gamma_n \end{vmatrix},$$

where γ_n, $n \geqslant 1$, the logarithmic coefficients, are defined in the series form

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n.$$

In particular, for a function given in (1), the logarithmic coefficients γ_n, $n = 1, 2, 3, 4$ are given as follows:

$$\gamma_1 = \frac{1}{2}a_2, \tag{5}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}{a_2}^2\right), \tag{6}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}{a_2}^3\right), \tag{7}$$

and

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + {a_2}^2a_3 - \frac{1}{2}{a_3}^2 - \frac{1}{4}{a_2}^4\right). \tag{8}$$

The logarithmic coefficients have great importance, for instance, these coefficients helped Kayumov [14] to solve Brennan's conjecture for conformal mapping and estimation of the logarithmic coefficients can be transferred to the Taylor coefficients of univalent functions via the Lebedev–Milin inequalities (see [9, 19–21] for details). Due to the great importance of logarithmic coefficients and the Hankel and Toeplitz determinants, some recent works on this problem that relate to the theory of univalent functions have been studied in [3–5, 12, 15, 16, 18, 23, 25, 29, 33, 36] but only a few papers have been published for the class of star-like functions with respect to other points. Motivated by these works, in this paper, we obtain the upper bounds of the Taylor coefficients $|a_n|$, $n = 2, 3, 4, 5$, logarithmic coefficients $|\gamma_n|$, $n = 1, 2, 3, 4$, and hence some cases of the Hankel determinant as well as Toeplitz determinant, whose both entries are logarithmic coefficients, i.e., $|H_{2,1}(F_f/2)|$, $|H_{2,2}(F_f/2)|$, $|T_{2,1}(\gamma_n)|$, and $|T_{2,2}(\gamma_n)|$ for the functions in the class $S^*_{SC}(\sin z)$ as defined in Definition 1.

2. Preliminary results

In this section, we give some lemmas to prove our main results.

Lemma 1. ([9]) *For a function $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in E$, the sharp inequality $|p_n| \leqslant 2$ holds for each $n \geqslant 1$. Equality holds for the function $p(z) = \frac{1+z}{1-z}$.*

Lemma 2. ([10]) *Let $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leqslant 2max\{1, |2\mu - 1|\}, \ 1 \leqslant k \leqslant n-1.$$

If $|2\mu - 1| \geqslant 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

Lemma 3. ([6]) *Let $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in E$ and $\alpha, \beta, \gamma \in \Re$. Then*

$$\left|\alpha {p_1}^3 - \beta p_1 p_2 + \gamma p_3\right| \leqslant 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|.$$

3. Main results

This section is devoted to the proof of our main results. We will now determine the upper bounds of the Taylor coefficients, logarithmic coefficients, and Hankel and Toeplitz determinants of logarithmic coefficients, respectively, as follows:

3.1. Taylor coefficients

Theorem 1. *If f is of the form (1) belongs to $S^*_{SC}(\sin z)$, then*

$$|a_2| \leqslant \frac{1}{2},$$
$$|a_3| \leqslant \frac{1}{2},$$
$$|a_4| \leqslant \frac{1}{4},$$

and

$$|a_5| \leqslant \frac{1}{2}.$$

Proof. Since $f \in S^{*}_{SC}(\sin z)$, from definition of subordination, there exists a Schwarz function υ with $\upsilon(0) = 0$ and $|\upsilon(z)| < 1$, and from (2) we have

$$\frac{zf'(z)}{h(z)} = 1 + \sin \upsilon(z), \; z \in E, \tag{9}$$

where $h(z) = \frac{f(z) - \overline{f(-\overline{z})}}{2}$.
Assuming that

$$p(z) = \frac{1 + \upsilon(z)}{1 - \upsilon(z)} = 1 + \sum_{n=1}^{\infty} k_n z^n \in P.$$

This leads to

$$\upsilon(z) = \frac{p(z) - 1}{p(z) + 1}.$$

Hence, from the right-hand side of (9), we obtain

$$1 + \sin \upsilon(z) = 1 + \frac{1}{2}k_1 z + \left(\frac{k_2}{2} - \frac{{k_1}^2}{4}\right) z^2 + \left(\frac{5{k_1}^3}{48} - \frac{k_1 k_2}{2} + \frac{k_3}{2}\right) z^3 + \left(\frac{k_4}{2} + \frac{5{k_1}^2 k_2}{16} - \frac{{k_2}^2}{4} - \frac{k_1 k_3}{2} - \frac{{k_1}^4}{32}\right) z^4 + \cdots.$$

On the other hand, since f of the form (1), this gives

$$zf'(z) = z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + 5a_5 z^5 + \cdots$$

and

$$h(z) = z + a_3 z^3 + a_5 z^5 + \cdots.$$

Further, we have from (9) that

$$\begin{aligned} & z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + 5a_5 z^5 + \cdots \\ & = \left(z + a_3 z^3 + a_5 z^5 + \cdots\right)\left[1 + \frac{1}{2}k_1 z + \left(\frac{k_2}{2} - \frac{{k_1}^2}{4}\right) z^2 + \left(\frac{5{k_1}^3}{48} - \frac{k_1 k_2}{2} + \frac{k_3}{2}\right) z^3 \right. \\ & \left. + \left(\frac{k_4}{2} + \frac{5{k_1}^2 k_2}{16} - \frac{{k_2}^2}{4} - \frac{k_1 k_3}{2} - \frac{{k_1}^4}{32}\right) z^4 + \cdots\right]. \end{aligned} \tag{10}$$

Expanding the series and comparing the coefficients of z^n, $n = 1, 2, 3, 4, 5$ on both sides of (10) yields

$$a_2 = \frac{k_1}{4}, \tag{11}$$

$$a_3 = \frac{1}{8}\left(2k_2 - {k_1}^2\right), \tag{12}$$

$$a_4 = \frac{1}{96}\left({k_1}^3 - 9k_1 k_2 + 12k_3\right), \tag{13}$$

and

$$a_5 = \frac{1}{64}\left(8k_4 + 3{k_1}^2 k_2 - 2{k_2}^2 - 8k_1 k_3\right). \tag{14}$$

Using triangle inequality and Lemma 1 in (11), we get

$$|a_2| \leqslant \frac{1}{2}.$$

Now, applying Lemma 2 in (12) and Lemma 3 in (13), respectively, implies that

$$|a_3| = \frac{1}{8}\left|2k_2 - {k_1}^2\right| \leqslant \frac{1}{4}\left[2\max\left\{1, \left|2\left(\frac{1}{2}\right) - 1\right|\right\}\right] = \frac{1}{2}$$

and

$$|a_4| = \frac{1}{96}\left|{k_1}^3 - 9k_1 k_2 + 12k_3\right| \leqslant \frac{1}{96}\left[2\left|1\right| + 2\left|9 - 2\left(1\right)\right| + 2\left|1 - 9 + 12\right|\right] = \frac{1}{4}.$$

Rearranging the terms in (14), we can rewrite it as

$$|a_5| = \frac{1}{64}\left|8\left(k_4 - \nu_1 k_1 k_3\right) - 2k_2\left(k_2 - \nu_2 {k_1}^2\right)\right|,$$

where $\nu_1 = 1$ and $\nu_2 = \frac{3}{2}$.
Consequently, by applying Lemma 1 and Lemma 2 as well as the triangle inequality, we obtain

$$|a_5| \leqslant \frac{1}{2}.$$

This completes the proof of Theorem 1.

3.2. Logarithmic coefficients

Theorem 2. *If f is of the form* (1) *belongs to* $S^*_{SC}(\sin z)$, *then*

$$|\gamma_1| \leqslant \frac{1}{4},$$
$$|\gamma_2| \leqslant \frac{1}{4},$$
$$|\gamma_3| \leqslant \frac{1}{8},$$

and

$$|\gamma_4| \leqslant \frac{7}{16}.$$

Proof. Putting (11)-(14) in (5)-(8), we obtain

$$\gamma_1 = \frac{k_1}{8}, \tag{15}$$

$$\begin{aligned}\gamma_2 &= \frac{1}{2}\left[\frac{1}{8}\left(2k_2 - {k_1}^2\right) - \frac{1}{2}\left(\frac{k_1}{4}\right)^2\right] \\ &= \frac{1}{8}\left(k_2 - \frac{5}{8}{k_1}^2\right),\end{aligned} \tag{16}$$

$$\begin{aligned}\gamma_3 &= \frac{1}{2}\left[\frac{1}{96}\left({k_1}^3 - 9k_1k_2 + 12k_3\right) - \left(\frac{k_1}{4}\right)\left(\frac{1}{8}\left(2k_2 - {k_1}^2\right)\right) + \frac{1}{3}\left(\frac{k_1}{4}\right)^3\right] \\ &= \frac{1}{128}\left(3{k_1}^3 - 10k_1k_2 + 8k_3\right),\end{aligned} \tag{17}$$

and

$$\begin{aligned}\gamma_4 &= \frac{1}{2}\left[\frac{1}{64}\left(8k_4 + 3{k_1}^2k_2 - 2{k_2}^2 - 8k_1k_3\right) - \frac{k_1}{4}\left(\frac{1}{96}\left({k_1}^3 - 9k_1k_2 + 12k_3\right)\right)\right. \\ &\quad \left. + \left(\frac{k_1}{4}\right)^2\left(\frac{1}{8}\left(2k_2 - {k_1}^2\right)\right) - \frac{1}{2}\left(\frac{1}{8}\left(2k_2 - {k_1}^2\right)\right)^2 - \frac{1}{4}\left(\frac{k_1}{4}\right)^4\right] \\ &= \frac{1}{6144}\left(384k_4 + 360{k_1}^2k_2 - 192{k_2}^2 - 480k_1k_3 - 59{k_1}^4\right).\end{aligned} \tag{18}$$

The bounds of $|\gamma_1|$, $|\gamma_2|$, and $|\gamma_3|$ follow from Lemma 1, Lemma 2, and Lemma 3, respectively.
On the other hand, rearranging the terms in (18), we get

$$\gamma_4 = \frac{1}{6144}\left(384\left(k_4 - \mu{k_2}^2\right) - k_1\left(\alpha{k_1}^3 - \beta k_1k_2 + \gamma k_3\right)\right),$$

where $\mu = \frac{1}{2}$, $\alpha = 59$, $\beta = 360$, and $\gamma = 480$.
Hence, implementing Lemma 2 and Lemma 3, we get the desired bound of $|\gamma_4|$. This completes the proof of Theorem 2.

3.3. Hankel determinant of logarithmic coefficients

Theorem 3. *If $f \in S^*_{SC}(\sin z)$ and has the series representation* (1)*, then*

$$|H_{2,1}(F_f/2)| \leqslant \frac{87}{1024}.$$

Proof. In view of (15)-(17), we have

$$\begin{aligned}H_{2,1}(F_f/2) &= \gamma_1\gamma_3 - {\gamma_2}^2 \\ &= \frac{k_1}{8}\left(\frac{1}{128}\left(3{k_1}^3 - 10k_1k_2 + 8k_3\right)\right) - \left(\frac{1}{8}\left(k_2 - \frac{5}{8}{k_1}^2\right)\right)^2 \\ &= \frac{1}{64}\left(\frac{3}{16}{k_1}^4 - \frac{10}{16}{k_1}^2k_2 + \frac{1}{2}k_1k_3 - {k_2}^2 + \frac{5}{4}{k_1}^2k_2 - \frac{25}{64}{k_1}^4\right) \\ &= \frac{1}{4096}\left(-13{k_1}^4 + 40{k_1}^2k_2 + 32k_1k_3 - {k_2}^2\right).\end{aligned} \tag{19}$$

Rearranging the terms in (19), it becomes

$$H_{2,1}\left(F_f/2\right)=\frac{1}{4096}\left(-k_1\left(\chi {k_1}^3-\lambda k_1k_2+\eta 32k_3\right)-{k_2}^2\right),$$

where $\chi=13,\ \lambda=40,$ and $\eta=-32$.
By applying the triangle inequality as well as Lemma 1 and Lemma 3, we get the desired inequality.

Theorem 4. *If $f\in S^*_{SC}(\sin z)$ and has the series representation* (1)*, then*

$$\left|H_{2,2}\left(F_f/2\right)\right|\leqslant\frac{33}{256}.$$

Proof. In view of (16)-(18), we can establish

$$\begin{aligned}H_{2,2}\left(F_f/2\right)&=\gamma_2\gamma_4-{\gamma_3}^2\\&=\frac{1}{8}\left(k_2-\frac{5}{8}{k_1}^2\right)\left(\frac{1}{6144}\left(384k_4+360{k_1}^2k_2-192{k_2}^2-480k_1k_3-59{k_1}^4\right)\right)\\&\quad-\left(\frac{1}{128}\left(3{k_1}^3-10k_1k_2+8k_3\right)\right)^2\\&=\frac{1}{393216}\left(3072k_2k_4+1440{k_1}^2{k_2}^2-1536{k_2}^3-832{k_1}^4k_2-1920{k_1}^2k_4\right.\\&\quad\left.+1248{k_1}^3k_3+79{k_1}^6-1536{k_3}^2\right).\end{aligned}\tag{20}$$

Further, rearranging the terms in (20), we can rewrite it in the following expression:

$$\begin{aligned}H_{2,2}\left(F_f/2\right)=\frac{1}{393216}&\left[\left(3072k_4\left(k_2-\frac{5}{8}{k_1}^2\right)-1536{k_2}^2\left(k_2-\frac{15}{16}{k_1}^2\right)\right.\right.\\&\left.\left.+{k_1}^3\left(79{k_1}^3-832k_1k_2+1248k_3\right)-1536{k_3}^2\right)\right].\end{aligned}$$

Hence, making use of Lemma 1, Lemma 2, and Lemma 3 yields the desired bound.

3.4. Toeplitz determinant of logarithmic coefficients

Theorem 5. *If $f\in S^*_{SC}(\sin z)$, then*

$$\left|{\gamma_1}^2-{\gamma_2}^2\right|\leqslant\frac{65}{256}.$$

Proof. Using (15) and (16), we obtain

$$\begin{aligned}{\gamma_1}^2-{\gamma_2}^2&=\frac{{k_1}^2}{64}-\frac{1}{64}\left(k_2-\frac{5}{8}{k_1}^2\right)^2\\&=\frac{1}{64}\left(\frac{5}{4}{k_1}^2\left(k_2-\frac{25}{80}{k_1}^2\right)+{k_1}^2-{k_2}^2\right).\end{aligned}\tag{21}$$

Applying the triangle inequality and Lemma 1 and Lemma 2, we get the desired inequality.

Theorem 6. *If* $f \in S^*_{SC}(\sin z)$ *, then*

$$|T_{2,2}(\gamma_n)| \leqslant \frac{11}{32}.$$

Proof. Making use of (16) and (17), upon simplification, we have

$$\begin{aligned} T_{2,2}(\gamma_n) &= {\gamma_2}^2 - {\gamma_3}^2 \\ &= \frac{1}{64}\left(k_2 - \frac{5}{8}{k_1}^2\right)^2 - \frac{1}{16384}\left(3{k_1}^3 - 10k_1k_2 + 8k_3\right)^2 \\ &= \frac{1}{16384}\left(256{k_2}^2 - 320{k_1}^2k_2 + 100{k_1}^4 - 9{k_1}^6 + 60{k_1}^4k_2 - 100{k_1}^2{k_2}^2 - 48{k_1}^3k_3 \right. \\ &\quad \left. +160k_1k_2k_3 - 64{k_3}^2\right) \end{aligned}$$

and equivalently,

$$\begin{aligned} T_{2,2}(\gamma_n) = \frac{1}{16384}&\left[256k_2\left(k_2 - \frac{5}{4}{k_1}^2\right) - 100{k_1}^2k_2\left(k_2 - \frac{3}{5}{k_1}^2\right) + 100{k_1}^4 - 9{k_1}^6 \right. \\ &\left. +k_3\left(-48{k_1}^3 + 160k_1k_2 - 64k_3\right)\right]. \end{aligned} \tag{22}$$

Applying Lemma 1, Lemma 2, and Lemma 3 on (22), we can obtain the desired bound.

4. Conclusion

This study was inspired by a number of previous studies. In this paper, we have obtained the upper bounds of some coefficient problems for functions in the class $S^*_{SC}(\sin z)$ including Taylor coefficients, logarithmic coefficients, and Hankel and Toeplitz determinants of logarithmic coefficients. The results provided in this paper perhaps could be the subject of further research related to the higher-order Hankel and Toeplitz determinants of logarithmic coefficients and other coefficient problems, for instance, the Fekete-Szegö functional, the Toeplitz and Hermitian-Toeplitz determinants, and the Krushkal inequality for functions from the subclass of star-like functions with respect to other points. Additionally, for another particular value of φ, several other classes of functions that are star-like with respect to symmetric conjugate points can also be studied.

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